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DER

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VAN

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EERSTE SECTIE

(Wiskunde - Natuurkunde - Scheikunde - Kristallenleer - Sterrenkunde - Weerkunde en Ingenieurswetenschappen.)

DEEL XI

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ERRATA.

Page	5,	line	12	from	top	replace	„edges"	by	„limits l_k"
„	14,	„	8	„	„	„	$7a$	„	$7c$
„	„ ,	„	9	„	„	„	$7b$	„	$7a$
„	16,	„	15	„	„	„	S_n	„	S_{n-1}
„	20,	„	1	„	„	„	19β	„	19α
„	21,	„	8	„	„	„	22γ, 22β	„	22π, 22γ

Geometrical deduction of semiregular from regular polytopes and space fillings

BY

Mrs. A. BOOLE STOTT.

Verhandelingen der Koninklijke Akademie van Wetenschappen te Amsterdam.

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Geometrical deduction of semiregular from regular polytopes and space fillings

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Introduction.

1. The object of this memoir is to give a method by which bodies having a certain kind of semiregularity may be derived from regular bodies in an Euclidean space of any number of dimensions; and space fillings of the former from space fillings of the latter.

These space fillings or nets for threedimensional space have been given in a paper entitled „Sulle reti di poliedri regolari e semiregolari e sulle corrispondenti reti correlative" by Mr. A. ANDREINI [1]), who deduced them by means of the angles of the different polyhedra. Photographs prepared for the stereoscope, taken from that paper, representing the various semiregular space fillings were sent to me by Prof. SCHOUTE to whom I desire to record here my thanks for the generous help he has given me during the whole course of this investigation. These photographs suggested a method by which at once the semiregular bodies and the manner in which they combine to fill fourdimensional space could be derived from regular polytopes and nets in that space. It will be seen that this method can be applied to spaces of any other number of dimensions.

The semiregularity considered here is that in which there is one kind of vertex and one length of edge [2]), and the symbols used

[1]) *Memorie della Società italiana della Scienze* (detta dei XL), serie 3a, tomo XIV.

[2]) So the greater part of the forms called semiregular here will have a degree of regularity less than $\frac{1}{2}$ in the scale of Mr. E. L. ELTE.

for the polyhedra of this description, almost the same as those given by ANDREINI, are as follows:

T, C, O, D, I indicate the five regular polyhedra, using the initial letters of their ordinary names Tetrahedron, Cube, Octahedron, Dodecahedron, Icosahedron; and if p_n indicates a regular polygon with n vertices we have

tT = truncated T	limited	by	$4p_6$	and	$4p_3$,
tC = ″ C	″	″	$6p_8$	″	$8p_3$,
tO = ″ O	″	″	$8p_6$	″	$6p_4$,
tD = ″ D	″	″	$12p_{10}$	″	$20p_3$,
tI = ″ I	″	″	$20p_6$	″	$12p_5$,
CO = C and O in equilibrium	″	″	$6p_4$	″	$8p_3$,
ID = I ″ D ″ ″	″	″	$12p_5$	″	$20p_3$,
RCO = combination of rhombic D, C and O	″	″	$18p_4$	″	$8p_3$,
RID = ″ ″ ″ Tr [1]), I ″ D	″	″	$12p_5$, $30p_4$	″	$20p_3$,
tCO = truncated [2]) CO	″	″	$6p_8$, $12p_4$	″	$8p_6$,
tID = ″ ID	″	″	$12p_{10}$, $30p_4$	″	$20p_6$.

Moreover we want:

$P_3, P_4,\ldots$ for threedimensional triangular prisms, square prisms (cubes), etc.

$P_C, P_O,\ldots$ for fourdimensional prisms on a cube, an octahedron,... as base,

P (3 ; 3) or simply (3 ; 3) for a prismotope [3]) with two groups of threedimensional prisms P_3 as limiting bodies,

P (6 ; 8) or simply (6 ; 8) for a prismotope having for limiting bodies six octagonal and eight hexagonal prisms, etc.

2. The transformation of the regular into the semiregular bodies and space fillings can be carried out by means of two inverse operations which may be called *expansion* and *contraction*.

In order to define these operations conveniently, the vertices, edges, faces, limiting bodies,... of a regular polytope are called

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[1]) By Tr we indicate the solid limited by 30 lozenges in planes through the edges of I or D normal to the lines joining the centre to the midpoint of each edge.

[2]) According to custom the word "truncated" is used here, though this body and the next one cannot be derived from the CO and the ID by truncation.

[3]) This body is also a „simplotope" as the describing polygons (placed here in planes perfectly normal to each other) are triangles (compare SCHOUTE's „Mehrdimensionale Geometrie", vol. II, p. 128).

In general a prismotope is generated in the following way:

Let S_p and S_q be two spaces of p and q dimensions having only one point in common; let P be a polytope in S_p, Q a polytope in S_q. Now move S_p with P in it parallel to itself, so as to make any vertex of P describe all the points of Q. Then P generates the prismotope. Here we have to deal only with the case of two planes ($p = q = 2$); by the symbol (6 ; 8) we will indicate the polytope limited by eight hexagonal and six octagonal prisms obtained in the indicated manner if we start from a hexagon and an octagon situated in two planes perfectly normal to each other.

its limits (l) and are denoted respectively by the symbols l_0, l_1, l_2, l_3...

I. The operations of expansion and contraction.

Definition of expansion.

3. Let O be the centre of a regular polytope and M_1, M_2, M_3.. the centres of its limits l_1, l_2, l_3... The operation of expansion e_k consists in moving the limits l_k to equal distances away from O each in the direction of the line $O\,M_k$ which joins O to its centre, the limits l_k remaining parallel to their original positions, retaining their original size, and being moved over such a distance that the two new positions of any vertex, which was common to two adjacent edges in the original polytope, shall be separated by a length equal to an edge.

The polytope determined by the new positions of the limits l_k will have the kind of semiregularity described above. The limits l_k are said to be the subject of expansion or briefly the *subject*; and the new polytope is denoted by the symbol of the original regular polytope preceded by the symbol e_k.

A few particular cases, in 2, 3 and 4 dimensions, will now be examined.

Examples of the e_1 expansion.

4. Here the edges (l_1) are the subject.

It is evident that this operation applied to any regular polygon changes it into a regular polygon having the same length of edge and twice as many sides. In Fig. 1*a* a square is changed into an octagon by the application of the e_1 expansion [1]).

Fig. 1*b* shews the e_1 expansion of a cube. The real movement of any edge AB is in the direction of the line OM_1 but that movement may be resolved into two. Thus instead of moving AB directly to the position A_1B_1 it might have been moved to $A'B'$ or to $A''B''$ and then to A_1B_1. If the movements of all the edges be thus resolved the result is the same as if the faces AC, AD... (Fig. 1*c*) of the original cube had been first transformed into octagons by an e_1 expansion of each in its own plane, and then moved

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[1]) In these drawings the *thick* lines represent edges of regular polytopes in their original or in new positions, the *thin* lines edges introduced by expansion.

away from the centre O until the edges $A'B'$ and $A''B''$, which correspond to an edge AB of the cube, are coincident and become the common edge of two octagons (transformed squares). It is to be noticed that as each vertex of a cube is common to three edges (three members of the subject) it takes three new positions, which, owing to the regularity of the cube, are the three vertices of an equilateral triangle. Thus the faces of the cube have been expanded into octagons and the vertices into triangles.

Fig. 2*a* shews the e_1 expansion of a tetrahedron. Each face is changed into a hexagon, each vertex into a triangle. Here again a vertex of the tetrahedron is common to three members of the subject; the result is a tT.

Fig. 2*b* shows the same expansion of an octahedron. Each face is changed into a hexagon; but each vertex into a square because in an octahedron each vertex belongs to four edges (four members of the subject); the result is a tO.

From these examples it is easy to find the e_1 expansion of an icosahedron and of a dodecahedron.

5. This investigation leads to the determination of the e_1 expansion applied to the fourdimensional polytopes. For instance in the C_8 each cube is transformed (in its own space) by the e_1 expansion and becomes a tC (Figs. 1*b* and 2*c*). These transformed cubes must be so adjusted that an edge which was in the C_8 common to three cubes [1]) is, in its new position, common to three transformed cubes. Again each vertex in a C_8 is common to four edges and must take four new positions which are the four vertices of a regular tetrahedron. Thus the vertex of the C_8 is expanded into a tetrahedron, which is said to be of vertex *import*. This tetrahedron might have been determined in another way; for four cubes meet in a vertex of a C_8 and in each the vertex is changed into a triangle; therefore a vertex of C_8 is replaced by a body limited by four triangles i. e. a tetrahedron.

The two kinds of limiting body of the new polytope $e_1 C_8$ are shewn in Fig. 2*c*; in Fig. 2*d* are shewn the limiting bodies of $e_1 C_5$.

In C_{16}, where six edges meet in a vertex, the e_1 expansion changes each tetrahedron into a tT (Fig. 2*a*) and each vertex into an octahedron (of vertex import) whose vertices are the six new positions of a vertex of the C_{16}.

Again in C_{24} eight edges meet in a vertex, so that the e_1 expan-

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[1]) In order to facilitate the application of the operation of expansion it is desirable to have at hand a table of incidences; this is provided on Table III.

sion here changes each octahedron into a tO (Fig. 2*b*) and each vertex into a cube (of vertex import) whose vertices are the eight new positions taken by a vertex of C_{24}.

In a similar manner the e_1 expansions of C_{120} and C_{600} may be determined.

6. *Rule.* These examples lead to the general rule for the e_1 expansion of a regular fourdimensional polytope P. The limiting bodies of P are transformed by the e_1 expansion and the vertices expanded into regular polyhedra each having as many vertices as there are edges meeting in a vertex of P.

Examples of the e_2 expansion.

7. As the faces are the subject in this expansion there can be no application to a single polygon in twodimensional space.

The e_2 expansion of a cube, an *RCO,* is shewn in figure 3*a*; there are three groups of faces:

1st :	squares	corresponding	to	the	faces	of	the	original	cube
2nd :	,,	,,	,,	,,	edges	,,	,,	,,	,,
3rd :	triangles	,,	,,	,,	vertices	,,	,,	,,	,,

In this expansion of any regular polyhedron the faces of the first group are like those of the original polyhedron; the faces of the second group are always squares, since they are determined by the two new positions of an edge of the original polyhedron; those of the third group are triangles, squares or pentagons according as a vertex of the original polyhedron belongs to three, four or five faces.

As the cube and the octahedron are reciprocal bodies, the number of vertices lying in a face of one being equal to the number of faces meeting in a vertex of the other, it follows that the e_2 expansion of the octahedron is also an *RCO* (Fig. 3*b*).

Again the tetrahedron is self reciprocal, the number of vertices lying in a face being equal to the number of faces meeting in a vertex; so in the e_2 expansion the faces of vertex import are, like the faces of the tetrahedron, equilateral triangles (Fig. 4).

The e_2 expansion of the icosahedron and dodecahedron, which are reciprocal bodies, is an *RID*.

8. The e_2 expansion of the C_8 transforms each cube into an *RCO* and, as in the C_8 each face is common to two cubes, so those faces in the *RCO* which are faces of the cubes in new positions must now be common to two *RCO*. In the C_8 each edge belongs to three faces, so in the new polytope each edge takes

three new positions which are the three parallel edges of a right prism on an equilateral triangular base.

This manner of determining the prism (expanded edge) bears the most direct relation to the particular expansion under consideration, namely that in which the faces are the subject; but it could have been determined otherwise. Thus in a C_8 three cubes meet in an edge and as each is changed into an RCO, its edges are changed into squares, so that instead of three coincident edges there are now three squares, the side faces of a right prism.

Again in the C_8 each vertex belongs to six faces and therefore must assume six positions. From this it is evident that the body taking the place in the new polytope of the vertex in the C_8 has six vertices and it remains to determine its faces.

In figure 5 are represented, in their true relative positions as far as threedimensional space will allow, two of the four RCO and two of the four P_3 which have taken the place of the four cubes and the four edges meeting in a vertex of the C_8. It shews that each RCO supplies a triangular face and each prism a triangular face — all equilateral — to the body that takes the place of the vertex of the C_8. This body therefore is a regular octahedron, four of whose faces are in contact with RCO and four with P_3.

The new polytope then, $e_2 C_8$, is limited by 8 RCO, 32 P_3 of edge import, 16 O of vertex import.

9. *Rule.* The rule for the e_2 expansion of a regular fourdimensional polytope P may be stated thus:

The limiting bodies of P are transformed by the e_2 expansion. The edges are expanded into prisms each having as many edges parallel to the axis as there are faces meeting in an edge of P. The vertices are expanded into bodies having two groups of faces, one kind of edge, and as many vertices as there are faces meeting in a vertex of P. One group of faces is supplied by the bases of the prisms of edge import and of these the number is equal to the number of edges meeting in a vertex of P; the other is supplied by the expanded vertices of the transformed limiting bodies, of which the number is equal to the number of limiting bodies meeting in a vertex of P.

Examples of the e_3 expansion.

10. Here the limiting bodies are the subject; and it is at once evident that this expansion applied to reciprocal fourdimensional

bodies, e. g. to C_8 and C_{16}, also to C_{120} and C_{600}, must produce the same result; while applied to a self reciprocal form it produces a polytope whose limiting bodies of vertex import are like the original limiting bodies and of the same number, and whose limiting bodies of face import are of the same number and kind as those of edge import.

Thus as in the C_8 each face belongs to two, each edge to three, and each vertex to four cubes, it follows that in the expansion each face takes two, each edge three, and each vertex four positions. The $e_3 C_8$ is therefore limited by 8 cubes of body import (cubes of the original C_8), 24 P_4 of face import, 32 P_3 of edge import, and 16 tetrahedra of vertex import (Fig. 6*a*). In the C_{16} each face belongs to two, each edge to four, each vertex to eight tetrahedra, so in the expansion each face takes two, each edge four, and each vertex eight positions and the $e_3 C_{16}$ is limited by 16 tetrahedra of body import, 32 P_3 of face import, 24 P_4 of edge import; and 8 cubes of vertex import (Fig. 6*b*). These two polytopes are alike except that the *imports* are *reciprocal.*

11. Generally there are four groups of limiting bodies:

1st: polyhedra of body import like the limiting bodies of the original cell,

2nd: prisms of face import defined by their bases (two positions of each face of the original cell),

3rd: prisms of edge import defined by their edges parallel to the axis (as many positions of an edge as there are bodies meeting in an edge of the original cell),

4th: polyhedra of vertex import having as many vertices as there are bodies meeting in a vertex of the original cell.

So in $e_3 C_5$ there are 10 T, 20 P_3; in C_{24} there are 48 O, 192 P_3.

This expansion of a C_{120} and a C_{600} (reciprocal cells) can easily be determined.

12. *Rule.* The rule for the e_3 expansion of a regular polytope P of fourdimensional space is as follows:

The limiting bodies of P are moved apart (untransformed).

The faces are replaced by prisms whose bases are parallel positions of a face of P. The edges are replaced by prisms each having as many edges parallel to the axis as there are limiting bodies meeting in an edge of P. Each vertex is replaced by a regular polyhedron, the number of whose vertices is equal to the number of limiting bodies meeting in a vertex of P.

Generalization.

13. The foregoing result may be generalized thus. If any set of limits e_r be the subject of expansion in a regular polytope P_n in a space of n dimensions the polytope P_n' defined by the new positions of the members of the subject has for its limits l'_{n-1}:

1st: a group consisting of the limits l_{n-1} of P_n transformed by the e_r expansion $(e_r\ l_{n-1})$,

2nd: a group of vertex import, each member of the group being determined by its vertices, the number of which is equal to the number of limits l_r meeting in a vertex of P_n and having one kind of edge. This polytope is regular in the e_1 and the e_{n-1} expansions. These two groups are the principle ones.

3rd: there are besides various kinds of prisms. Those of edge import (1-import) are determined by the new positions of an edge of P_n and the number of these positions is equal to the number of limits l_r meeting in an edge of P_n. The prisms of face import (2-import) are determined each by the new positions of a face of P_n, and the number of these is equal to the number of limits l_r meeting in a face of P_n and so on. The whole series of prisms is as follows: 1-import, 2-import, ...r—1-import.

Combination of operations.

14. The expansions described above have been applied to regular bodies according to the definition given on page 5, transforming them into bodies possessing a particular kind of semiregularity.

The question now arises: can these semiregular bodies be transformed by the application of any further expansion without having lost the kind of semiregularity defined above?

It is evident in the first place that a movement of all the edges or of all the faces would produce bodies with edges of different lenghts. But an inspection of the transformed bodies in three-dimensional space (Figs. 1 *b*, 2 *a* and 2 *b*) shows that in each of the polyhedra tC, tT and tO there are two groups of faces, each of which taken alone defines the polyhedron: one group corresponds to the faces (expanded), the other to the vertices (expanded) of the original polyhedron, and these two groups differ as to a particular characteristic.

The members of the first group are in contact with members of the same group; the members of the second are separated by at least the length of an edge from members of their own group. As

the operation of expansion applied to a set of limits has the effect of separating any two adjacent members, it follows that the first group can, the second group cannot, be made the subject of expansion.

For instance in $e_1 C$ (Fig. *7a*) the triangles cannot be moved away from the centre without increasing the length of the edges joining them, but the octagons may be moved away from the centre until the edge AB common to two has assumed two new positions $A'B'$, $A''B''$ which are the opposite sides of a square. The new positions of the octagons define a polyhedron having the required kind of semiregularity. [1])

15. This double operation may be denoted by the symbol $e_2 e_1 C$ where it is understood that the faces forming the subject of the e_2 expansion are only those which have taken the place of faces in the original cube. Similarly the interpretation of the symbol $e_1 e_2 C$ is that the e_2 expansion is applied to a cube and that the subject of further expansion is composed of those faces which have taken the place of edges in the original cube. This is shewn in Fig. *7b* where the group of 12 squares (corresponding to the edges of the original cube) form the subject of expansion. These two figures *7a* and *7b* show that

$$e_1 e_2 C = e_2 e_1 C = tCO$$

and it is evident that the order in which the operations are applied to any regular polyhedron is indifferent, for the two operations could have been carried out simultaneously.

In Fig. *7c* is shewn the result of the double operation $e_2 e_1 O$ applied to an octahedron. This is also a tCO.

If the double operation be applied to a I and an D the result in both cases will be a tID.

This body and the tCO are incapable of further expansion.

16. Thus it appears that three expansions can be applied to the cube, octahedron, dodecahedron, icosahedron, namely e_1, e_2, $e_1 e_2$. But more can be done with the tetrahedron owing to the fact that it is self reciprocal.

Fig. *7d* and *7e* show respectively the result of the $e_2 e_1$ and the $e_1 e_2$ expansion applied to a tetrahedron, and the result in both cases is a tO which can be further expanded into a tCO (Fig. *7c*). Thus the self reciprocity of the tetrahedron allows an expansion which cannot be carried out in the other four polyhedra. The

[1]) Here the group of octagons may be called the „independent" variable, while the triangles, which are transformed into hexagons, are the „dependent" ones.

combination of operations may be applied in the same way to the cells of fourdimensional space as one or two examples will show.

17. *Case* $e_1\, e_2\, C_8$. — The e_2 operation applied to a C_8 produces a polytope limited by 8 RCO, 32 P_3, 16 O (Fig. 5). The symbol $e_1\, e_2\, C_8$ directs that the new subject of expansion comprising those limiting bodies in $e_2\, C_8$ which correspond to edges of C_8, i. e. the 32 P_3, shall, themselves unchanged, be carried away from the centre (of the $e_2\, C_8$).

These P_3 in their new positions define the polytope sought. This movement changes the RCO and the O. Each RCO was derived from a cube by the e_2 expansion; the new expansion e_1 carries out the group of 12 squares (corresponding to the edges of the cube), thereby producing a tCO (Fig. 7*b*). In order to determine the change in the octahedron of vertex import it is only necessary to observe that four of its faces (those in contact with bases of P_3) are still in contact with them and are only changed in position, while the other four (those which were in contact with RCO) are changed into hexagons in contact with tCO. Thus the octahedron is changed into a tT. The effect on a single octahedron is the same as if its alternate faces had been made the subject of expansion (Fig. 8).

18. *Case* $e_2\, e_3\, C_8$. — The result of applying the e_3 operation to a C_8 is a polytope limited by cubes (original cubes of the C_8), P_4 of face import, P_3 of edge import, and tetrahedra of vertex import (Fig. 6*a*). The symbol e_2 directs that the square prisms of face import shall be moved away from the centre of $e_3\, C_8$, they themselves remaining unchanged except in position. These in their new positions define the new polytope and it only remains to determine in what manner their movement has modified the remaining limiting bodies of the $e_3\, C_8$. This can be seen at once in a drawing. In figure 9*a* are shewn seven limiting bodies of the $e_3\, C_8$; one is a cube of the original C_8, after having been separated by the e_3 movement from the adjacent cubes; three are cubes of face import interposed by the same movement between the cubes of the C_8; three are P_3 of edge import, their bases being faces of a tetrahedron of vertex import. The symbol e_2 directs that the cubes of face import are to be moved out. The result is shewn in figure 9*b*; the original cube is changed into an RCO, the P_3 into a P_6 and the T into a tT. It is necessary to bear in mind that only one limiting body of any polytope can be in threedimensional space at a time, and in representing several at once in it there must be either distortion of the limiting bodies or separation of faces and edges which ac-

tually coincide. Moreover the direction of the real movement cannot be represented; but valid conclusions may be drawn from diagrams such as these, if the mind always distinguishes between the actual and the apparent relation of parts.

These two examples suffice to show how the result of the combination of operations may be applied to the fourdimensional cells. There are seven expansions of each:

$$e_1, \quad e_2, \quad e_3, \quad e_1 e_2, \quad e_1 e_3, \quad e_2 e_3, \quad e_1 e_2 e_3,$$

but owing to the reciprocity of some of the figures these are not all different.

Thus it appears that in any expansion a set of limits, which define the body and which is such that each member is in contact with other members of the same set, may be made the subject of expansion.

Definition of contraction.

19. In each of the expansions e_1, e_2, e_3 . . . the resulting semiregular polytope may be reduced to the regular one from which it was derived, by an inverse operation which may be called contraction.

Here the limits which formed the subject of the expansion are moved towards the centre and brought back to their original positions.

The direct operation separates the members of the subject; the inverse operation brings them again into contact, annihilating the edges introduced by expansion. In both positions they define the polytope of which they are the limits.

The conditions necessary to the inverse operation are: 1st, the limits forming the subject must define the polytope; 2nd, no two members of the subject can be in contact before contraction.

The polytopes of vertex import always satisfy these conditions and can be made the subject of contraction. The symbol c is used to denote contraction. The import of the limits forming the subject is shown by means of subscripts, as in expansion.

Examples of contraction.

20. The inverse operation will be made clear by one or two examples.

In figure 10 the square $A\ B\ C\ D$ has been expanded by the e_1 operation; the edges of vertex import in the resulting octagon have been made the subject of the inverse operation, that is, they have been moved nearer to the centre so far that the edges of the original square are annihilated, and the final result is the square

E F G H, denoted by the symbol $c_0\, e_1\, S$ where S is the square *A B C D*.

In figure 10*b* is shown a cube transformed by the e_1 operation, i. e. an $e_1\, C$; the triangles of vertex import are brought nearer to the centre by the c_0 operation and the result is a *CO* whose symbol is now $c_0\, e_1\, C$.

Again, the *tCO* may be considered in two ways. It may be deduced from either the octahedron or the cube (compare Figs. 7*a* and 7*b*), so it may be denoted by $e_1\, e_2\, C$ or $e_1\, e_2\, O$. Though the identity of these results may be expressed in the form of an equation: $e_1\, e_2\, C = e_1\, e_2\, O$, it must still be borne in mind that the imports are different. Let each of these symbols be preceded by c_0. What are the results? If the *tCO* has been derived from the cube, the hexagons are of vertex import; if, on the other hand, it has been derived from the octahedron, the octagons are of vertex import. Thus the symbol $c_0\, e_1\, e_2\, C$ indicates that the hexagons, and the symbol $c_0\, e_1\, e_2\, O$ that the octagons, are the subject of the inverse operation whence $c_0\, e_1\, e_2\, C = tO$ (Fig. 7*c*), $c_0\, e_1\, e_2\, O = tC$ (Fig. 7*a*). But the octagons correspond to the faces of the cube and the hexagons to the faces of the octahedron, so that $c_0\, e_1\, e_2\, C = c_2\, e_1\, e_2\, O$, $c_0\, e_1\, e_2\, O = c_2\, e_1\, e_2\, C$.

21. An example will show the combination of inverse operations. The *tCO* derived from a cube (Fig. 11*a*) may be reduced to an octahedron by moving the squares and the hexagons nearer to the centre; the *tCO* derived from an octahedron (Fig. 11*b*) may be reduced to a cube by moving the squares and the octagons nearer to the centre.

These operations are denoted respectively by the equations

$$c_0\, c_1\, e_1\, e_2\, C = O \quad , \quad c_0\, c_1\, e_1\, e_2\, O = C.$$

22. In figure 5 are shown, the limiting bodies of an $e_2\, C_8$. If the octahedra of vertex import be made the subject of the inverse operation, the following changes will take place: each P_3, separating two neighbouring octahedra, is reduced to two coincident triangles. This annihilates the edges of the prism parallel to the axis. But these are the edges of the original C_8 in the new positions due to expansion and if these be annihilated each *RCO* will be reduced to an octahedron. Thus the new body is a C_{24}, eight of whose limiting bodies are compressed *RCO*, while sixteen are of vertex import in the expansion $e_2\, C_8$.

As in the enumeration of the polytopes and the nets given in the three Tables only the c_0 appears, c_0 has been replaced by c.

Partial operations.

23. It has been seen that in both expansion and contraction it is a necessary condition that the subject of operation shall define the polytope both before and after the movement.

In expansion, each member of the subject must be in contact with other members. In contraction, each member must be separated from the other members by at least the length of an edge.

It sometimes happens that one of these conditions is satisfied by a *group* consisting of the alternate members of a set of limits. Such a group may then be made the subject of expansion or contraction. If the members be in contact, they may be made the subject of expansion; if they be not in contact, they may be made the subject of contraction.

24. Thus, an octahedron is defined by a group of four alternate triangles, but each of these triangles is in contact with the other three, so that these four may be made the subject of expansion. This partial operation, which changes the octahedron into a truncated tetrahedron, is denoted by the symbol $\frac{1}{2} e_2 O$. So $\frac{1}{2} e_2 O = tT$.

Again, a CO whose symbol is $c_0 e_1 C$ is defined by a group of four alternate triangles. Each of these is separated from the others by the length of an edge. This group may therefore be made the subject of the c operation, which changes the CO into a T. So $\frac{1}{2} c_0 c_0 e_1 C = T$.

It may be remarked that the partial contraction $\frac{1}{2} c_o$ can never take place without a previous complete contraction c_o.

25. The corresponding case in fourdimensional space is expressed by the symbol $\frac{1}{2} c_0 c_0 e_1 C_8$. This indicates that first, the edges of the C_8 are made the subject of expansion; second, the sixteen tetrahedra of vertex import are made the subject of contraction; third, a group of eight alternate tetrahedra are made the subject of still further contraction. This last partial movement changes the cubes of the C_8 into tetrahedra and annihilates eight of the tetrahedra of vertex import, thus changing the C_8 into a C_{16}, eight of whose limiting tetrahedra are derived from the limiting cubes of the C_8, the remaining eight being of vertex import. So $\frac{1}{2} c_0 c_0 e_1 C_8 = C_{16}$.

These examples suffice to show in what manner and under what conditions the partial operations may be applied.

II. Application to space fillings.

Expansion applied to the nets.

26. A space filling or net in any space S_n may be considered as a polytope with an infinite number of limiting spaces of n dimensions in a space S_{n+1} of one dimension higher. [1]) According to this view the operations of expansion and contraction and their combinations may be applied to nets; but the fact that the net is a particular case of a polytope modifies the manner in which the operation is to be applied.

Expansion has been defined as a movement of any set of limits away from the centre of a polytope. This movement in general separates the members of the subject.

In a polytope in S_n with an infinite number of n—1-dimensional limits (a net) the centre is at an infinite distance in a direction normal to the space S_n of the net and no movement away from the centre can separate the limits forming the subject, in other words can expand the net. Now it has been shewn that the real movement taking place in an expansion may be resolved into two, one of which transforms the limits each in its own space and the other adjusts those transformed limits. In this way the operation can be applied to the special case under consideration. Thus if the e_1 expansion be applied to a net of squares (Fig. 12) they are transformed into overlapping octagons and then the octagons must be moved apart until an edge which was common to two squares becomes common to two octagons.

This adjustment leaves a gap A_1 A_2 A_3 A_4 (vertex gap) between the octagons corresponding to the vertex A common to four squares. Thus the transformed net of squares is composed of two constituents, octagons corresponding to the squares, and squares corresponding to vertices of the original net.

27. In threedimensional space there is only one regular space filling i. e. the net NC of cubes. The net $N(O, T)$ of octahedra and tetrahedra is semiregular.

If the e_1 expansion be applied to a net of cubes each cube is transformed into a tC. These will overlap and must be moved apart until an edge which was common to four cubes becomes common to four tC (Fig. 13) By this adjustment octahedral gaps (vertex gaps) are left at the vertices. So the net $e_1 NC$ is formed of tC and O.

In order to determine the octahedra it is necessary to observe that as a vertex of the original net belongs to six edges, i. e.

[1]) See the quoted paper of Andreini, art. 47.

six members of the subject, each vertex takes six new positions, forming the six vertices of an octahedron (see rule, art. 6) whose eight faces are supplied by the expanded vertices of the eight cubes meeting in a vertex of the original net.

28. The application to fourdimensional space is simple.

For instance if the e_1 expansion be applied to a net of C_8 each C_8 is changed into an $e_1 C_8$ (Fig. 2*c*), two adjacent ones having a tC in common. As a vertex in the net C_8 belongs to eight edges (eight members of the subject) each vertex takes eight new positions which are the eight vertices of a C_{16}.

The limiting bodies of this C_{16} may be identified as follows. In the net C_8 each vertex is surrounded by 16 members. Each vertex of a C_8 is changed by expansion into a tetrahedron, so that the vertex gap in the net is surrounded by 16 tetrahedra, the limiting bodies of a C_{16}. Thus by the e_1 expansion a net of C_8 has been converted into a net $e_1 NC_8$ of two constituents, $e_1 C_8$ and C_{16}, in which two adjacent $e_1 C_8$ have a tC in common, while an $e_1 C_8$ and a C_{16} have a tetrahedron in common.

29. Again the e_2 expansion may be applied to a plane net. In this case the constituents of the net are moved apart until an edge assumes two positions, the opposite sides of a square, and the vertex gap is a polygon with as many vertices as there were constituents meeting in a point in the original net; figure 14 (*a* and *b*) shews this with regard to a net of triangles.

If the e_2 operation be applied to a net of squares, it moves apart the squares and the result is again a net of squares; but they are not all of the same kind, some being the squares of the original net, some of edge import, others of vertex import (Fig. 15). From this simple example it may be seen that the e_n expansion applied to a net of measure polytopes in n-dimensional space produces again a net of measure polytopes; but the latter is composed of constituents with different imports, and the subject of any further expansion must be suitably chosen. For instance if the $e_1 e_2$ expansion be applied to a net of squares the subject of the e_1 expansion comprises only those squares of edge import introduced by the e_2 expansion in a net of squares (Fig. 15*b*). The result is that the squares of the subject remain unchanged except in position. Those of vertex import and those corresponding to the squares of the original net are changed into octagons of different imports. The corresponding double expansion of the net of triangles is shewn in figure 14*c*.

30. If the e_2 expansion be applied to a net of cubes each cube

is changed into an *RCO*. Four of these are shewn in Fig. 16 after having been adjusted so that a face which was common to two cubes becomes common to two *RCO*.

This adjustment leaves edge gaps and vertex gaps.

As an edge belongs to four and a vertex to twelve faces (members of the subject) the edge gap is defined by four new parallel positions of an edge and the vertex gap by twelve new positions of a vertex. Therefore the first is filled by a square prism (a cube) and the second by a *CO*. In the *CO* the triangles are supplied by triangular faces of the eight *RCO* (expanded cubes) and the squares by the bases of the six prisms (expanded edges) surrounding the gap. Thus the net of cubes is changed by the e_2 transformation into a net $e_2 NC$ with the three constituents *RCO*, *C* and *CO* (A. 20) [1].

The e_2 expansion may be applied to a net $N(O,T)$ of O and T by taking either the group of O or the group of T as independent variable, and the faces of that group as subject. Whichever group is chosen, its faces in their original position define the net $N(O,T)$, in their final position the new net. Thus if the e_2 expansion be applied to the O each O is changed into an *RCO* (Fig. 3*b*) whose triangular faces are in contact with the untransformed tetrahedra. The vertices of each O are now changed into squares (Fig. 3*b*) and as six octahedra meet in a vertex of $N(O,T)$ the vertex gap is a cube. Thus the new net $e_2 N(\underline{O},T)$ has three constituents *RCO*, *C*, *T* (Fig. 17) (A. 19).

In figure 18 is shewn the result $e_1 N(\underline{O},T) = e_1 N(O,\underline{T})$ of the e_1 expansion applied either to the octahedra or to the tetrahedra of the net (O,T).

31. In fourdimensional space an example is given of the e_2 expansion $e_2 NC_{24}$. Each C_{24} is changed into an $e_2 C_{24}$ limited by 24 *RCO*, 96 P_3, 24 *CO* (see rule, art. 9 and Fig. 19π). [2]

The *RCO* are transformed octahedra, the P_3 are expanded edges, and the *CO* expanded vertices. When the transformed C_{24} are adjusted so that an octahedron which in the regular net is common to two C_{24} is changed into an *RCO* common to two $e_2 C_{24}$, there are edge gaps and vertex gaps.

In order to facilitate the determination of these gaps it will be well to state clearly the manner in which the three kinds of limiting bodies are mutually arranged in the $e_2 C_{24}$.

[1]) This means Fig. 20 in ANDREINI's memoir quoted in art. 1. In order to facilitate comparison a table of threedimensional nets is given on plate III.

[2]) Here and in the following figures π means "principal" constituent, while α, β, etc. stand for the polytopes filling the vertex gap, the edge gap, etc.

A shaded face $A_1B_1C_1$ common to two RCO (in Fig. 19) is the new position of a face ABC common to two octahedra in C_{24}; A_1B_1, A_2B_2, A_3B_3 are three new positions of an edge AB of the C_{24}, and the two positions A_1B_1, A_2B_2 in the RCO are identical with the two positions A_1B_1, A_2B_2 in the prism. Again, the vertices of the CO are the 12 positions taken by a vertex A of the C_{24} of which four $A_1\,A_2\,A_4\,A_5$ are identical with four $A_1\,A_2\,A_4\,A_5$ in the RCO.

In the net of C_{24} an edge is common to four and a vertex to 32 faces (members of the subject), so that the edge gap is defined by four positions of an edge and the vertex gap by 32 positions of a vertex. The limiting bodies surrounding these two gaps may be found in the following manner. Four C_{24} meet in an edge and eight in a vertex of the net C_{24}. In each, the edge is changed into a P_3 and the vertex into a CO. Thus among the limiting bodies surrounding the edge and vertex gaps there must be four P_3 in parallel positions in the former and 8 CO in the latter.

Now in the original net two adjacent C_{24}, let us say M & N, have a common octahedron, or it may be said that two octahedra, limiting bodies of two adjacent C_{24}, coincide. So in the transformed net two adjacent $e_2\,C_{24}$ have an RCO (transformed octahedron) in common; or it may be said that two RCO, limiting bodies of two adjacent $e_2\,C_{24}$, M & N, coincide.

Thus the RCO (Fig. 19π) represents two coincident limiting bodies, one belonging to M and the other to N. In each the face $(A_1\,B_1, A_2\,B_2)$ is in contact with a P_3 and these two P_3 can have no other point in common, or else the polytopes M and N, having already one common limiting body, an RCO, would coincide.

Thus two adjacent P_3 surrounding the edge gap have a square face in common. It remains now to seek a polytope which satisfies the following conditions. It must be determined by four parallel positions of an edge and have amongst its limiting bodies four parallel P_3 of which any adjacent two have a square face in common.

A fourdimensional prism on a tetrahedral base is the only body which satisfies these conditions, so that the limiting bodies are 4 P_3, 2 T (Fig. 19β).

Each of the tetrahedra is determined by its vertices i. e. four positions assumed by the end point of an edge of the net C_{24} and is therefore of vertex import.

As 16 edges meet in a vertex of the net C_{24}, there are 16 of these tetrahedra surrounding the vertex gap.

The limiting bodies of the polytope which must fill the vertex

gap are therefore 16 tetrahedra and 8 CO (Fig. 19β). The regular net of C_{24} has thus been transformed into one of three constituents:

(1) $e_2 C_{24}$ (limited by RCO, P_3, CO), Fig. 19π

(2) P_T, Fig. 19β

(3) $c e_1 C_8$ (limited by 8 CO, 16 T), Fig. 19α.

In this net two polytopes (π) have an RCO, a (π) and a (β) have a P_3, a (π) and an (α) have a CO, and an (α) and a (β) have a T in common.

The e_3 expansion applied to a block of cubes.

32. The figure 20 shews the result $e_3 NC$ clearly. It has already been remarked that this expansion leads to a block of cubes of different kinds, some having face import (a), some edge import (b), and some vertex import (c).

In figure 21 is shewn the result of the operation $e_1 e_3 NC$; the cubes corresponding to those of the original net are changed into tC; the cubes of edge import (subject of the second operation e_1) remain cubes; those of face and vertex import are changed respectively into P_8 and RCO (A. 22).

The e_3 expansion applied to a net of C_{16}.

33. Each C_{16} is expanded according to the rule and produces a polytope limited by T, P_3, P_4, C (Fig. 22π).

When these are adjusted, so that tetrahedra which were common to two C_{16} are common to two $e_3 C_{16}$, there are face, edge, and vertex gaps; these are defined respectively by three parallel positions of a face, 12 parallel positions of an edge, and 96 positions of a vertex; since in the NC_{16} a face is common to three, an edge to 12, and a vertex to 96 tetrahedra (members of the subject). It remains only to determine the limiting bodies surrounding these gaps.

34. In order to find those of the face gap the three new parallel positions of the face ABC are represented by the triangles $A_1 B_1 C_1$, $A_2 B_2 C_2$, $A_3 B_3 C_3$ (Fig. 23).

It follows from the definition of expansion that the lines $A_1 A_2$, $A_2 A_3$, $A_3 A_1$ . . . . . are normal to the face ABC and equal to an edge. Thus the face gap is surrounded by two groups of three P_3; one group consists of the P_3: $A_1 B_1 C_1 A_2 B_2 C_2$, $A_2 B_2 C_2 A_3 B_3 C_3$, $A_3 B_3 C_3 A_1 B_1 C_1$ of face import and the other of $A_1 A_2 A_3 B_1 B_2 B_3$, $B_1 B_2 B_3 C_1 C_2 C_3$, $C_1 C_2 C_3 A_1 A_2 A_3$ of edge import.

The members of each group are in triangular contact with members of the same and in square contact with members of the other group.

This polytope, called a simplotope, is a special case of a group of polytopes called prismotopes [1]).

Two kinds of limiting bodies surrounding the edge gap have now been found, i. e. square prisms due to the transformed C_{16} (Fig. 22γ) and P_3 due to the expanded face (Fig. 22β); there are six of the former and eight of the latter, since six C_{16} and eight faces meet in an edge of NC_{16}. As the axes of these 14 prisms are parallel, the body must be a fourdimensional prism whose base is a CO of vertex import (since its vertices are the 12 positions taken by the end point of an edge).

The vertex gap is surrounded by cubes (π) and CO (β), and there are 24 of each since 24 C_{16} and 24 edges meet in a vertex of NC_{16}.

Thus there are four constituents in the new net $e_3 NC_{16} : e_3 C_{16}$, prismotope (3 ; 3), P_{CO}; and a polytope $e_2 C_{16}$ limited by 24 C, 24 CO.

The manner in which these different bodies are in contact is indicated by the imports in the drawings and by the vertical lines.

35. Two examples are given in order to show how a second operation may be applied to the result of a single expansion (Figs. 24 & 25).

Let it be desired to apply the e_1 expansion to the net obtained above. Here those constituents taking the place of *edges* in the original NC_{16} are the subject and must be moved unchanged into new positions. Thus the edge gap in the new net is like that in the e_3 expansion (compare Figs. 22β & 24β).

Moreover those limiting bodies of edge import in the transformed C_{16} and in the prismotope (face gap) must also remain unchanged (compare the parts π and γ of Fig. 22 and Fig. 24).

The tetrahedra (Fig. 22π) are transformed by the e_1 expansion into tT (Fig. 24π).

A careful examination of the manner in which the P_3 of face import and the cube of vertex import in the same polytope (π) are in contact with the tetrahedra will show in what manner they must be changed (see Fig. 24π). From these may be traced the changes in the face gap (γ) and vertex gap (α).

36. If it be desired to apply the e_2 expansion to $e_3 NC_{16}$ the

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[1]) Compare the foot note [3]) in art. 1.

face gap remains unchanged (Figs 22γ and 25γ), as well as the limiting body of face import in the $e_3 C_{16}$ (π).

The tetrahedron (Fig. 22π) is changed by the e_2 expansion into a CO (Fig. 25π) and again the manner in which the other limiting bodies of this polytope are affected by the change can be traced by an examination of the manner in which they are connected with the tetrahedra.

The changes in the edge and vertex gaps can also be traced (compare Figs. 22 and 25).

The polytope of vertex import in Fig. 25 is remarkable, as it is limited by 48 semiregular polyhedra *of the same kind.*

The e_4 expansion.

37. The e_4 expansion applied to a net of C_8, C_{16} or C_{24} separates the adjacent constituents by a distance equal to an edge. Thus two neighbouring members of a block are separated by a fourdimensional prism whose two opposite bases are the two limiting bodies that coincided in the regular net. The net of C_8 so treated results in another net of C_8 of different imports.

The net of C_{16} transformed by the e_4 expansion leads to the following result. The C_{16} are separated, so that instead of two having a tetrahedron in common they are separated by a distance equal to an edge.

In other words the tetrahedron common to two adjacent C_{16} has assumed two parallel positions, the bases of a fourdimensional prism (Fig. 26δ).

The side limiting bodies of this fourdimensional prism are four P_3 (of face import). As three C_{16} meet in a face in the net of C_{16} each face must assume three positions which define a prismotope (3 ; 3) (Fig. 26γ).

Again six C_{16} meet in an edge of the net, therefore each edge takes six positions, i. e. the new positions are the side edges of a fourdimensional prism on an octahedral base (β). It may be seen by (π), (δ), (γ) and (β) that only one of these four polytopes possesses a limiting body with vertex import, i. e. the one filling the edge gap (β), so that the vertex gap is surrounded by octahedra, and as in the net of C_{16} there are 24 edges meeting in a vertex it follows that 24 octahedra surround the vertex gap; that is, it is a C_{24}. This new net evidently may also be obtained by applying the e_4 expansion to the net NC_{24}.

38. The foregoing investigation leads to the following conclusion as to the nets of fourdimensional space.

If the edges are the subject there are only vertex gaps.

If the faces are the subject there are edge and vertex gaps.

If the limiting bodies are the subject there are face, edge, and vertex gaps.

If the constituents are the subject there are body, face, edge, and vertex gaps.

The vertex gaps are filled by polytopes determined by their vertices. Their limiting bodies are regular or semiregular polyhedra.

The edge gaps are filled by fourdimensional prisms determined by edges parallel to their axes. Their bases are either regular or semiregular polyhedra and their other limiting bodies are prisms.

The face gaps are filled by prismotopes determined by parallel positions of a face and are limited by two groups of prisms.

The body gaps are filled by fourdimensional prisms determined by two parallel positions of a regular or semiregular polyhedron.

Contraction applied to the nets.

39. One or two examples will suffice to shew the application of this process to the nets.

If in the net $e_1 N(O,T)$ (Fig. 18) (A. 24) the CO corresponding to the vertices of the original octahedra be made the subject of contraction, the tO are reduced to CO, the tT to O, while the CO remain unchanged. Thus $ce_1 N(O,T)$ denotes a net composed of O and CO (A. 18).

40. In the net $e_2 NC_{24}$ (Fig. 19) the polytopes filling the vertex gap (α) may be made the subject of contraction, when the following changes take place. The polytope α remains unchanged except in position; the prism β is reduced to a tetrahedron common to two of the polytopes α; the CO of π remain unchanged while the RCO are reduced to cubes. Thus the net of three constituents is reduced to one of two constituents, one limited by $8\,CO$ and $16\,T$, the other by $24\,C$ and $24\,CO$.

Tables.

41. The chief results of this memoir are tabulated in the Tables I and II.

Table I gives the 48 polytopes of expansion (the regular polytopes included) and the 42 polytopes of contraction. The first set has

been numbered from 1 to 48; if p stands for any number, p' of the second set is obtained by application of the operation $c\,(=c_0)$ to p of the first set. The first set consists of 39 different polytopes; the second set contains only eight new ones.

Table II gives the 48 nets of expansion (the regular nets included) and of the nets of contraction only the seven new ones, so altogether $39 + 7$ i. e. 46 fourdimensional nets.

Table III gives the nets of threedimensional space and a table of incidences.

Number	Symbol of expansion	Limiting bodies & import: body	face	edge	vertex	
			Expansion.			
		5	10	10	5	
1	C_5	T	—	—	—	
2	e_1C_5	tT	—	—	T	
3	e_2C_5	CO	—	P_3	O	
4	e_3C_5	T	P_3	P_3	T	
5	$e_1e_2C_5$	tO	—	P_3	tT	
6	$e_1e_3C_5$	tT	P_6	P_3	CO	= 7
7	$e_2e_3C_5$	CO	P_3	P_6	tT	= 6
8	$e_1e_2e_3C_5$	tO	P_6	P_6	tO	
		24	96	96	24	
25	C_{24}	O	—	—	—	
26	e_1C_{24}	tO	—	—	C	= 21
27	e_2C_{24}	RCO	—	P_3	CO	
28	e_3C_{24}	O	P_3	P_3	O	
29	$e_1e_2C_{24}$	tCO	-	P_3	tC	
30	$e_1e_3C_{24}$	tO	P_6	P_3	RCO	= 31
31	$e_2e_3C_{24}$	RCO	P_3	P_6	tO	= 30
32	$e_1e_2e_3C_{24}$	tCO	P_6	P_6	tCO	
			Contraction.			
		5	10	10	5	
2′	ce_1C_5	O	—	—	T	= 3′
3′	ce_2C_5	T	—	—	O	= 2′
4′	ce_3C_5	—	—	—	T	= 1
5′	$ce_1e_2C_5$	tT	—	—	tT	
6′	$ce_1e_3C_5$	O	P_3	—	CO	= 3
7′	$ce_2e_3C_5$	T	—	—	tT	= 2
8′	$ce_1e_2e_3C_5$	tT	P_3	-	tO	= 5
		24	96	96	24	
26′	ce_1C_{24}	CO	—	—	C	= 19
27′	ce_2C_{24}	C	—	—	CO	= 19
28′	ce_3C_{24}	—	—	—	O	= 25
29′	$ce_1e_2C_{24}$	tC	—	—	tC	
30′	$ce_1e_3C_{24}$	CO	P_3	—	RCO	= 27
31′	$ce_2e_3C_{24}$	C	—	—	tO	= 26
32′	$ce_1e_2e_3C_{24}$	tC	P_3	—	tCO	= 29

Number	Symbol of expansion	Limiting bodies & import: body	face	edge	vertex	
			Expansion.			
		8	24	32	16	
9	C_8	C	—	—		
10	e_1C_8	tC	—	—	T	
11	e_2C_8	RCO	—	P_3	O	
12	e_3C_8	C	P_4	P_3	T	= 20
13	$e_1e_2C_8$	tCO	—	P_3	tT	
14	$e_1e_3C_8$	tC	P_8	P_3	CO	= 23
15	$e_2e_3C_8$	RCO	P_4	P_6	tT	= 22
16	$e_1e_2e_3C_8$	tCO	P_8	P_6	tO	= 24
		120	720	1200	600	
33	C_{120}	D	—	—		
34	e_1C_{120}	tD	—	—	T	
35	e_2C_{120}	RID	—	P_3	O	
36	e_3C_{120}	D	P_5	P_3	T	= 44
37	$e_1e_2C_{120}$	tID	—	P_3	tT	
38	$e_1e_3C_{120}$	tD	P_{10}	P_3	CO	
39	$e_2e_3C_{120}$	RID	P_5	P_6	tT	
40	$e_1e_2e_3C_{120}$	tID	P_{10}	P_6	tO	= 48
			Contraction.			
		8	24	32	16	
10′	ce_1C_8	CO	—	—	T	= 19′
11′	ce_2C_8	O	—	—	O	= 25
12′	ce_3C_8	—	—	—	T	= 17
13′	$ce_1e_2C_8$	tO	—	—	tT	= 21′
14′	$ce_1e_3C_8$	CO	P_4	—	CO	= 19
15′	$ce_2e_3C_8$	O	—	—	tT	= 18
16′	$ce_1e_2e_3C_8$	tO	P_4	—	tO	= 21
		120	720	1200	600	
34′	ce_1C_{120}	ID	—	—	T	= 43′
35′	ce_2C_{120}	I	—	—	O	= 42′
36′	ce_3C_{120}	—	—	—	T	= 41
37′	$ce_1e_2C_{120}$	tI	—	—	tT	= 45′
38′	$ce_1e_3C_{120}$	ID	P_5	—	CO	= 43
39′	$ce_2e_3C_{120}$	I	—	—	tT	= 42
40′	$ce_1e_2e_3C_{120}$	tI	P_5	—	tO	= 45

Number	Symbol of expansion	Limiting bodies & import: body	face	edge	vertex	
			Expansion.			
		16	32	24	8	
17	C_{16}	T	—	—	—	
18	e_1C_{16}	tT	—	—	O	
19	e_2C_{16}	CO	—	P_4	CO	
20	e_3C_{16}	T	P_3	P_4	C	= 12
21	$e_1e_2C_{16}$	tO	—	P_4	tO	= 26
22	$e_1e_3C_{16}$	tT	P_6	P_4	RCO	= 15
23	$e_2e_3C_{16}$	CO	P_3	P_8	tC	= 14
24	$e_1e_2e_3C_{16}$	tO	P_6	P_8	tCO	= 16
		600	1200	720	120	
41	C_{600}	T	—	—	—	
42	e_1C_{600}	tT	—	—	I	
43	e_2C_{600}	CO	—	P_5	ID	
44	e_3C_{600}	T	P_3	P_5	D	= 36
45	$e_1e_2C_{600}$	tO	—	P_5	tI	
46	$e_1e_3C_{600}$	tT	P_6	P_5	RID	
47	$e_2e_3C_{600}$	CO	P_3	P_{10}	tD	
48	$e_1e_2e_3C_{600}$	tO	P_6	P_{10}	tID	= 40
			Contraction.			
		16	32	24	8	
18′	ce_1C_{16}	O	—	—	O	= 25
19′	ce_2C_{16}	T	—	—	CO	= 10′
20′	ce_3C_{16}	—	—	—	C	= 9
21′	$ce_1e_2C_{16}$	tT	—	—	tO	= 13′
22′	$ce_1e_3C_{16}$	O	P_3	—	RCO	= 11
23′	$ce_2e_3C_{16}$	T	—	—	tC	= 10
24′	$ce_1e_2e_3C_{16}$	tT	P_3	—	tCO	= 13
		600	1200	720	120	
42′	ce_1C_{600}	O	—	—	I	= 35′
43′	ce_2C_{600}	T	—	—	ID	= 34′
44′	ce_3C_{600}	—	—	—	D	= 33
45′	$ce_1e_2C_{600}$	tT	—	—	tI	= 37′
46′	$ce_1e_3C_{690}$	O	P_3	-	RID	= 35
47′	$ce_2e_3C_{600}$	T	—	—	tD	= 34
48′	$ce_1e_2e_3C_{600}$	tT	P_3	—	tID	= 37

Number	Symbol of expansion	Transformed original constituent	Gaps: body	Gaps: face	Gaps: edge	Gaps: vertex	
	Expansion.						
1	NC_8	C_8	—	—	—	—	
2	e_1NC_8	e_1C_8	—	—	—	C_{16}	
3	e_2NC_8	e_2C_8	—	—	P_O	C_{24}	
4	e_3NC_8	e_3C_8	—	(4 ; 4)	P_{CO}	ce_1C_8	
5	e_4NC_8	C_8	P_C	(4 ; 4)	P_C	C_8	
6	$e_1e_2NC_8$	$e_1e_2C_8$	—	—	P_O	e_1C_{16}	
7	$e_1e_3NC_8$	$e_1e_3C_8$	—	(4 ; 8)	P_{CO}	e_2C_{16}	
8	$e_1e_4NC_8$	e_1C_8	P_{tC}	(4 ; 8)	P_C	e_3C_8	= 11
9	$e_2e_3NC_8$	$e_2e_3C_8$	—	(4 ; 4)	P_{tO}	$ce_1e_2C_8$	
10	$e_2e_4NC_8$	e_2C_8	P_{RCO}	(4 ; 4)	P_{RCO}	e_2C_8	
11	$e_3e_4NC_8$	e_3C_8	P_C	(8 ; 4)	P_{tC}	e_1C_8	= 8
12	$e_1e_2e_3NC_8$	$e_1e_2e_3C_8$	—	(4 ; 8)	P_{tO}	e_1C_{24}	
13	$e_1e_2e_4NC_8$	$e_1e_2C_8$	P_{tCO}	(4 ; 8)	P_{RCO}	$e_2e_3C_8$	= 15
14	$e_1e_3e_4NC_8$	$e_1e_3C_8$	P_{tC}	(8 ; 8)	P_{tC}	$e_1e_3C_8$	
15	$e_2e_3e_4NC_8$	$e_2e_3C_8$	P_{RCO}	(8 ; 4)	P_{tCO}	$e_1e_2C_8$	= 13
16	$e_1e_2e_3e_4NC_8$	$e_1e_2e_3C_8$	P_{tCO}	(8 ; 8)	P_{tCO}	$e_1e_2e_3C_8$	
	Contraction.						
49	ce_1NC_8	ce_1C_8	—	—	—	C_{16}	
50	$ce_1e_2NC_8$	$ce_1e_2C_8$	—	—	—	e_1C_{16}	

Number	Symbol of expansion	Transformed original constituent	Gaps: body	Gaps: face	Gaps: edge	Gaps: vertex	
	Expansion.						
17	NC_{16}	C_{16}	—	—	—	—	
18	e_1NC_{16}	e_1C_{16}	—	—	—	C_{24}	
19	e_2NC_{16}	e_2C_{16}	—	—	P_C	e_2C_{16}	
20	e_3NC_{16}	e_3C_{16}	—	(3 ; 3)	P_{CO}	e_2C_{16}	
21	e_4NC_{16}	C_{16}	P_T	(3 ; 3)	P_O	C_{24}	= 37
22	$e_1e_2NC_{16}$	$e_1e_2C_{16}$	—	—	P_C	$e_1e_2C_{16}$	
23	$e_1e_3NC_{16}$	$e_1e_3C_{16}$	—	(3 ; 6)	P_{CO}	e_2C_{24}	
24	$e_1e_4NC_{16}$	e_1C_{16}	P_{tT}	(3 ; 6)	P_O	e_3C_{24}	= 43
25	$e_2e_3NC_{16}$	$e_2e_3C_{16}$	—	(3 ; 3)	P_{tC}	$ce_1e_2C_{24}$	
26	$e_2e_4NC_{16}$	e_2C_{16}	P_{CO}	(3 ; 3)	P_{RCO}	e_2C_{24}	= 42
27	$e_3e_4NC_{16}$	e_3C_{16}	—	(6 ; 3)	P_{tO}	$e_1e_2C_{16}$	= 40
28	$e_1e_2e_3NC_{16}$	$e_1e_2e_3C_{16}$	P_T	(3 ; 6)	P_{tC}	$e_1e_2C_{24}$	
29	$e_1e_2e_4NC_{16}$	$e_1e_2C_{16}$	P_{tO}	(3 ; 6)	P_{RCO}	$e_1e_3C_{24}$	
30	$e_1e_3e_4NC_{16}$	$e_1e_3C_{16}$	P_{tT}	(6 ; 6)	P_{tO}	$e_1e_3C_{24}$	= 46
31	$e_2e_3e_4NC_{16}$	$e_2e_3C_{16}$	P_{CO}	(6 ; 3)	P_{tCO}	$e_1e_2C_{24}$	= 45
32	$e_1e_2e_3e_4NC_{16}$	$e_1e_2e_3C_{16}$	P_{tO}	(6 ; 6)	P_{tCO}	$e_1e_2e_3C_{24}$	= 48
	Contraction.						
51	ce_2NC_{16}	ce_2C_{16}	—	—	—	e_2C_{16}	
52	$ce_1e_2NC_{16}$	$ce_1e_2C_{16}$	—	—	—	$e_1e_2C_{16}$	
53	$ce_1e_3NC_{16}$	$ce_1e_3C_{16}$	—	(3 ; 3)	—	e_2C_{24}	
54	$ce_2e_3NC_{16}$	$ce_2e_3C_{16}$	—	—	—	$ce_1e_2C_{24}$	
55	$ce_1e_2e_3NC_{16}$	$ce_1e_2e_3C_{16}$	—	(3 ; 3)	—	$c_1e_2C_{24}$	

Number	Symbol of expansion	Transformed original constituent	Gaps: body	Gaps: face	Gaps: edge	Gaps: vertex	
	Expansion.						
33	NC_{24}	C_{24}	—	—	—	—	
34	e_1NC_{24}	e_1C_{24}	—	—	—	C_8	
35	e_2NC_{24}	e_2C_{24}	—	—	P_T	ce_1C_8	
36	e_3NC_{24}	e_3C_{24}	—	(3 ; 3)	P_O	C_{24}	
37	e_4NC_{24}	C_{24}	P_O	(3 ; 3)	P_T	C_{16}	= 21
38	$e_1e_2NC_{24}$	$e_1e_2C_{24}$	—	—	P_T	e_1C_8	
39	$e_1e_3NC_{24}$	$e_1e_3C_{24}$	—	(3 ; 6)	P_O	e_2C_8	
40	$e_1e_4NC_{24}$	e_1C_{24}	P_{tO}	(3 ; 6)	P_T	e_3C_8	= 27
41	$e_2e_3NC_{24}$	$e_2e_3C_{24}$	—	(3 ; 3)	P_{tT}	$ce_1e_2C_8$	
42	$e_2e_4NC_{24}$	e_2C_{24}	P_{RCO}	(3 ; 3)	P_{CO}	e_2C_{16}	= 26
43	$e_3e_4NC_{24}$	e_3C_{24}	P_O	(6 ; 3)	P_{tT}	e_1C_{16}	= 24
44	$e_1e_2e_3NC_{24}$	$e_1e_2e_3C_{24}$	—	(3 ; 6)	P_{tT}	$e_1e_2C_8$	
45	$e_1e_2e_4NC_{24}$	$e_1e_2C_{24}$	P_{tCO}	(3 ; 6)	P_{CO}	$e_1e_3C_8$	= 31
46	$e_1e_3e_4NC_{24}$	$e_1e_3C_{24}$	P_{tO}	(6 ; 6)	P_{tT}	$e_2e_3C_8$	= 30
47	$e_2e_3e_4NC_{24}$	$e_2e_3C_{24}$	P_{RCO}	(6 ; 3)	P_{tO}	$e_1e_2C_{16}$	
48	$e_1e_2e_3e_4NC_{24}$	$e_1e_2e_3C_{24}$	P_{tCO}	(6 ; 6)	P_{tO}	$e_1e_2e_3C_{16}$	= 32

TABLE OF NETS IN S_3.

Andreini number	Symbol	Transformed original constituents	Gaps face	Gaps edge	Gaps vertex	Andreini number	Symbol	Transformed original constituents	Gaps face	Gaps edge	Gaps vertex
	NC	C				12	$N(O,\dot{T})$	O,T			
17	e_1NC	tC			O	24	$e_1N(O,T)$	tO,tT			CO
20	e_2NC	RCO		P_4	CO	19	$e_2N(\underline{O},T)$	RCO,T			C
	e_3NC	C	P_4	P_4	C	23	$e_1e_2N(\underline{O},T)$	tCO,tT			tC
21	e_1e_2NC	tCO		P_4	tO	14	$e_1e_2N(O,\underline{T})$	tO,tO			tO
22	e_1e_3NC	tC	P_8	P_4	RCO	15	$\frac{1}{2}e_2N(\underline{O},T)$	tT,T			T
22	e_2e_3NC	RCO	P_4	P_8	tC						
24^{bis}	$e_1e_2e_3NC$	tCO	P_8	P_8	tCO						
18	ce_1NC	CO			O						

TABLE OF INCIDENCES.

	Single polytopes C_5	C_8	C_{16}	C_{24}	C_{120}	C_{600}	Nets C_8	C_{16}	C_{24}
Cells meeting in a body . .							2	2	2
" " " " face . . .							4	3	3
" " " an edge . .							8	6	4
" " " a vertex. .							16	24	8
Bodies " " " face . . .	2	2	2	2	2	2	4	3	3
" " " an edge . .	3	3	4	3	3	5	12	12	6
" " " a vertex .	4	4	8	6	4	20	32	96	24
Faces " " an edge . . .	3	3	4	3	3	5	6	8	4
" " " a vertex. .	6	6	12	12	6	30	24	96	32
Edges " " " " . .	4	4	6	8	4	12	8	24	16

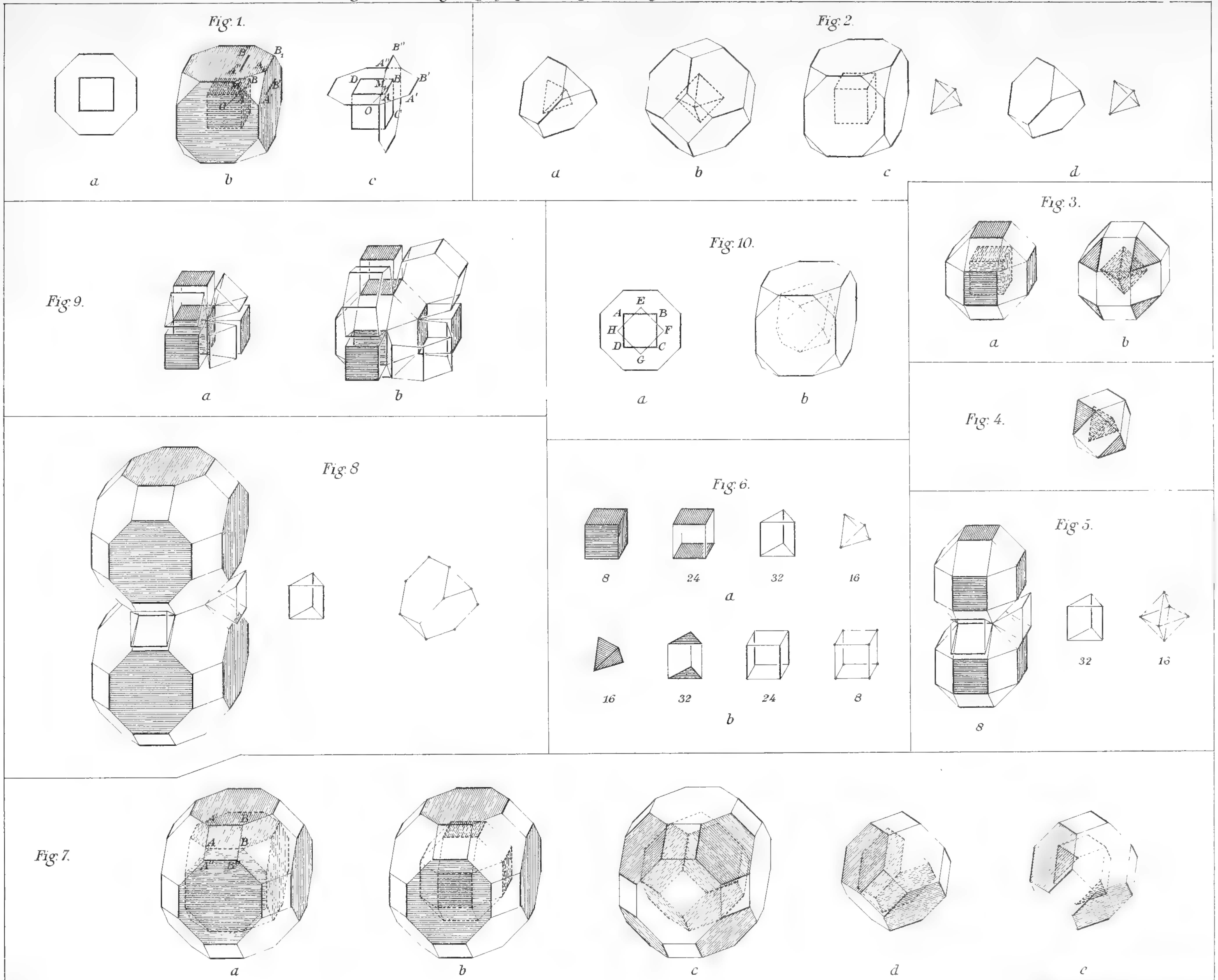
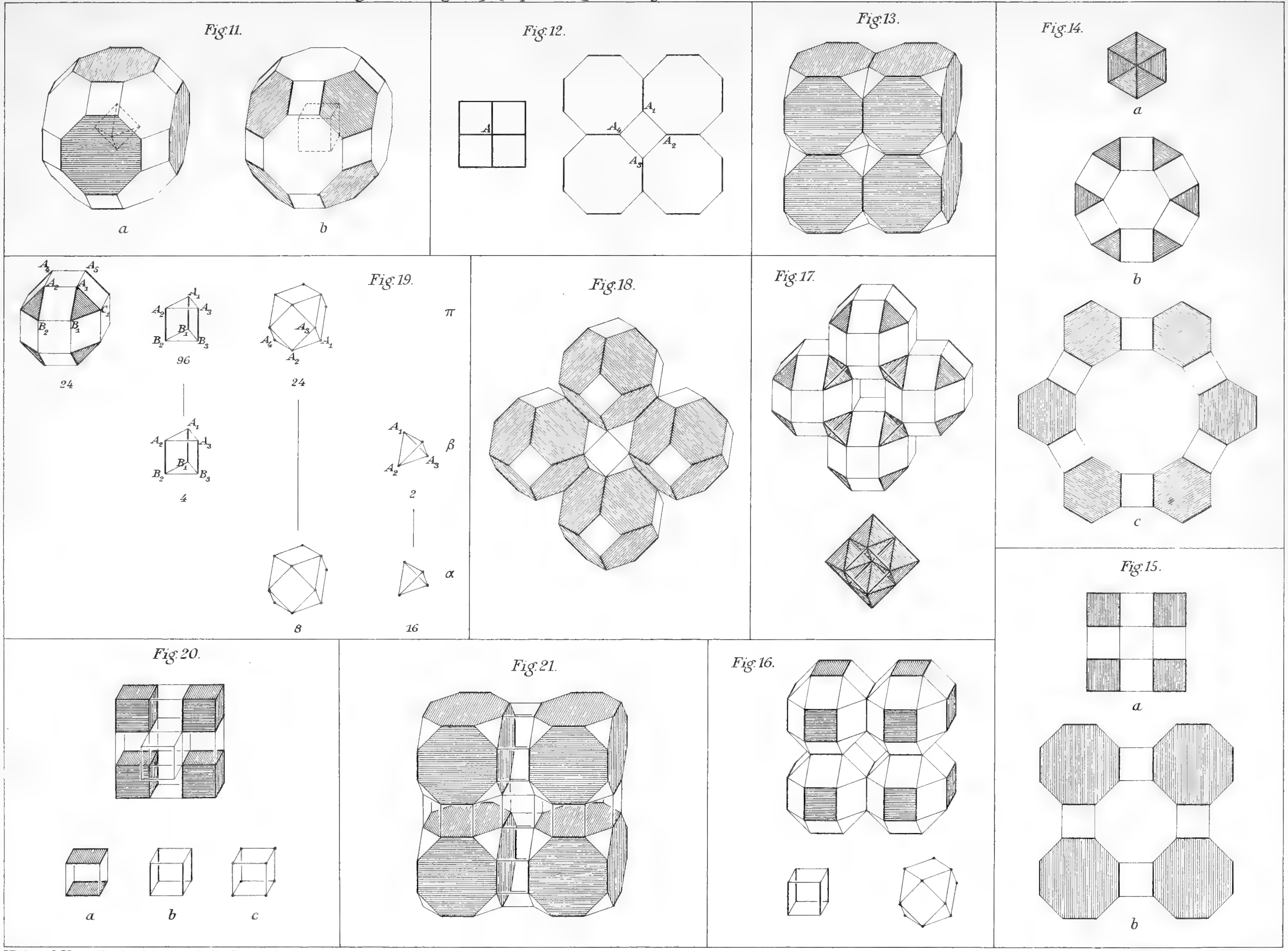


Fig. 1.
a
b
c
Fig. 2.
a
b
c
d
Fig. 3.
a
b
Fig. 4.
Fig. 5.
8
32
16
Fig. 6.
8
24
32
16
a
16
32
24
8
b
Fig. 7.
a
b
c
d
e
Fig. 8
Fig. 9.
a
b
Fig. 10.
a
b



J. Bytel lith. P. J. Mulder impr. Leiden.

Fig. 22.

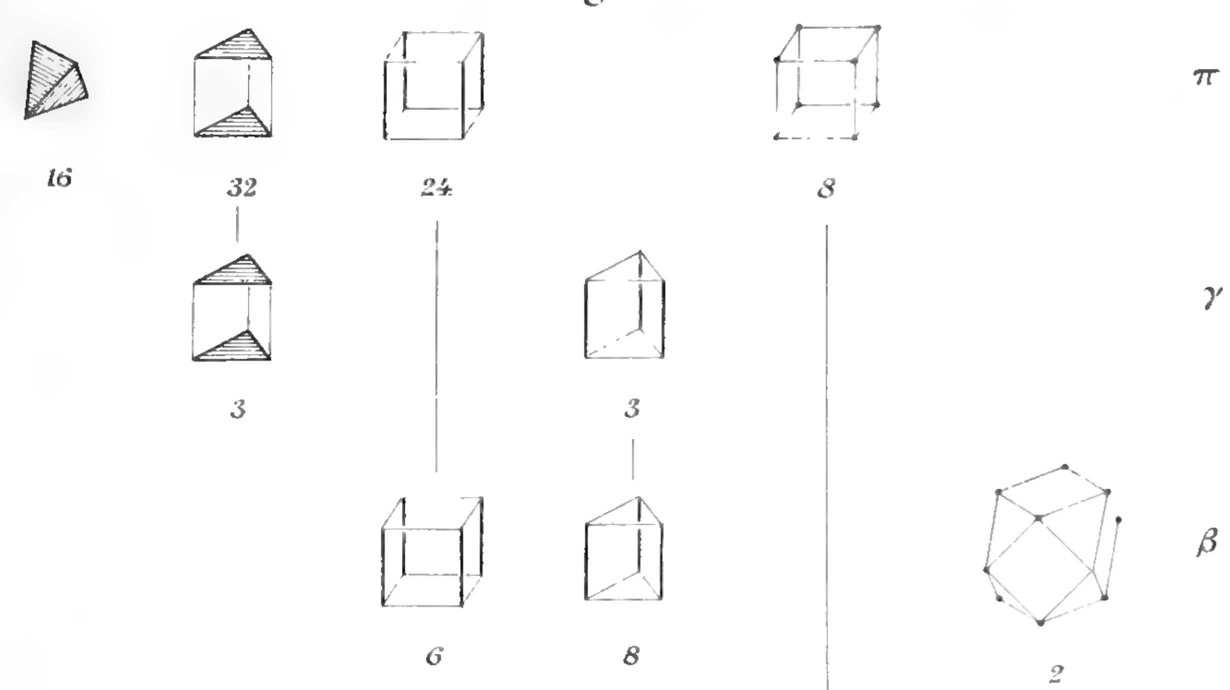
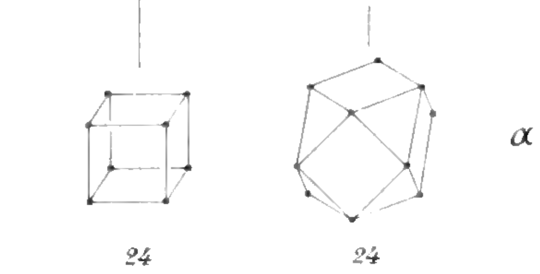



Fig. 23

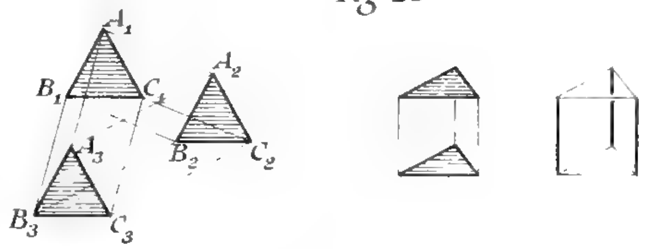


Fig. 25.

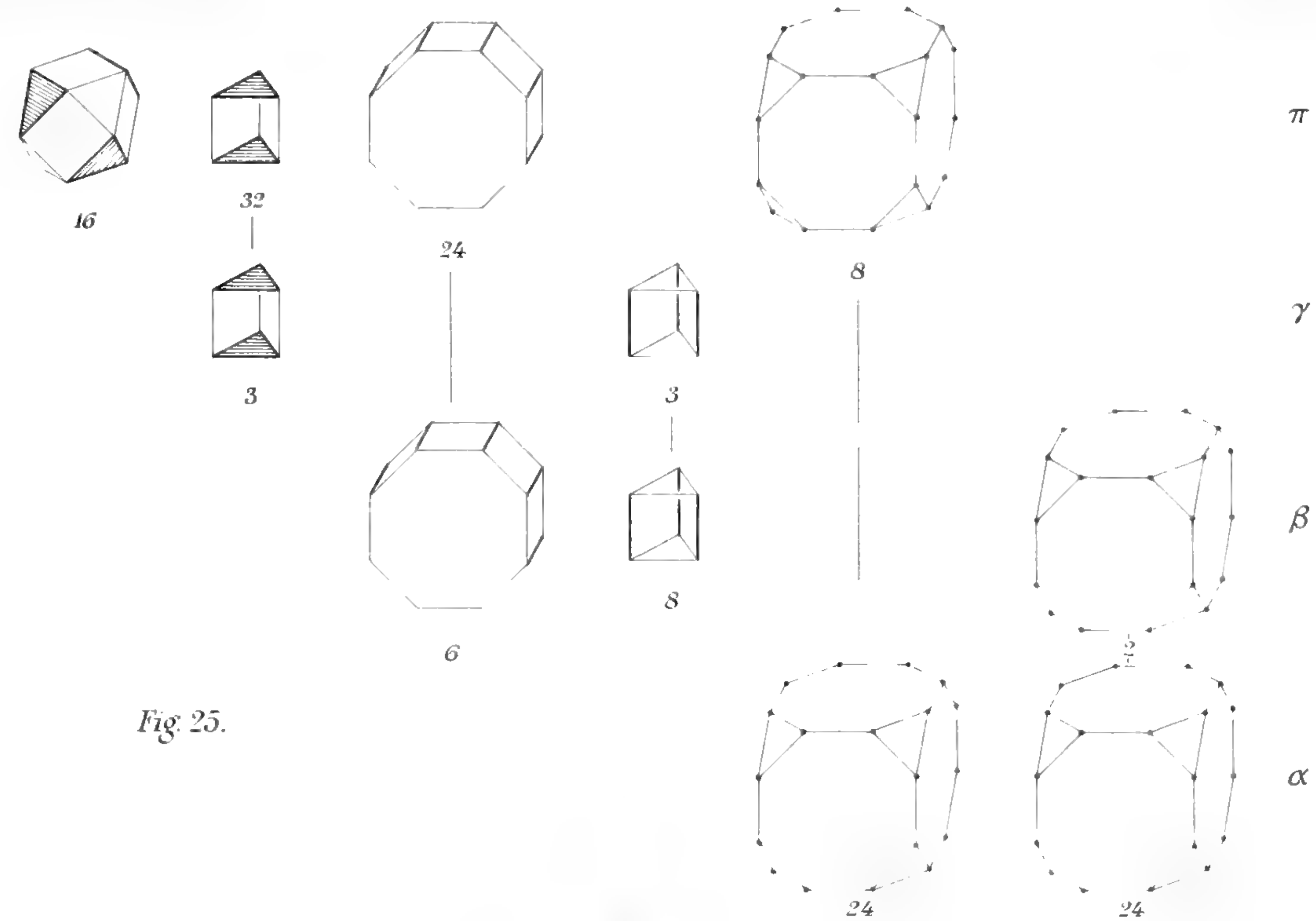


Fig. 24.

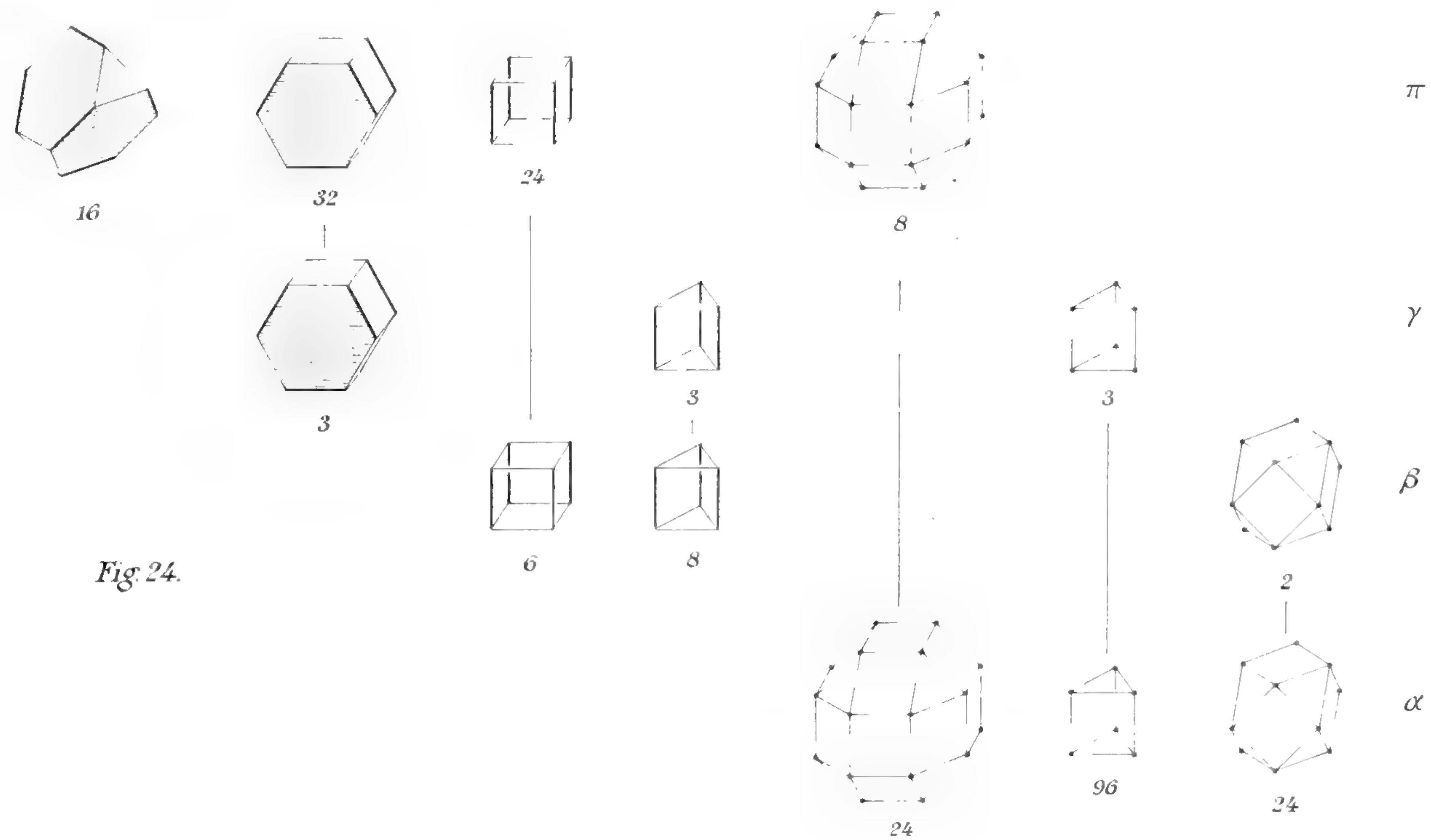
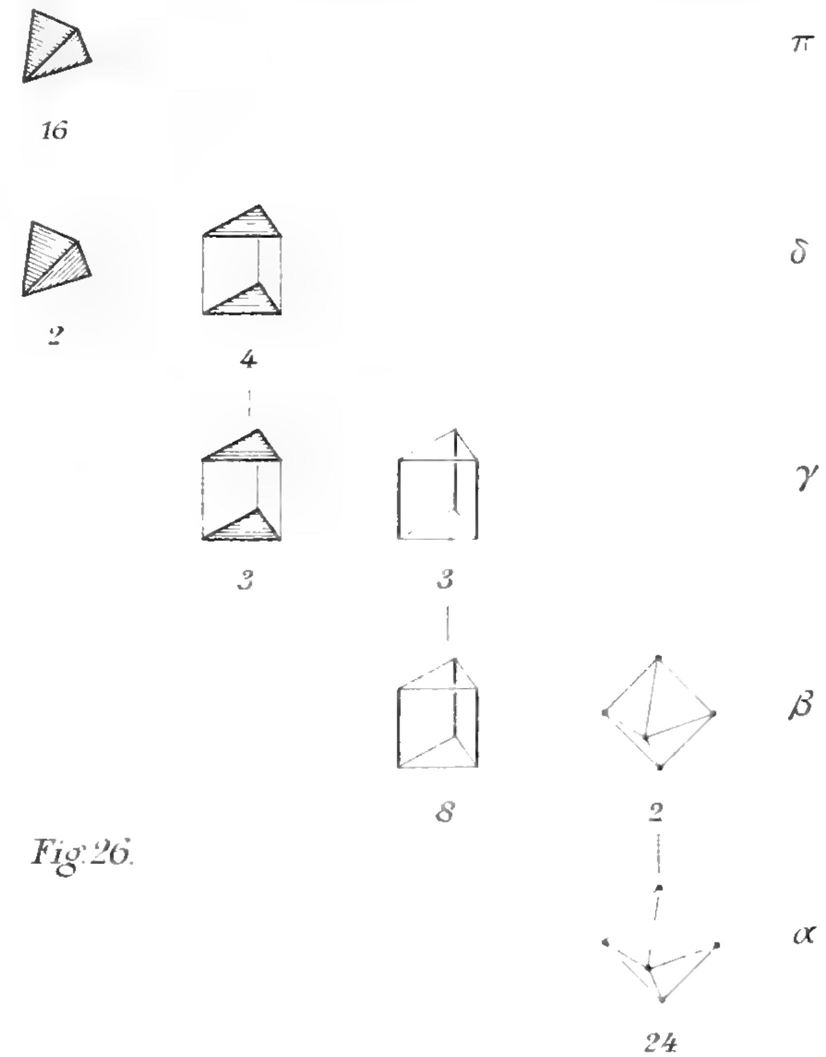


Fig. 26.



GETIJCONSTANTEN VOOR PLAATSEN LANGS DE KUSTEN EN BENEDENRIVIEREN IN NEDERLAND BEREKEND UIT DE WATERSTANDEN VAN HET JAAR 1906.

DOOR

M. H. VAN BERESTEYN.

Verhandelingen der Koninklijke Akademie van Wetenschappen te Amsterdam.
(EERSTE SECTIE.)
DEEL XI. N°. 2.

(Met één kaart).

AMSTERDAM,
JOHANNES MÜLLER.
Februari 1911.

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LITERATUUR.

G. H. Darwin. Scientific Papers. Volume I Oceanic Tides (1907).

Prof. Dr. C. Börgen. Die harmonische Analyse der Gezeitenbeobachtungen (1885).

Ableitung der harmonischen Konstanten der Gezeiten aus drei täglichen Wasserstandsablesungen zu bestimmten Stunden, nebst Bearbeitung dreijähriger Beobachtungen zu Kameron. (Methode von Dr. van der Stok). Annalen der Hydrografie und Maritimen Meteorologie Heft X, XI 1903.

M. Lévy. Leçons sur la Théorie des Marées (1898).

Ph. Hatt. Des Marées.

Prof. Dr. H. G. van de Sande Bakhuyzen. Over de getijden te Helder, IJmuiden en Hoek van Holland.
Verhandelingen Kon. Ac. v. Wet. 26 Jan. 1895.

Dr. J. P. van der Stok. Studiën over getijden in den Indischen Archipel. I en II. Tijdschrift K. I. v. I. afdeeling Ned.-Indië. 1890/91, 1891/92.

Etudes des Phénomènes de Marée sur les côtes néerlandaises. I & II.
Kon. Ned. Met. Inst. n°. 90.

De eenige publicatie's waarin getijconstanten van plaatsen aan de Nederlandsche kusten vermeld worden, zijn de hiervoren onder „Literatuur" genoemde verhandelingen van Prof. Dr. H. G. VAN DE SANDE BAKHUYZEN en Dr. J. P. VAN DER STOK.

In de eerste zijn nagenoeg de constanten van alle elementaire getijden, waarin de getijkromme ontbonden wordt, gegeven; terwijl in de laatste (n°. I) over een tijdperk van 18 jaar de constanten van de getijden der zonnegroep en het halfdaagsche Maangetij M_2 uit de waarnemingen van waterstanden te 2—8—14—20 uur voor Katwijk, Harlingen en Urk, berekend zijn.

Waar dus voor betrekkelijk weinig plaatsen de constanten bekend waren, was het wel van belang deze voor meerdere havens te berekenen en de bestaande gegevens aan te vullen.

Niet alleen voor het practische doel: het samenstellen van getijtafels, welke aangeven den tijd en hoogte van hoog- en laagwater voor komende jaren of wel de voorspelling van de waterstanden voor de uren van den dag. Hierin wordt trouwens, dank zij den overheerschenden invloed van het halfdaagsche Maangetij M_2 zeer voldoende voorzien door de „getijtafels" bewerkt bij den Algemeenen Dienst van den Waterstaat volgens de methode van den Oud-Hoofdingenieur-Directeur van den Waterstaat H. E. DE BRUYN.

Maar ook uit een theoretisch oogpunt is de kennis van deze constanten van groot gewicht. De vorm toch van onze kusten is zoodanig, dat talrijke vraagstukken over de voortplanting en interferentie van golven zich daarbij voordoen en dus de uitkomsten der theoretische oplossing van een gegeven vraagstuk met die der waarneming vergeleken kunnen worden.

Bovendien kunnen wanneer de constanten over meerdere jaren bekend zijn systematische afwijkingen opgespoord worden, zooals die, welke voorkomen in de H „constante" van het halfdaagsche Maangetij M_2. Deze is n.l. niet constant maar verandert met de lengte van den klimmenden knoop der maansbaan, zoodat de afwijkingen van het gemiddelde over 18 jaar van deze „constante"

duidelijk te voorschijn komen bij de groote amplitude van M_2 in de zuidelijke kustplaatsen van ons land.

Deze omstandigheid, reeds vermeld in „Ebbe und Fluth" door Hugo Lenz, doet zien dat de reductie coefficient fm_2 om uit de constante M_2, uit een waarnemingsgroep gedurende eene zekere periode berekend, de amplituden voor een bepaald jaar te bepalen, onjuiste uitkomsten kan geven.

Om dus de constanten der getijbewegingen zoo nauwkeurig mogelijk te verkrijgen is het wel noodzakelijk deze over meerdere jaren te berekenen en zal eene geregelde publicatie van deze geenszins nutteloos zijn. Integendeel, meer in bijzonderheden leeren kennen, de eigenaardigheden, en de veranderingen die de getijbeweging langs onze kusten heeft of ondergaat.

Hoewel aanvankelijk het voornemen bestond de analyse's der getijkrommen te verrichten uit waarnemingen op 24 uur per dag gedurende een jaar, — op deze wijze zijn berekend de constanten in de genoemde verhandeling van Prof. v. d. Sande Bakhuyzen — bleek na eenige proefberekeningen, dat voor de meeste getijden zelfs de kleinere nagenoeg dezelfde uitkomsten verkregen worden, wanneer men de waarnemingen van de waterstanden op de 8 equidistante uren 2—5—8—11—14—17—20—23 gebruikt.

Deze proefberekeningen zijn verricht voor de drie getijkrommen te Hansweert, Brouwershaven en Delfzijl van het jaar 1900 en waarvan de uitkomsten verzameld zijn in bijlage 1.

Zoowel voor de waarnemingen op de 24 uren 0—23 als voor die op de genoemde 8 uren zijn de rangschikkingen der waterstanden geschied naar de methode van Darwin. (Sc. P. Vol. I). Voor de laatste groep van waterstanden ondergingen de reductiefactoren van de amplituden en de correctie's aan de phase eene kleine verandering (zie n°. 22).

Voor Hansweert zijn op deze methode voor de 8 uur waarnemingen ook bepaald de combinatiegetijden MS en $2\,MS$. Bovendien op eene andere wijze, omschreven in n°. 15. De uitkomsten daar mede verkregen leidden er toe de rangschikking naar de Darwinsche methode van deze getijden als ook van $2\,SM$ niet meer te volgen voor de overige plaatsen.

De constanten, die nu bepaald zijn, uit de twee waarnemingsgroepen 2—8—14—20. . . .(I) en 5—11—17—23. . . .(II) waar deze bekend waren zijn:

$$\begin{array}{lll} A_0, S_1, & \multirow{2}{*}{S_2} \left\{ \begin{array}{l} cos\,(k_{s2} - 60^\circ),\ P,\ K_1,\ K_2,\ T,\ S_{a1.2.3.4},\ M_2,\ M_4 \text{ en } M_6 \\ sin\,(k_{s2} - 60^\circ),\ k_r,\ k',\ k'',\ k_t,\ k_{sa1.2.3.4},\ k_{m2},\ k_{m4},\ k_{m6}. \end{array} \right. \\ \text{en } \quad k_{s1} & \end{array}$$

In aanmerking moet genomen worden, dat de periode van één jaar onvoldoende is voor deze constanten en men minstens 4 jaar waarnemingen hebben moet om de storende getijden in voldoende mate te elimineeren.

Wat het getij M_2 betreft, maakt men door dit te bepalen uit een der beide groepen eene geringe doch constante fout (zie n°. 16) en is bovendien de constante M_2 aan eene periodieke verandering onderhevig zooals boven reeds is gezegd.

De constanten bepaald uit de waterstanden op de 8 uren 2—5—8—11—14—17—20—23 (III) zijn die van de getijden, genoemd in onderstaande staat en ontleend aan DARWIN Sc. P. I.

A_0 = gemiddelde waterstand:

	Spoed.			
S_1	$\gamma - \eta$	=	15°.	per mid. zonneuur.
S_2	$2\,(\gamma - \eta)$	=	30.	
P	$\gamma - 2\,\eta$	=	14.9589314	
K_1	γ	=	15.0410686	
K_2	$2\,\gamma$	=	30.0821372	
T	$2\,\gamma - 3\,\eta$	=	29.9589314	
R	$2\,\gamma - \eta$	=	30.0410686	
S_{a1}	η	=	0.0410686	
S_{a2}	$2\,\eta$	=	0.0821372	
S_{a3}	$3\,\eta$	=	0.1232058	
S_{a4}	$4\,\eta$	=	0.1642744	
M_1	$\gamma - \sigma$	=	14.4920521	
M_2	$2\,(\gamma - \sigma)$	=	28.9841042	
M_3	$3\,(\gamma - \sigma)$	=	43.4761563	
M_4	$4\,(\gamma - \sigma)$	=	57.9682084	
M_6	$6\,(\gamma - \sigma)$	=	86.9523126	
M_8	$8\,(\gamma - \sigma)$	=	115.9364168	
N	$2\,\gamma - 3\,\sigma + \omega$	=	28.4397296	
L	$2\,\gamma - \sigma - \omega$	=	29.5284788	
ν	$2\,\gamma - 3\,\sigma - \omega + 2\,\eta$	=	28.5125830	
λ	$2\,\gamma - \sigma + \omega - 2\,\eta$	=	29.4556254	
O	$\gamma - 2\,\sigma$	=	13.9430356	

	Spoed.			
Q	$\gamma - 3\,\sigma + \omega$	=	13°.3986609	per mid. zonneuur.
J	$\gamma + \sigma - \omega$	=	15.5854433	
MS	$4\,\gamma - 2\,\sigma - 2\,\eta$	=	58.9841042	
$2\,\acute{M}S$	$2\,\gamma - 4\,\sigma + 2\,\eta$	=	27.9682084	
$2\,SM$	$2\,\gamma + 2\,\sigma - 4\,\eta$	=	31.0158958	
Mm	$\sigma - \omega$	=	0.5443747	
Mf	$2\,\sigma$	=	1.0980330	
MSf	$2\,(\sigma - \eta)$	=	1.0158958	

en de waarde van $S_4\ cos\,(k_{s4} - 120^\circ)$.

$$\left.\begin{array}{l} \gamma = 15^\circ.0410686 = \text{hoeksnelheid der aardrotatie} \\ \sigma = \ \ 0^\circ.5490165 = \text{gemidd. maansbeweging} \\ \eta = \ \ 0^\circ.0410686 = \text{gemidd. zonsbeweging} \\ \omega = \ \ 0^\circ.0046418 = \text{gemidd. beweging maansperigeum} \end{array}\right\}\ \text{per mid. } S \text{ uur.}$$

Voor de beteekenis der hierboven door letters aangeduide getijden moge verwezen worden naar de werken over de Harmonische Analyse der getijden; in het bijzonder naar de verhandeling van Dr. J. P. VAN DER STOK: De Harmonische Analyse der getijden in het Tijdschrift van het Kon. Inst. v. Ing. Afd. Ned.-Indië 1890/91.

Behalve de numerieke opgave der constanten (bijlagen 2, 3, 4) is in dit stuk alleen een korte opgave der uit verschillende bronnen samengestelde wijze van berekening gegeven en alleen wat op deze betrekking heeft. Vervolgens een kaart (bijlage 5) waarop de afstanden der peilschalen bij de verschillende plaatsen in *K. M.* zijn aangegeven. De cijfers, die deze afstanden aangeven, zijn loodrecht op de verbindingslijn van twee punten geplaatst.

1. Voor de volgende plaatsen zijn getijconstanten bepaald, voor de cursief gedrukte plaatsen werd eene volledige analyse uitgevoerd; voor de overige zijn alleen de getijden, vermeld in n°. 2, en de maangetijden M_2, M_4 en M_6 berekend.

	λ	φ	Δt
Ostende	0ᵘ.20	51°.2	0ᵘ.20
Wielingen	0.22	51.4	—0.11
Vlissingen	0.24	51.4	—0.09
Neuzen	0.26	51.3	—0.07
Hansweert	0.27	51.4	—0.06
Veere	0.24	51.5	—0.09
Wemeldinge	0.27	51.5	—0.06
Zierikzee	0.26	51.6	—0.07
Brouwershaven	0.26	51.7	—0.07
Bruinisse	0.27	51.7	—0.06
Steenbergsche Vliet	0.29	51.7	—0.04
Willemstad	0.30	51.7	—0.03
Moerdijk	0.31	51.7	—0.02
Willemsdorp	0.31	51.8	—0.02
Mond der Donge	0.32	51.8	—0.01
's-Gravendeel	0.31	51.7	—0.02
Dordrecht	0.31	51.8	—0.02
Alblasserdam	0.31	51.9	—0.02
Puttershoek	0.31	51.8	—0.02
Spijkenisse	0.29	51.9	—0.04
Hellevoetsluis	0.28	51.8	0.00
Hoek van Holland	0.27	52.0	—0.06
Maassluis	0.28	51.9	—0.05
Vlaardingen	0.29	51.9	—0.04
Rotterdam	0.30	51.9	—0.03
Krimpen a/d. Lek	0.31	51.9	—0.02
Streefkerk	0.32	51.9	—0.01
Schoonhoven	0.33	52.0	0.00
Vreeswijk	0.34	52.0	0.01
Scheveningen	0.28	52.1	—0.05
Katwijk	0.28	52.2	—0.05
IJmuiden	0.30	52.5	—0.03
Helder	0.32	53.0	—0.01
Vlieland	0.34	53.3	0.01
Enkhuizen	0.35	52.7	0.02
Oranjesluizen	0.33	52.4	0.00

	λ	φ	Δt
Nijkerk..........	$0^u.36$	$52^\circ.3$	$0^u.04$
Elburg...........	0.39	52.5	0.06
Urk.............	0.37	52.7	0.04
Schokland	0.39	52.7	0.06
Kraggenburg	0.40	52.7	0.07
Lemmer..........	0.38	52.8	0.05
Stavoren..........	0.35	52.9	0.02
Hindeloopen.......	0.36	52.9	0.03
Harlingen.........	0.36	53.2	0.03
Roptazijl	0.36	53.3	0.03
Zoutkamp.........	0.41	53.3	0.08
Delfzijl..........	0.46	53.3	0.13
Nieuw-Statenzijl....	0.48	53.3	0.15

λ = Lengte der plaats in uren oostelijk van Greenwich.
φ = N. breedte der plaats.
Δt = verschil in uren tusschen den plaatselijken tijd en den tijd, dien het uurwerk aangeeft.

De waterstanden te 2 en 8 uur voor- en namiddag zijn ontleend aan de: „Verzamelingstabellen der waterhoogten'' volgens de bladen der registreerende peilschalen voor het jaar 1906 bewerkt door den Algemeenen Dienst van den Waterstaat; die van 5 en 11 uur voor- en namiddag eveneens aan bovengenoemde bladen.

Voor Ostende zijn de waterstanden bepaald uit: Diagrammes des Variations de niveau de la mer observées à l'extrémité de l'Estacade Est du chenal d'entrée au port pendant l'année 1906. (Ministère des Finances et des Travaux publics.)

Met uitzondering van Hansweert, waar de registreerende peilschaal van 4 Maart tot 19 April ontbrak, kwamen geene belangrijke onderbrekingen van de waterstanden voor. Waar somtijds door een of andere stoornis van den getijmeter waterstanden ontbraken, zijn zij gegist in overeenstemming met waterstanden van naburige plaatsen of wel, werd eene getijkromme geconstrueerd uit eenige bekende standen en aldus de ontbrekende bepaald.

Als begintijdstip werd aangenomen 1 Januari 1906 te 12 uur 's middags en de uren gerekend van 0...23.

Zoo dat 2—5—8—11 namiddag = 2—5—8—11
2—5—8—11 voormiddag = 14—17—20—23.

De periode der waarnemingen: 1 Jan. 1906 0^u—4 Jan. 1907 0^u.

De tijd, die de uurwerken der getijmeters aangeven, is aangenomen te zijn die, welke bepaald wordt door den meridiaan $0^{u}.33$ oostelijk van Greenwich, en welke ongeveer overeenstemt met Amsterdamsche tijd. Dit is met alle het geval met uitzondering van Ostende en Hellevoetsluis.

In Ostende wijst het uurwerk Greenwich tijd, in Hellevoetsluis plaatselijke tijd aan, en zijn voor de reductie op Amsterdamsche tijd de correcties aan de uit de waarneming afgeleide „constante" k aangebracht (n°. 24), en daarom zijn de constanten k onmiddellijk onderling vergelijkbaar.

Een getijtafel voorspeld met gebruikmaking van achterstaande constanten k, geeft dus de tijdstippen van hoog- en laagwater in Amsterdamsche tijd.

2. De getijden van korte periode:

$$S_1, S_2, P, K_1, K_2, T \text{ en } R$$

en die van lange periode

$$Sa_{1,2,3,4}$$

werden bepaald volgens eene methode, ontleend aan en samengesteld uit de aangehaalde werken van G. H. Darwin, pp. 221—237. Prof. Dr. C. Börgen, J. P. van der Stok.

3. Een korte opgave van de betrekkingen tusschen de maandgemiddelden der waterstanden te 2—5—8—11—14—17—20—23 uur en de componenten der bovengenoemde getijden moge hieronder volgen. Men vindt deze voor de 2—8—14—20 uur waterstanden terug in het aangehaalde werk van van der Stok, terwijl voor de 5—11—17—23 uur waterstanden deze betrekkingen eenvoudig zijn af te leiden uit de voor deze getijden gegeven spoed of verandering per middelbaar zonneuur.

Voor de bepaling der maandgemiddelden der waterstanden werd het jaar verdeeld in 12 maanden van 30 dagen (zie Darwin p. 224) n.l.

Maand	aanvangende		Maand	aanvangende	
0	1	Januari	VI	3(2)	Juli
I	31	„	VII	2(1)	Augustus
II	3(2)	Maart	VIII	1(31 Aug.)	September
III	2(1)	April	IX	2(1)	October
IV	3(2)	Mei	X	1(31 Oct.)	November
V	2(1)	Juni	XI	2(1)	December.

De cijfers in ( ) zijn de begindata voor schrikkeljaren.

Voor iedere maand m ($m = 0, \mathrm{I} \ldots \mathrm{XI}$) wordt berekend de gemiddelde som der waterstanden te 2—5—8—11—14—17—20—23 uur, welke gemiddelde sommen respectievelijk zijn voor te stellen door:

$$h_m^{\;2,\,5,\,8,\,11,\,14,\,17,\,20,\,23}$$

en waarmede de volgende combinatie's gevormd worden.

$$\left.\begin{aligned}
h_m^{\;\mathrm{I}l} &= \tfrac{1}{2}\left[\, h_m^{\;2} - h_m^{\;14} \right] \\
h_m^{\;\mathrm{II}l} &= \tfrac{1}{2}\left[\, h_m^{\;8} - h_m^{\;20} \right] \\
h_m^{\;\mathrm{III}l} &= \tfrac{1}{4}\left[\, (h_m^{\;2} + h_m^{\;14}) - (h_m^{\;8} + h_m^{\;20}) \right] \\
h_m^{\;\mathrm{IV}l} &= \tfrac{1}{4}\left[\, (h_m^{\;2} + h_m^{\;14}) + (h_m^{\;8} + h_m^{\;20}) \right]
\end{aligned}\right\} \ldots\ldots (1)$$

$$\left.\begin{aligned}
h_m^{\;\mathrm{I}r} &= \tfrac{1}{2}\left[\, h_m^{\;5} - h_m^{\;17} \right] \\
h_m^{\;\mathrm{II}r} &= \tfrac{1}{2}\left[\, h_m^{\;11} - h_m^{\;23} \right] \\
h_m^{\;\mathrm{III}r} &= \tfrac{1}{4}\left[\, (h_m^{\;5} + h_m^{\;17}) - (h_m^{\;11} + h_m^{\;23}) \right] \\
h_m^{\;\mathrm{IV}r} &= \tfrac{1}{4}\left[\, (h_m^{\;5} + h_m^{\;17}) + (h_m^{\;11} + h_m^{\;23}) \right]
\end{aligned}\right\} \ldots\ldots (2)$$

Uit de 12 waarden $h_m^{\;n}$ ($m = 0, \mathrm{I} \ldots \mathrm{XI}$ $n = \mathrm{I} \ldots \mathrm{IV}$) zijn op de gewone wijze te berekenen voor beide groepen van combinatie's (1) en (2) de volgende uitdrukkingen:

(Zie voor de wijze van berekening: Darwin, Sc. P. I. pp. 54—55. Börgen, Harmonische Analyse der Gezeitenbeobachtungen 1885 p. 40 e. a.)

$$a_{\mathrm{I}} = \tfrac{1}{12}\left[\, h_m^{\;\mathrm{I}} \,\right]$$

$$b_{\mathrm{II}} = \tfrac{1}{12}\left[\, h_m^{\;\mathrm{II}} \,\right]$$

$$a_{\mathrm{III}} = \tfrac{1}{12}\left[\, h_m^{\;\mathrm{III}} \,\right]$$

$$a_{\mathrm{IV}} = \tfrac{1}{12}\left[\, h_m^{\;\mathrm{IV}} \,\right]$$

$$\begin{matrix} A_1^{\;\mathrm{I}} = \\ B_1^{\;\mathrm{I}} = \end{matrix} \;\tfrac{1}{6}\left[\, h_m^{\;\mathrm{I}} \left\{\begin{matrix} \cos \\ \sin \end{matrix}\right\} 30^\circ.m \,\right]$$

$$\begin{matrix} A_1^{\;\mathrm{II}} = \\ B_1^{\;\mathrm{II}} = \end{matrix} \;\tfrac{1}{6}\left[\, h_m^{\;\mathrm{II}} \left\{\begin{matrix} \cos \\ \sin \end{matrix}\right\} 30^\circ.m \,\right]$$

$$\begin{matrix} A_1^{\;\mathrm{III}} = \\ B_1^{\;\mathrm{III}} = \end{matrix} \;\tfrac{1}{6}\left[\, h_m^{\;\mathrm{III}} \left\{\begin{matrix} \cos \\ \sin \end{matrix}\right\} 30^\circ.m \,\right]$$

$$\begin{matrix} A_2^{\;\mathrm{III}} = \\ B_2^{\;\mathrm{III}} = \end{matrix} \;\tfrac{1}{6}\left[\, h_m^{\;\mathrm{III}} \left\{\begin{matrix} \cos \\ \sin \end{matrix}\right\} 60^\circ.m \,\right]$$

$$\begin{matrix} A_p^{\;\mathrm{IV}} = \\ B_p^{\;\mathrm{IV}} = \end{matrix} \;\tfrac{1}{6}\left[\, h_m^{\;\mathrm{IV}} \left\{\begin{matrix} \cos \\ \sin \end{matrix}\right\} 30^\circ.p.m \right] \quad (p = 1, 2, 3, 4)$$

resp. te voorzien van de letters l en r, al naarmate de 2—8—14—20 uur of wel de 5—11—17—23 uur waterstanden gebezigd zijn.

4. *Correctie's.* Aan de combinatie's $h_m^{\mathrm{I},\,\mathrm{II},\,\mathrm{IV}}$ of aan de daarmede berekende waarden behoeven geen correctie's aangebracht te worden.

De maximum invloed toch, op $h_m^{\mathrm{I},\,\mathrm{II}}$ bedraagt voor een storend getij met amplitude $= H.$

„ spoed $= \sigma_h.$

$$\Delta\, h_m = \pm \frac{H}{n} \frac{\sin 12\, n\sigma_h}{\sin 12\sigma_h} \sin 6\, \sigma_h. \quad (n = 30).$$

en wordt voor de halfdaagsche getijden, waarvoor σ_h ongeveer 30° is, zeer gering.

Voor de enkeldaagsche getijden (σ_h ongeveer 15°), kan Δh_m relatief groot worden, b. v. voor het maansdeclinatie getij O vindt men

$$\Delta h_m = 0.0528\; O \;\; (O = \text{amplitude})$$

Op onze kusten is dit getij ongeveer 10 cM. Bovendien heeft deze invloed een periode van ongeveer een $^1/_2$ jaar in de combinatie's $h_m^{\mathrm{I},\,\mathrm{II}}$, terwijl die van de getijden P en K_1 eene jaarlijksche periode in dezelfde combinatie's vertoonen en is dus een correctie niet aangebracht.

De maximum invloed op h_m^{IV} voor een zeker getij (H, σ_h) is

$$\Delta\, h_m = \pm \frac{H}{n} \frac{\sin 12\, n\, \sigma_h}{\sin 12\, \sigma_h} \cos 6\, \sigma_h \cos 3\, \sigma_h$$

en wordt dus zeer gering zoowel voor halfdaagsche als voor ééndaagsche getijden.

De maximum invloed op de combinatie h_m^{III} is voor een zeker getij (H, σ_h)

$$\Delta_h\, h_m^{\mathrm{III}} = \frac{H}{n} \frac{\sin 12\, n\, \sigma_h}{\sin 12\, \sigma_h} \cos 6\, \sigma_h \sin 3\, \sigma_h = C_h\, H$$

zoodat alleen de invloed in aanmerking komt van halfdaagsche getijden en is van deze alleen de invloed van het halfdaagsche Maangetij M_2 berekend.

(Zie Darwin Sc. P. I. p. 227, v. d. Stok Etudes des Phénomènes de Marées enz. p. 6).

De correctie aan de combinatie h_m^{III} is voor te stellen door:

$$\Delta_h\, h_m^{\mathrm{III}l} = -\, C_h\, H \sin\left[11\sigma_h - \varepsilon_h + 29.12\, \sigma_h + 24\sigma_h\,(30m + x)\right]$$
$$\Delta_h\, h_m^{\mathrm{III}r} = -\, C_h\, H \sin\left[14\sigma_h - \varepsilon_h + 29.12\, \sigma_h + 24\sigma_h\,(30m + x)\right]$$

waarin ε_h = phase van het getij op 1 Januari 0^{u}. en voor

m	x	m	x
0	0	VI	3
I	0	VII	3
II	1	VIII	3
III	2	IX	4
IV	2	X	4
V	3	XI	5.

Voor M_2 worden deze uitdrukkingen:

	$\Delta_{m2}\, h_m^{\mathrm{III}l}$ =	$\Delta_{m2}\, h_m^{\mathrm{III}r}$ =
m = 0	$0.0156\, M_2 \sin(325 - \varepsilon_{m2})$	$0.0156\, M_2 \sin(52 - \varepsilon_{m2})$
I	$\sin(314 - \varepsilon_{m2})$	$\sin(41 - \varepsilon_{m2})$
II	$\sin(278 - \varepsilon_{m2})$	$\sin(5 - \varepsilon_{m2})$
III	$\sin(266 - \varepsilon_{m2})$	$\sin(353 - \varepsilon_{m2})$
IV	$\sin(230 - \varepsilon_{m2})$	$\sin(317 - \varepsilon_{m2})$
V	$\sin(219 - \varepsilon_{m2})$	$\sin(306 - \varepsilon_{m2})$
VI	$\sin(208 - \varepsilon_{m2})$	$\sin(270 - \varepsilon_{m2})$
VII	$\sin(172 - \varepsilon_{m2})$	$\sin(259 - \varepsilon_{m2})$
VIII	$\sin(160 - \varepsilon_{m2})$	$\sin(247 - \varepsilon_{m2})$
IX	$\sin(124 - \varepsilon_{m2})$	$\sin(211 - \varepsilon_{m2})$
X	$\sin(113 - \varepsilon_{m2})$	$\sin(200 - \varepsilon_{m2})$
XI	$\sin(102 - \varepsilon_{m2})$	$\sin(189 - \varepsilon_{m2})$

Men kan nu aan ieder der 12 waarden van h_m^{III} bovenstaande correctie's aanbrengen (zie v. D. STOK a. w.) of wel aan de daarmede berekende uitdrukkingen: $A_{1.2}^{\mathrm{III}}$ $B_{1.2}^{\mathrm{III}}$ a_{III} (zie DARWIN Sc. P. I. t. a. p.).

Door de analyse van $\Delta_m\, h_m^{\mathrm{III}}$ waarin de *sin* en *cos* der veranderlijke hoek in getallenwaarde is bepaald vindt men:

$$\Delta_{m2}\; a_{\mathrm{III}l} = -\,0.0055\, M_2 \sin(\varepsilon_{m2} + 159)$$
$$\Delta_{m2}\; a_{\mathrm{III}r} = +\,0.0055\, M_2 \cos(\varepsilon_{m2} + 162)$$
$$\Delta_{m2}\, A_1^{\mathrm{III}l} = +\,0.0116\, M_2 \sin(\varepsilon_{m2} + 176)$$
$$\Delta_{m2}\, B_1^{\mathrm{III}l} = -\,0.0154\, M_2 \sin(\varepsilon_{m2} + 73)$$
$$\Delta_{m2}\, A_2^{\mathrm{III}l} = -\,0.0021\, M_2 \sin(\varepsilon_{m2} + 24)$$
$$\Delta_{m2}\, B_2^{\mathrm{III}l} = -\,0.0044\, M_2 \sin(\varepsilon_{m2} + 62)$$

$$\Delta_{m2} A_1^{\text{III}r} = -0.0116\, M_2 \cos(\varepsilon_{m2} + 179)$$
$$\Delta_{m2} B_1^{\text{III}r} = +0.0154\, M_2 \cos(\varepsilon_{m2} + 76)$$
$$\Delta_{m2} A_2^{\text{III}r} = +0.0021\, M_2 \cos(\varepsilon_{m2} + 27)$$
$$\Delta_{n2} B_2^{\text{III}r} = +0.0044\, M_2 \cos(\varepsilon_{m2} + 65)$$

In alle deze uitdrukkingen is M_2 de amplitude van het getij M_2 voor de periode waarover de waarnemingen genomen zijn. Men kan deze amplitude voor die periode berekenen of wel uit de eenmaal bekende „constante” M_2 vermenigvuldigd met een reductiefactor (f_{m2}), bepalen.

Evenzoo de phase ε_{m2} op den 1e dag der waarnemingen onmiddelijk uit deze afgeleid of wel uit de constante k_{m2} herleid op den stand van het fictieve hemellichaan op den 1en dag der waarnemingen te 0 uur.

5. In aansluiting van de uitdrukkingen in n°. 3, zij nu gesteld de op eene der beide wijzen voor M_2 gecorrigeerde waarde van:

$$\begin{array}{lcl} a_{\text{III}l} & \ldots\ldots\ldots\ldots & c_{\text{III}} \\ a_{\text{III}r} & \ldots\ldots\ldots\ldots & d_{\text{III}} \\ A_{1.2}^{\text{III}} & \ldots\ldots\ldots\ldots & C_{1.2}^{\text{III}} \\ B_{1.2}^{\text{III}} & \ldots\ldots\ldots\ldots & D_{1.2}^{\text{III}} \end{array}$$

6. De betrekkingen tusschen de combinatie's h van de waterstanden te 2—8—14—20 uur, (index l) en de componenten der getijden bovengenoemd, zijn dan

$$\begin{array}{l} a_{\text{I}l} = \\ b_{\text{II}l} = \end{array} \; S_1 \left\{ \begin{array}{l} \cos \\ \sin \end{array} \right\} (k_{s1} - 30^\circ)$$

$$\begin{array}{l} e_l = (A_1^{\text{II}l} - B_1^{\text{I}l}) = \\ f_l = (A_1^{\text{I}l} + B_1^{\text{II}l}) = \end{array} \; \frac{2P}{F_1} \left\{ \begin{array}{l} \sin \\ \cos \end{array} \right\} (\varepsilon_p - 15^\circ)$$

$$\begin{array}{l} c_l = (A_1^{\text{II}l} + B_1^{\text{I}l}) = \\ d_l = (A_1^{\text{I}l} - B_1^{\text{II}l}) = \end{array} \; \frac{2K_1}{F_1} \left\{ \begin{array}{l} \sin \\ \cos \end{array} \right\} (\varepsilon_{k1} - 45^\circ)$$

$$c_{\text{III}} = S_2 \cos(k_{s2} - 60^\circ)$$

$$\begin{array}{l} C_1^{\text{III}l} = \\ D_1^{\text{III}l} = \end{array} \; \frac{T}{F_1} \left\{ \begin{array}{l} \cos \\ -\sin \end{array} \right\} (\varepsilon_t - 45^\circ) + \frac{R}{F_1} \left\{ \begin{array}{l} \cos \\ \sin \end{array} \right\} (\varepsilon_r - 75^\circ)$$

$$\begin{array}{l} C_2^{\mathrm{III}l} = \\ \\ D_2^{\mathrm{III}l} = \end{array} \frac{K_2}{F_2}\left\{\begin{array}{l} \cos \\ \\ \sin \end{array}\right\}(\varepsilon_{k2} - 90^\circ)$$

$$\begin{array}{l} A_p^{\mathrm{IV}l} = \\ \\ B_p^{\mathrm{IV}l} = \end{array} \frac{S_{ap}}{F_r}\left\{\begin{array}{l} \cos \\ \\ \sin \end{array}\right\}(\varepsilon_{sap} - p.\ 15^\circ) \quad (p = 1, 2, 3, 4)$$

en eindelijk de gemiddelde waterstand A_o uit:

$$a_{\mathrm{IV}l} = A_o.$$

7. Voor de combinatie's met de waterstanden te 5—11—17—23 uur (index r):

$$\begin{array}{l} a_{\mathrm{I}r} = \\ \\ b_{\mathrm{II}r} = \end{array} S_1\left\{\begin{array}{l} \cos \\ \\ \sin \end{array}\right\}(k_{s1} - 75^\circ)$$

$$\begin{array}{l} e_r = A_1^{\mathrm{II}r} - B_1^{\mathrm{I}r} = \\ \\ f_r = A_1^{\mathrm{I}r} + B_1^{\mathrm{II}r} = \end{array} \frac{2P}{F_1}\left\{\begin{array}{l} \sin \\ \\ \cos \end{array}\right\}(\varepsilon_p - 60^\circ)$$

$$\begin{array}{l} c_r = A_1^{\mathrm{II}r} + B_1^{\mathrm{I}r} = \\ \\ d_r = A_1^{\mathrm{I}r} - B_1^{\mathrm{II}r} = \end{array} \frac{2K_1}{F_1}\left\{\begin{array}{l} \sin \\ \\ \cos \end{array}\right\}(\varepsilon_{k1} - 90^\circ)$$

$$d_{\mathrm{III}} = S_2 \sin(k_{s2} - 60^\circ).$$

$$\begin{array}{l} C_1^{\mathrm{III}r} = \\ \\ D_1^{\mathrm{III}r} = \end{array} \frac{T}{F_1}\left\{\begin{array}{l} \sin \\ \\ \cos \end{array}\right\}(\varepsilon_t - 45^\circ) + \frac{R}{F_1}\left\{\begin{array}{l} \sin \\ \\ -\cos \end{array}\right\}(\varepsilon_r - 75^\circ)$$

$$\begin{array}{l} C_2^{\mathrm{III}r} = \\ \\ D_2^{\mathrm{III}r} = \end{array} \frac{K_2}{F_2}\left\{\begin{array}{l} \cos \\ \\ \sin \end{array}\right\}(\varepsilon_{k2} - 180^\circ)$$

$$\left.\begin{array}{l} A_p^{\mathrm{IV}l} = \\ \\ B_p^{\mathrm{IV}l} = \end{array}\right. \frac{S_{ap}}{F_p} \left\{\begin{array}{l} cos \\ \\ sin \end{array}\right. (\varepsilon_{sap} - p.\ 15^\circ) \left.\right\} (p = 1, 2, 3, 4)$$

en de gemiddelde waterstand A_o

$$a_{\mathrm{IV}r} = A_o.$$

8. De in de uitdrukkingen van 5 en 6 voorkomend factor $\frac{1}{F_p}$ is de verkleining die de amplitude van het getij ondergaat door de gemiddelde som over 30 dagen te nemen. Zij is te berekenen uit:

$$\frac{1}{F_p} = \frac{sin\ 12 \times 30 \times p \times \eta}{30\ sin\ 12 \times p \times \eta} \quad (\eta = 0^\circ.0410686).$$

Men vindt:

$$\begin{aligned} log\ F_1 &= 0.00478 \\ log\ F_2 &= 0.01939 \\ log\ F_3 &= 0.04419 \\ log\ F_4 &= 0.08000 \end{aligned}$$

Met de factoren F_p behooren nu de uit de waarnemingen afgeleide amplituden (waar noodig) vermenigvuldigd te worden om de juiste waarden te verkrijgen.

9. Uit 6 en 7 kunnen nu afgeleid worden de betrekkingen tusschen de componenten en de combinatie's der waterstanden op de 8 equidistante uren

2—5—8—11—14—17—20—23

n.l.

$$\tfrac{1}{2}(a_{\mathrm{IV}l} + a_{\mathrm{IV}r}) = A_0$$

$$\tfrac{1}{2}(a_{\mathrm{IV}l} - a_{\mathrm{IV}r}) = S_4\ cos\ (k_{s4} - 120^\circ).$$

$$\left.\begin{array}{l} a_{\mathrm{I}l} + a_{\mathrm{I}r} = \\ \\ b_{\mathrm{II}l} + b_{\mathrm{II}r} = \end{array}\right. 2\ cos\ 22^\circ.5\ S_1 \left\{\begin{array}{l} cos \\ \\ sin \end{array}\right. (k_{s1} - 52^\circ.5) \left.\right\}$$

$$\left.\begin{array}{l} c_{\mathrm{III}} = \\ \\ d_{\mathrm{III}} = \end{array}\right. S_2 \left\{\begin{array}{l} cos \\ \\ sin \end{array}\right. (k_{s2} - 60^\circ) \left.\right\}$$

$$\left.\begin{aligned}(e_l + e_r) &= \\ (f_l + f_r) &= \end{aligned}\right\} \frac{4\,P}{F_1} \cos 22^\circ.5 \left\{\begin{matrix} sin \\ \\ cos \end{matrix}\right\} (\varepsilon_p - 37^\circ.5)$$

$$\left.\begin{aligned}(c_l + c_r) &= \\ (d_l + d_r) &= \end{aligned}\right\} 4 \cos 22^\circ.5 \frac{K_1}{F_1} \left\{\begin{matrix} sin \\ \\ cos \end{matrix}\right\} (\varepsilon_{k1} - 67^\circ.5)$$

$$\left.\begin{aligned}C_1^{\mathrm{III}r} - D_1^{\mathrm{III}l} &= \\ C_1^{\mathrm{III}l} + D_1^{\mathrm{III}r} &= \end{aligned}\right\} \frac{2\,T}{F_1} \left\{\begin{matrix} sin \\ \\ cos \end{matrix}\right\} (\varepsilon_t - 45^\circ)$$

$$\left.\begin{aligned}C_1^{\mathrm{III}r} + D_1^{\mathrm{III}l} &= \\ C_1^{\mathrm{III}l} - D_1^{\mathrm{III}r} &= \end{aligned}\right\} \frac{2\,R}{F_1} \left\{\begin{matrix} sin \\ \\ cos \end{matrix}\right\} (\varepsilon_r - 75^\circ)$$

$$\left.\begin{aligned}C_2^{\mathrm{III}r} + D_2^{\mathrm{III}l} &= \\ C_2^{\mathrm{III}l} - D_2^{\mathrm{III}r} &= \end{aligned}\right\} \frac{2\,K_2}{F_2} \left\{\begin{matrix} sin \\ \\ cos \end{matrix}\right\} (\varepsilon_{k2} - 90^\circ)$$

$$\left.\begin{aligned}B_p^{\mathrm{IV}l} + B_p^{\mathrm{IV}r} &= \\ A_p^{\mathrm{IV}l} + A_p^{\mathrm{IV}r} &= \end{aligned}\right\} \frac{2\,S_{ap}}{F_p} \left\{\begin{matrix} sin \\ \\ cos \end{matrix}\right\} (\varepsilon_{sap} - p\,15^\circ) \quad (p = 1.\,2.\,3.\,4).$$

10. Berekening van de getijden der M groep, de combinatiegetijden MS, $2\,MS$, $2\,SM$ en MSf.

Wat de berekening van de getijden $M_{1.2\ldots8}$ betreft, deze is verricht volgens de methode van v. d. Stok (zie v. d. Stok a. w. Tijds. K. I. v. I. 1891/'92. Börgen Ann. der Hydr. 1903).

Daartoe werden de waterstanden op één der uren 2—5....23 gerangschikt volgens M „uren". Deze „uren" — in zulk een uur doorloopt de fictieve maan een boog van 15° — zijn te berekenen volgens de uitdrukking:

$$15^\circ (\tau + \alpha) = 15^\circ\, t - (\sigma - \eta)\, t - 24\,(\sigma - \eta)\, i \quad . \quad . \quad (1)$$

waarin

τ = een der M „uren" 0....23.
α = eene waarde die tusschen $+0.5$ en -0.5 getijuur varieert.
t = een der S uren 2—5....23.
η, σ = gemiddelde beweging van zon en maan.
i = aantal dagen na den 1[en] dag (1[e] dag = 0).

Voor een constant S uur t kunnen nu voor iederen dag van het jaar ($i = 0...364$) de daarmede overeenkomende M „uren" bepaald worden.

Waren in plaats van voor ieder S uur de met het S uur 12 correspondeerende M „uren" τ berekend en de waterstanden gerangschikt volgens deze τ's dan komt deze methode neer op die van DARWIN. (Zie Sc. P. I. p. 216), toegepast op enkele uren.

De gemiddelde som der onder het uur τ van (1) gerangschikte waterstanden te t uur S tijd is voor te stellen door:

$$\frac{1}{n}\Sigma^p\Sigma\, M_p \cos\{15^\circ p(\tau+\alpha)-\varepsilon_{mp}\} \quad p=1...8 \quad . \quad . \quad (2)$$

wanneer n het aantal der onder het uur τ voorkomende waterstanden is; en daar α over de n waarnemingen gelijkmatig verdeeld zal zijn kan men (2) schrijven

$$\Sigma^p \frac{M_p}{F_p} \cos\{15^\circ p\,\tau-\varepsilon_{mp}\}. \quad . \quad . \quad . \quad . \quad (3)$$

De factoren F_p waarmede de door de berekening gevonden amplituden $\left(\frac{M_p}{F_p}\right)$ vermenigvuldigd moeten worden, zijn te bepalen uit:

$$\frac{1}{F_p}=\frac{1}{11}\frac{\sin 0.55\times 15^\circ p}{\sin 0.05\times 15^\circ p}$$

Men vindt

$$\begin{aligned}
F_1 &= 1.0034 & \log F_1 &= 0.00149\\
F_2 &= 1.0138 & \log F_2 &= 0.00593\\
F_3 &= 1.0315 & \log F_3 &= 0.01348\\
F_4 &= 1.0570 & \log F_4 &= 0.02408\\
F_6 &= 1.1350 & \log F_6 &= 0.05492\\
F_8 &= 1.2586 & \log F_8 &= 0.09982
\end{aligned}$$

11. Door de rangschikking der waterstanden volgens M „uren", welke uren alleen bepaald worden voor een constant S uur t door i en $(\sigma-\eta)$ volgens (1), worden tevens, behalve de getijden der M serie, die getijden opgenomen, waarvan de verandering per etmaal van het argument een veelvoud van $(\sigma-\eta)$ is; dus die getijden die na eene semi-lunaire periode dezelfde phase ten opzichte van S_2 innemen n.l. de combinatie getijden

$$\begin{array}{l} 2\,SM \\ \;\;MS \\ 2\,MS \\ \;\;MSf \end{array}$$

12. De invloed van deze getijden op de gemiddelde som der waterstanden onder een zeker M „uur" en die op onze kusten een belangrijke rol in de getijbeweging spelen, kan op deze wijze bepaald worden:

De spoed van het getij $2\,SM$ is:

$$\begin{aligned} \sigma_{2sm} &= 2\,\gamma + 2\,\sigma - 4\,\eta \\ &= 2 \times 15^\circ + 2\,(\sigma-\eta) \end{aligned}$$

waaruit volgt de invloed op den waterstand te t uur van den i^{en} dag (1^e dag $= 0$)

$$(2\,SM)\cos\{30^\circ\,t + 2\,(\sigma-\eta)\,t + 24 \times 2\,(\sigma-\eta)\,i - \varepsilon_{2\,sm}\}.$$

In verband met (1) kan deze uitdrukking geschreven worden in den vorm

$$(2\,SM)\cos\{30^\circ\,(\tau+\alpha) - \varepsilon'_{2sm}\}$$

waarin $\varepsilon'_{2sm} = 60^\circ\,t - \varepsilon_{2\,sm}$ en dus de invloed op de gemiddelde som $h^{(t)}_\tau$ der onder het M uur τ gerangschikte waterstanden

$$\frac{(2\,SM)}{F_2}\cos\{30^\circ\,\tau - \varepsilon'_{2sm}\}.$$

De spoed van het getij MS (MS, σ_{ms}, ε_{ms}) is:

$$\begin{aligned} \sigma_{ms} &= 4\,\gamma - 2\,\sigma - 2\,\eta \\ &= 4 \times 15^\circ - 2\,(\sigma-\eta). \end{aligned}$$

De invloed dus op den i^{en} dag te t uur.

$$(MS)\cos\{60^\circ\,t - 2\,(\sigma-\eta)\,t - 24 \times 2\,(\sigma-\eta)\,i - \varepsilon_{ms}\}.$$

Na substitutie van (1) in dit argument wordt dit:

$$(MS)\cos(30^\circ(\tau+\alpha)-\varepsilon'_{ms}) \qquad \varepsilon'_{ms}=\varepsilon_{ms}-60^\circ t$$

en de invloed op de gemiddelde som $h^{(t)}_\tau$.

$$\frac{(MS)}{F_2}\cos(30^\circ\tau-\varepsilon'_{ms})$$

Op overeenkomstige wijze vindt men voor den invloed van $2\,MS\,(2\,MS,\sigma_{2ms},\varepsilon_{2ms})$ waarvan de spoed

$$\begin{aligned}\sigma_{2ms}&=2\gamma-4\sigma+2\eta\\&=2\times15^\circ-4(\sigma-\eta)\end{aligned}$$

op de gemiddelde som $h^{(t)}_\tau$.

$$\frac{2\,MS}{F_4}\cos(60^\circ\tau-\varepsilon'_{2ms}) \qquad \varepsilon'_{2ms}=\varepsilon_{2ms}+60^\circ t.$$

Eindelijk voor het getij $MSf\ (MSf,\sigma_{msf},\varepsilon_{msf})$ waarin

$$\sigma_{msf}=2(\sigma-\eta)$$

de invloed op de gemiddelde som $h^{(t)}_\tau$

$$\frac{MSf}{F_2}\cos\{30^\circ\tau-\varepsilon'_{msf}\} \qquad \varepsilon'_{msf}=30^\circ t-\varepsilon_{msf}.$$

13. Recapituleerende, vindt men voor de gemiddelde som $h^{(t)}_\tau$, wanneer A_0 = de gemiddelde waterstand en de invloed van storende getijden met uitzondering van het halfdaagsche zonnetij S_2 buiten rekening latende, onderstaande uitdrukking:

$$\begin{aligned}h^{(t)}_\tau=A_0&+S_2\cos(30^\circ t-k_{s2})\\&\left.\begin{aligned}&+\sum^{p}\frac{M_p}{F_p}\cos(15^\circ p\tau-\varepsilon_{mp})\\&+\frac{(2\,SM)}{F_2}\cos(30^\circ\tau-\varepsilon'_{2sm})\\&+\frac{(MS)}{F_2}\cos(30^\circ\tau-\varepsilon'_{ms})\\&+\frac{(2\,MS)}{F_4}\cos(60^\circ\tau-\varepsilon'_{2ms})\\&+\frac{(MSf)}{F_2}\cos(30^\circ\tau-\varepsilon'_{msf})\end{aligned}\right\}\end{aligned}$$

14. Nu zijn achtereenvolgens onder de M „uren" $\tau = 0\ldots 23$ afzonderlijk gerangschikt de waterstanden te $t = 2—5—8\ldots 23$ uur en dus bepaald de gemiddelde sommen:

$$h_\tau^{(2)} \quad h_\tau^{(8)} \quad h_\tau^{(14)} \quad h_\tau^{(20)}$$

$$h_\tau^{(5)} \quad h_\tau^{(11)} \quad h_\tau^{(17)} \quad h_\tau^{(23)}.$$

Daarmede kunnen de volgende combinatie's gevormd worden:

$$\begin{aligned}
H_\tau^{\mathrm{I}l} &= \tfrac{1}{4}\left[(h_\tau^{(2)} + h_\tau^{(14)}) + (h_\tau^{(8)} + h_\tau^{(20)})\right]\\
H_\tau^{\mathrm{II}l} &= \tfrac{1}{4}\left[(h_\tau^{(2)} + h_\tau^{(14)}) - (h_\tau^{(8)} + h_\tau^{(20)})\right]\\
H_\tau^{\mathrm{I}r} &= \tfrac{1}{4}\left[(h_\tau^{(5)} + h_\tau^{(17)}) + (h_\tau^{(11)} + h_\tau^{(23)})\right]\\
H_\tau^{\mathrm{II}r} &= \tfrac{1}{4}\left[(h_\tau^{(5)} + h_\tau^{(17)}) - (h_\tau^{(11)} + h_\tau^{(23)})\right]
\end{aligned}$$

Gemakkelijk is nu na te gaan, dat, wanneer men in de uitdrukking van $h_\tau^{(t)}$ in n°. 13 achtereenvolgens $t = 2, 14, 8, 20$ enz. stelt, in aanmerking nemende de uitdrukkingen van $\varepsilon' = q\, 30°\, t \pm \varepsilon$ in n°. 12, men vindt:

$$\begin{aligned}
H_\tau^{\mathrm{I}l} &= A_0 + \overset{p}{\Sigma} \frac{M_p}{F_p} \cos(15°\, p\, \tau - \varepsilon_{mp})\\
&\quad + \frac{(2SM)}{F_2} \cos(30°\, \tau - \varepsilon'_{2sm}) \qquad \varepsilon'_{2sm} = 120° - \varepsilon_{2sm}\\
H_\tau^{\mathrm{II}l} &= \qquad S_2 \cos(k_{s2} - 60°)\\
&\quad + \frac{(MS)}{F_2} \cos(30°\, \tau - \varepsilon'_{ms}) \qquad \varepsilon'_{ms} = \varepsilon_{ms} - 60°\\
&\quad + \frac{(2MS)}{F_4} \cos(60°\, \tau - \varepsilon'_{2ms}) \qquad \varepsilon'_{2ms} = \varepsilon_{2ms} + 60°\\
&\quad + \frac{(MSf)}{F_2} \cos(30°\, \tau - \varepsilon'_{msf}) \qquad \varepsilon'_{msf} = -(\varepsilon_{msf} - 60°)
\end{aligned}$$

en

$$\begin{aligned}
H_\tau^{\mathrm{I}r} &= A_0 + \overset{p}{\Sigma} \frac{M_p}{F_p} \cos(15°\, p\, \tau - \varepsilon_{mp})\\
&\quad - \frac{(2SM)}{F_2} \cos(30°\, \tau - \varepsilon'_{2sm}) \qquad \varepsilon'_{2sm} = 120° - \varepsilon_{2sm}
\end{aligned}$$

$$
\begin{aligned}
H^{IIr}_{\tau} = \quad & S_2 \sin (k_{s2} - 60^\circ) \\
& + \frac{(MS)}{F_2} \cos (30^\circ\, \tau - \varepsilon''_{ms}) \qquad \varepsilon''_{ms} = \varepsilon_{ms} - 150^\circ \\
& + \frac{(2MS)}{F_4} \cos (60^\circ\, \tau - \varepsilon''_{2ms}) \qquad \varepsilon''_{2ms} = \varepsilon_{2ms} + 150^\circ \\
& + \frac{(MSf)}{F_2} \cos (30^\circ\, \tau - \varepsilon''_{msf}) \qquad \varepsilon''_{msf} = -\varepsilon_{msf} + 150^\circ .
\end{aligned}
$$

15. Door de analyse van de 24 waarden H_τ op de gewone wijze de componenten van de p^e orde ($A_p\, B_p$) bepalende, vindt men ten slotte:

voor de groep der 4 uren (2—8—14—20)
A_0^{II} = gemiddelde waterstand

$$
\begin{array}{l} A_1^{II} = \\ \\ B_1^{II} = \end{array} \quad \frac{M_1}{F_1} \left[\begin{array}{l} \cos \\ \\ \sin \end{array} \quad \varepsilon_{m1} \right] +
$$

$$
\begin{array}{l} A_2^{II} = \\ \\ B_2^{II} = \end{array} \quad \frac{M_2}{F_2} \left[\begin{array}{l} \cos \\ \\ \sin \end{array} \quad \varepsilon_{m2} \right] + \frac{(2SM)}{F_2} \left[\begin{array}{l} \cos \\ \\ \sin \end{array} \quad (120^\circ - \varepsilon_{2sm}) \right]
$$

$$
\begin{array}{l} A_3^{II} = \\ \\ B_3^{II} = \end{array} \quad \frac{M_3}{F_3} \left[\begin{array}{l} \cos \\ \\ \sin \end{array} \quad \varepsilon_{m3} \right]
$$

$$
\begin{array}{l} A_4^{II} = \\ \\ B_4^{II} = \end{array} \quad \frac{M_4}{F_4} \left[\begin{array}{l} \cos \\ \\ \sin \end{array} \quad \varepsilon_{m4} \right]
$$

$$
\begin{array}{l} A_6^{II} = \\ \\ B_6^{II} = \end{array} \quad \frac{M_6}{F_6} \left[\begin{array}{l} \cos \\ \\ \sin \end{array} \quad \varepsilon_{m6} \right]
$$

$$
\begin{array}{l} A_8^{II} = \\ \\ B_8^{II} = \end{array} \quad \frac{M_8}{F_8} \left[\begin{array}{l} \cos \\ \\ \sin \end{array} \quad \varepsilon_{m8} \right]
$$

en:

$$A_0^{\mathrm{II}l} = \quad S_2 \cos (k_{s2} - 60^\circ)$$

$$\begin{matrix} A_2^{\mathrm{II}l} = \\ \\ B_2^{\mathrm{II}l} = \end{matrix} \quad \frac{(MS)}{F_2} \left[\begin{matrix} \cos \\ \\ \sin \end{matrix} \; (\varepsilon_{ms} - 60^\circ) \right] + \frac{(MSf)}{F_2} \left[\begin{matrix} \cos \\ \\ \sin \end{matrix} \; (60^\circ - \varepsilon_{msf}) \right]$$

$$\begin{matrix} A_4^{\mathrm{II}l} = \\ \\ B_4^{\mathrm{II}l} = \end{matrix} \quad \frac{(2MS)}{F_4} \left[\begin{matrix} \cos \\ \\ \sin \end{matrix} \; (\varepsilon_{2ms} + 60^\circ) \right]$$

en voor de groep der waterstanden te 5—11—17—23 uur

$A_0^{\mathrm{I}r}$ = gemiddelde waterstand.

$$\begin{matrix} A_{1,3,4,6,8}^{\mathrm{I}r} = \\ \\ B_{1,3,4,6,8}^{\mathrm{I}r} = \end{matrix} \quad \frac{M_{1,3,4,6,8}}{F_{1,3,4,6,8}} \left[\begin{matrix} \cos \\ \\ \sin \end{matrix} \; \varepsilon_{m1,3,4,6,8} \right]$$

$$\begin{matrix} A_2^{\mathrm{I}r} = \\ \\ B_2^{\mathrm{I}r} = \end{matrix} \quad \frac{M_2}{F_2} \left[\begin{matrix} \cos \\ \\ \sin \end{matrix} \; \varepsilon_{m2} \right] - \frac{(2\,SM)}{F_2} \left[\begin{matrix} \cos \\ \\ \sin \end{matrix} \; (120^\circ - {}_{2sm}) \right]$$

en

$$A_0^{\mathrm{II}r} = S_2 \sin (k_{s2} - 60^\circ)$$

$$\begin{matrix} A_2^{\mathrm{II}r} = \\ \\ B_2^{\mathrm{II}r} = \end{matrix} \quad \frac{(MS)}{F_2} \left[\begin{matrix} \cos \\ \\ \sin \end{matrix} \; (\varepsilon_{ms} - 150^\circ) \right] + \frac{(MSf)}{F_2} \left[\begin{matrix} \cos \\ \\ \sin \end{matrix} \; (150^\circ - \varepsilon_{msf}) \right]$$

$$\begin{matrix} A_4^{\mathrm{II}r} = \\ \\ B_4^{\mathrm{II}r} = \end{matrix} \quad \frac{(2\,MS)}{F_4} \left[\begin{matrix} \cos \\ \\ \sin \end{matrix} \; (\varepsilon_{2ms} + 150^\circ) \right]$$

En uit de combinatie der beide groepen l en r voor de componenten der getijden van de M serie:

$$\left.\begin{array}{l}\frac{1}{2}(A_p^{\mathrm{I}l} + A_p^{\mathrm{I}r}) = \\ \\ \frac{1}{2}(B_p^{\mathrm{I}l} + B_p^{\mathrm{I}r}) = \end{array}\right. \frac{M_p}{F_p}\begin{bmatrix}\cos \\ & \varepsilon_{mp} \\ \sin\end{bmatrix} \quad (p = 1, 2, 3, 4, 6, 8)$$

voor die der combinatiegetijden:

$$\left.\begin{array}{l}\frac{1}{2}(A_2^{\mathrm{I}r} - A_2^{\mathrm{I}l}) = \\ \\ \frac{1}{2}(B_2^{\mathrm{I}r} - B_2^{\mathrm{I}l}) = \end{array}\right. \frac{(2\,SM)}{F_2}\begin{bmatrix}\cos \\ & (\varepsilon_{2sm} - 120^\circ) \\ \sin\end{bmatrix}$$

$$\left.\begin{array}{l}\frac{1}{2}(A_2^{\mathrm{II}l} - B_2^{\mathrm{II}r}) = \\ \\ \frac{1}{2}(A_2^{\mathrm{II}r} + B_2^{\mathrm{II}l}) = \end{array}\right. \frac{(MS)}{F_2}\begin{bmatrix}\cos \\ & (\varepsilon_{ms} - 60^\circ) \\ \sin\end{bmatrix}$$

$$\left.\begin{array}{l}\frac{1}{2}(A_2^{\mathrm{II}l} + B_2^{\mathrm{II}r}) = \\ \\ \frac{1}{2}(A_2^{\mathrm{II}r} - B_2^{\mathrm{II}l}) = \end{array}\right. \frac{(MSf)}{F_2}\begin{bmatrix}\cos \\ & (\varepsilon_{msf} - 60^\circ) \\ \sin\end{bmatrix}$$

$$\left.\begin{array}{l}\frac{1}{2}(A_4^{\mathrm{II}l} + B_4^{\mathrm{II}r}) = \\ \\ \frac{1}{2}(B_4^{\mathrm{II}l} - A_4^{\mathrm{II}r}) = \end{array}\right. \frac{(2\,MS)}{F_4}\begin{bmatrix}\cos \\ & (\varepsilon_{2ms} + 60^\circ) \\ \sin\end{bmatrix}$$

Uit het bovenstaande blijkt, dat uit de enkele rangschikking der waterstanden volgens M „uren" de voornaamste combinatiegetijden bepaald kunnen worden bij geschikte keuze der waarnemingstijdstippen. Vallen deze op 4 equidistante uren van een dag, dan kan alleen het getij 2 MS berekend worden, terwijl 2 SM niet van M_2 kan bevrijd worden en evenmin MSf van MS en omgekeerd, tenzij een der beide getijden 2 SM of M_2 en MSf en MS bekend is.

16. *Correctie's.*

De componenten A en B uit de M rangschikking der waterstanden verkregen, kunnen met uitzondering van A_8, B_8, gebruikt worden voor de bepaling der getijden vermeld in n°. 10 zonder eenige correctie voor storende getijden aan te brengen.

Het grootste van deze — S_2 — is uit den aard der methode geëlimineerd. Voor de overige N, L, O is de invloed onderzocht door de waarden $\begin{matrix}\cos \\ \cdot \\ \sin\end{matrix}$ $24\,\sigma i$ in de kolommen $\tau = 0 \ldots 23$ in te

schrijven en aldus te bepalen den invloed op h_τ. Hoewel deze invloed van eenige beteekenis kan zijn op ééne der 24 waarden h_τ, blijkt alleen dat voor N eene correctie aan A_8, B_8 behoort aangebracht te worden, zoowel voor de groep der 2—8—14—20 uur, als voor die der waterstanden te 5—11—17—23 uur en dus eveneens voor de combinatie der beide groepen n.l.

$$\Delta A_8 = -0.11 f_n N \cos(\varepsilon_n + 144^\circ)$$

$$\Delta B_8 = 0.11 f_n N \sin(\varepsilon_n + 144^\circ)$$

f_n = reductiefactor; zie DARWIN Sc. P. I p. 46.

Wanneer voor de bepaling van M_2 de 4 uren 2—8—14—20 of wel die te 5—11—17—23 gebruikt worden dan zou men aan dit getij correctie's moeten aanbrengen voor $2SM$.

Daar dit in verhouding tot M_2 zeer klein blijkt te zijn kunnen de correctie's in plaats van aan de componenten A_2 en B_2 onmiddellijk aan de amplitude R' en phase ε'_{m2} aangebracht worden:

$$\Delta R = \pm f m_2 (2SM) \cos(\varepsilon_{2sm} + \varepsilon'_{m2} + 60^\circ)$$

$$\Delta \varepsilon = \mp f m_2 \frac{(2SM)}{R_{m2}} \sin(\varepsilon_{2sm} + \varepsilon'_{m2} + 60^\circ)$$

(bovenste teekens voor de bepaling uit 2—8—14—20 uur, onderste voor die uit 5—11—17—23 uur.)

Door $f_{m2} = 1$ te stellen in verband met de kleine waarde van $2SM$ en in aanmerking te nemen, dat de som van het astronomisch gedeelte der argumenten $= 2\pi$ en dus $\varepsilon_{m2} + \varepsilon_{2sm}$ = constante, kunnen deze correctie's als constant beschouwd worden over verschillende jaren. Men vindt dus (M_2, ε_{m2}) steeds te groot of te klein uit de 4 waterstanden per dag op tijdintervallen van 6 uur.

Voor de combinatie getijden zijn evenmin correctie's aan de componenten A, B aan te brengen. Het grootste getij dat hier invloed kan uitoefenen, het getij M_2, is door de verschillen te bepalen van de gemiddelde som der onder hetzelfde M „uur" τ gerangschikte waterstanden op verschillende uren, nagenoeg geëlimineerd.

Daar echter onder hetzelfde uur τ niet de juiste waarde van de τ ordinaat der M sinusoïde is geplaatst, maar die welke tusschen $\tau + 0.5$ en $\tau - 0.5$ liggen, kan de eliminatie niet volkomen zijn bij een betrekkelijk gering aantal waarnemingen.

17. De combinatiegetijden $2SM$, MS, $2MS$, MSf, zijn dus berekend zonder eenige correctie aan de componenten aan te brengen. De afwijkingen, die de amplitude en phase van MS op deze wijze berekend vertoonden, met die, verkregen uit 24 en 8 waterstanden per dag op de methode van DARWIN voor de 3 plaatsen Hansweert, Brouwershaven en Delfzijl 1900 uitgevoerd, deden eene nog niet geëlimineerde invloed vermoeden bij toepassing der laatste methode.

Inderdaad blijkt er aan de componenten van MS op de methode van DARWIN verkregen eene niet onbelangrijke correctie voor M_2 noodzakelijk te zijn.

Op de volgende wijze kan deze bepaald worden:

De „spoed" van MS per middelbaar S uur is:

$$\begin{aligned}\sigma_{m} &= 4\,\gamma - 2\,(\sigma + \eta) \\ &= 60^\circ - 2\,(\sigma - \eta).\end{aligned}$$

Zoodat de bepaling der met het S uur 12 van een zekeren dag i overeenkomende MS „uur" τ (zie DARWIN p. 237) kan verricht worden naar de uitdrukking

$$(1) \ldots\ldots 60^\circ\,(\tau + \alpha) = 60^\circ\,12 - 2\,(\sigma - \eta).\,12 - 2\,(\sigma - \eta)\,24\,i.$$
$$(\alpha = \pm\,0.5)$$

en is de waterstand op het S uur $12 - t$ van dienzelfden dag, die op het MS „uur"

$$\tau - t = \tau' \qquad \text{(zie n}^\circ\text{. 22)}$$

dus:
$$60^\circ\,(\tau' + \alpha) = 60^\circ.\,12 - 60^\circ\,t - 2\,(\sigma - \eta).\,12 - $$
$$- 2.24\,(\sigma - \eta)\,i \ldots (2)$$

De verandering van het argument van M_2 per middelbaar S uur is

$$\begin{aligned}\sigma_{m2} &= 2\,\gamma - 2\,\sigma \\ &= 30^\circ - 2\,(\sigma - \eta) \ldots (3)\end{aligned}$$

zoodat de invloed van dit getij op den i^{en} dag (1^e dag $= 0$) te $(12 - t)$ uur S tijd, wanneer

$R =$ de amplitude

$\varepsilon_{m2} =$ phase op den 1^{en} dag der waarnemingen te 0 uur

bedraagt:

$$I = R\,cos\,\{30^\circ\,(12 - t) - 2(\sigma - \eta)\,(12 - t) - 2\,(\sigma - \eta)\,24\,i - \varepsilon_{m2}\}$$

$$= R \cos \left[\{ 60^\circ\, 12 - 60^\circ t - 2(\sigma - \eta)\, 12 - 2(\sigma - \eta)\, 24 i \} + \right.$$
$$\left. + \{ 30^\circ t + 2(\sigma - \eta)\, t - \varepsilon_{m2} \} \right]$$

of volgens (2) en (3)

$$I = R \cos \{ 60^\circ (\tau' + \alpha) + (60^\circ - \sigma_{m2})\, t - \varepsilon_{m2} \}.$$

Zijn nu de waterstanden op de 24 uren van een dag (0 . . . 23) volgens de methode van DARWIN in de MS „uren" gerangschikt, dan varieert in de kolom τ', α van $+ 0.5$ tot $- 0.5$ en t van $- 11$ tot $+ 12$ $\{12 - t = 0 - 23\}$ bij een groot aantal waarnemingen.

De gemiddelde invloed van M_2 op de gemiddelde som der waterstanden onder het MS „uur" τ' is dan:

$$i = \frac{1}{24} \sum_{t'=-11}^{t=12} \int_{-0.5}^{0.5} R \cos \{ 60^\circ (\tau' + \alpha) + (60^\circ - \sigma_{m2})\, t - \varepsilon_{m2} \}\, d\alpha \quad (4)$$

$$= R \times \frac{1}{F_4} \frac{\sin 24 (60^\circ - \sigma_{m2})\, ^1/_2}{24 \sin (60^\circ - \sigma_{m2})\, ^1/_2} \cos \{ 60^\circ \tau - \varepsilon_{m2} + {}^1/_2 (60^\circ - \sigma_{m2}) \} \quad (5)$$

(F_4 = reductiefactor, zie n°. 21)

en op de componenten

$$\begin{matrix} A_4 = \\ \\ B_4 = \end{matrix} \frac{1}{12} \left[\sum_{\tau=0}^{\tau=23} h\tau \begin{matrix} \cos \\ \\ \sin \end{matrix} 60^\circ \tau \right]$$

$$\begin{matrix} i_a \\ \\ i_b \end{matrix} = R \frac{1}{F_4} \frac{\sin 24 (60^\circ - \sigma_{m2})\, ^1/_2}{24 \sin (60^\circ - \sigma_{m2})\, ^1/_2} \times \begin{matrix} \cos \\ \\ \sin \end{matrix} \left\{ \varepsilon_{m2} - {}^1/_2 (60^\circ - \sigma_{m2}.) \right\}$$

Opgemerkt moge worden dat na substitutie van σ_{ms} voor σ_{m2} in (5) men den invloed van het getij MS verkrijgt en daarmede de reductiefactor F_{ms} (zie DARWIN p. 240).

en de correctie Θ_{ms} aan de phase ε_{ms}

nl.
$$F_{ms} = F_4 \frac{24 \sin (60^\circ - \sigma_{ms})\, ^1/_2}{\sin 24 (60^\circ - \sigma_{ms})\, ^1/_2}$$

$$\Theta_{ms} = + {}^1/_2 (60^\circ - \varepsilon_{ms})$$

en daar $$F_p = \frac{\frac{15^\circ}{2} p}{\sin \frac{15^\circ}{2} p} \qquad \text{(zie n°. 21)}$$

$$^1/_2 \, (15^\circ p - \sigma) = 7^\circ.5 \, p \, \beta$$

$$F_{ms} = \frac{7^\circ.5 \, p}{\sin 7^\circ.5 \, p} \, \frac{24 \sin 7^\circ.5 \, p \, \beta}{\sin 24 \times 7^\circ.5 \, p \, \beta} \quad (p = 4)$$

$$\Theta_{ms} = + \, 7^\circ.5 \, p \, \beta.$$

Overeenkomstig die, in eene hieronder gevonden uitdrukking. (n°. 22).

18. Op dezelfde wijze kan de invloed van M_2 op de componenten van MS bepaald worden bij toepassing der methode van Darwin op de 8 waterstanden te 2—5—8—11—14—17—20—23 uur voor de berekening van MS.

Men neme slechts in aanmerking dat de uurreeks van een dag niet is $12 - t$ voor $t = -11 \, ... + 12$
maar

$$12 - (10 - 3\,q) \text{ voor } q = 0 \, ... \, 7$$

en dus in (4) te stellen

$$t = 10 - 3\,q.$$

De gemiddelde invloed voor M_2 wordt dan:

$$\frac{1}{8} R \sum_{q1=0}^{q2=7} \int_{-0.5}^{0.5} \cos \{60^\circ(\tau' + \alpha) + (60 - \sigma_{m2})(10 - 3q) - \varepsilon_{m2}\} \, d\alpha$$

of:

$$\frac{1}{F_4} \frac{\sin 8 \, (60^\circ - \sigma_{m2}) \, ^3/_2}{\sin (60^\circ - \sigma_{m2}) \, ^3/_2} R \cos \{60^\circ \tau' - \varepsilon_{m2} - {}^1/_2 \, (60^\circ - \sigma_{m2})\}$$

en de gemiddelde invloed op de componenten A_4, B_4

$$\begin{array}{l} i_a = \\ \\ i_b = \end{array} \frac{1}{F_4} \frac{\sin 8 \, (60^\circ - \sigma_{m2}) \, ^3/_2}{8 \sin (60^\circ - \sigma_{m2}) \, ^3/_2} R \begin{array}{c} \cos \\ \\ \sin \end{array} \{\varepsilon_{m2} + {}^1/_2 \, (60^\circ - \sigma_{m2})\}.$$

Substitueert men hierin voor σ_{m2}: σ_{ms}, dan verkrijgt men de reductie die de amplitude en de verandering die de phase ε ondergaat, bij toepassing der methode van DARWIN, wanneer de 8 genoemde waterstanden gebruikt worden.

De correctie's aan de componenten van MS aan te brengen voor M_2 zijn dan

$$\Delta A = -i_a$$
$$\Delta B = -i_b$$

De getallenwaarde van σ_{m2} invoerende vindt men ten slotte bij de Darwinsche rangschikking van MS voor

24 uren, de correctie's:

$$\Delta A_4 = -0.0314\, f_{m2}\, M_2 \cos(\varepsilon_{m2} - 15°.5)$$
$$\Delta B_4 = -0.0314\, f_{m2}\, M_2 \sin(\varepsilon_{m2} - 15°.5) \qquad [\log\ coeff. = 8.49737]$$

voor 8 uren:

$$\Delta A_4 = -0.0347\, f_{m2}\, M_2 \cos(\varepsilon_{m2} + 15°.5)$$
$$\Delta B_4 = -0.0347\, f_{m2}\, M_2 \sin(\varepsilon_{m2} + 15°.5) \qquad [\log\ coeff. = 8.54078]$$

19. Deze correctie wordt niet vermeld in het handboek van DARWIN. Toch kan, waar M_2 groot is, de invloed aanzienlijk zijn en is de bepaling van MS uit waarnemingen gedurende één jaar onvoldoende.

B. v. werd gevonden voor Hansweert 1900 zonder correctie

uit 24 uur waarnemingen 369 dagen	k_{ms} = 80°.9	MS = 4.43 cM.
8 ,, ,, ,, ,,	= 109°.2	= 6.53 ,,
8 ,, ,, volgens n°. 15	= 201°.1	= 3.28 ,,

en

uit 24 uur waarnemingen (met correctie)	k_{ms} = 193.0	MS = 2.97 cM.
8 ,, ,, ,,	193.0	2.89 ,,
8 ,, ,, volgens n°. 15	201.1	3.28 ,,

dus onderling veel beter overeenstemmende waarden.

20. De overige getijden van korte periode, waar deze berekend zijn, werden bepaald volgens de methode van DARWIN (zie DARWIN pp. 216 e. v.)

Daartoe werden voor de verschillende getijden de met het zonneuur 12 van iederen dag van het jaar correspondeerende getijuren $\tau = 0 \ldots 23$ bepaald, zóó, dat voor eenen bepaalden dag i het uur 12 zonnetijd $=$ het getijuur $(\tau \pm 0.5)$.

Dit kan geschieden volgens de uitdrukking:

$$15^\circ p\,(\tau \pm 0.5) = 12\,\sigma + 24\,\sigma \times i$$

waarin σ de „spoed'' van het getij per middelbaar zonneuur, $p =$ 1, 2 enz., afhangende van de orde van het getij.

Zie verder voor de berekening dezer uren τ:

Börgen Ann. der Hydrografie 1903 Heft X p. 444.

v. d. Stok. Studiën over getijden in den Indischen Archipel II. Tijdschrift van het K. I. v. I. Afdeeling Ned. Indië 1891/92.

21. Bij de oorspronkelijke methode voor de scheiding der verschillende getijden, werden de waarnemingen op de 24 uren van een dag zoodanig gerangschikt, dat een waterstand op een zeker uur van een bepaalden dag geplaatst werd in de kolom van een der met dat S uur overeenkomende 24 getijuren τ, binnen de grenzen van een $^1/_2$ getijuur.

De waterstand te t uur S tijd werd dus geacht te zijn waargenomen op het getijuur: $\tau \pm \alpha$ (α van $+ 0.5$ tot $- 0.5$).

Zie Darwin pp. 48 e. v.

Lévy Theorie des Marées pp. 81 e. v.

Börgen Die harmonische Analyse der Gezeitenbeobachtungen pp. 42 e. v.

Over een groot aantal waarnemingen kan men nu aannemen dat de afwijkingen van het juiste getijuur τ gelijkmatig verdeeld zijn tusschen de grenzen $+ 0.5$ en $- 0.5$ uur en wordt dus de gemiddelde waarde der functie: $y = R \cos (n\tau - \varepsilon)$

$$y_{gem.} = \int_{\tau - 0.5}^{\tau + 0.5} R \cos (n\,\tau - \varepsilon)\, d\,\tau$$

$$= \frac{\sin {}^1/_2\, n}{{}^1/_2\, n}\, R \cos (n\tau - \varepsilon).$$

De juiste amplitude ondergaat hierdoor een verkleining bepaald door den factor $F = \dfrac{{}^1/_2\, n}{\sin {}^1/_2\, n}$, waarmede de uit de waarnemingen afgeleide amplitude moet vermenigvuldigd worden.

$n = 15^\circ p$ stellende, vindt men voor:

$$
\begin{aligned}
p = 1 \quad & \log F_1 = 0.00124 \\
2 \quad & \log F_2 = 0.00498 \\
3 \quad & \log F_3 = 0.01122 \\
4 \quad & \log F_4 = 0.02003 \\
6 \quad & \log F_6 = 0.04561 \\
8 \quad & \log F_8 = 0.08250
\end{aligned}
$$
(Börgen, Die h. A. der Gez. p. 48.)

Voor alle getijden van dezelfde orde p,
($p = 1$ voor enkel daagsche
$= 2$ „ half daagsche enz.)
zijn deze factoren constant terwijl door deze gelijkmatige verdeeling der afwijkingen tusschen $+ 0.5$ en $- 0.5$ uur, van het juiste getijuur geene correctie aan de phase ε behoeft aangebracht te worden.

22. Deze gelijkmatige verdeeling der afwijkingen van het juiste getijuur τ heeft niet plaats bij de methode van Darwin, waarbij zooals boven reeds is vermeld, alleen het S uur 12 binnen de grenzen van een half uur overeenstemt met een zeker getijuur τ. Voor de overige 24 uur wordt dan aangenomen, dat het getijuur $\tau \pm r$ samenvalt met het S uur $12 - r$ van zekeren dag.

Gedurende een dag wordt de „spoed" van het getij gelijkgesteld aan die van den middelbaren zon. Men maakt daardoor een zekere fout want is σ de spoed van het getij per middelbaar zonneuur dan is

$$1\ S \text{ uur} = (1 - \beta) \text{ getijuur} \left\{ \beta = 1 - \frac{\sigma}{15^\circ p}; \quad p = 1, 2 \ldots n \right\}$$

en zijn de waarnemingen niet verricht op het getijuur:

$$\tau \pm r \pm \alpha$$

maar op het uur:

$$\tau \pm r \pm \alpha \mp r\beta.$$

Voor waarnemingen op 24 uren van een dag neemt r de waarden aan van $- 12 \ldots + 11$ en is de grootste positieve afwijking van het juiste getijuur τ

$$(0.5 + 12\ \beta) \text{ getijuur}$$

de grootste negatieve afwijking

$$-(0.5 + 11\ \beta) \text{ getijuur.}$$

Nu worden de waterstanden zóó gerangschikt, dat men in een zelfde kolom plaatst, die, welke op het getijuur $(\tau \pm r) = \tau' =$ constant, kunnen geacht te zijn waargenomen.

Over een groot aantal waarnemingen verkrijgt bij constante τ'

τ	alle waarden tusschen	$0 \ldots 23$
r	„ „ „	$-12 \ldots +11$
α	„ „ „	$-0.5 \ldots +0.5$

en is de gemiddelde waarde van de functie

$$R \cos (n\tau' - r)$$

voor $\tau' = \tau' \pm \alpha \mp r\beta$, met bovengenoemde variaties van α en r te stellen:

$$y = \frac{1}{24} \sum_{r=-11}^{r=12} \int_{-0.5}^{+0.5} R \cos \{ n(\tau' + \alpha + r\beta) - \varepsilon \} \, d\alpha$$

$$= \frac{\sin \frac{n}{2}}{\frac{n}{2}} \frac{\sin 24 \frac{n\beta}{2}}{24 \sin \frac{n\beta}{2}} R \cos \{ n(\tau + \tfrac{\beta}{2}) - \varepsilon \}$$

Om dus de juiste amplitude en de juiste phase ε te vinden moet men de eerste vermenigvuldigen met den

factor $$F_r^{(24)} = \frac{\frac{n}{2}}{\sin \frac{n}{2}} \frac{24 \sin \frac{n\beta}{2}}{\sin 24 \frac{n\beta}{2}} \quad \ldots\ldots\ldots \quad (1)$$

en aan de berekende phase de correctie

$$\Theta_r^{(24)} = + \frac{n\beta}{2} \quad \ldots\ldots\ldots \quad (2)$$

aanbrengen.

Zijn de gemiddelde waarden der ordinaten y voor eene zekere rangschikking op bovengenoemde methode bepaald uit de waterstanden op de 8 equidistante uren $2-5 \ldots 23$ dan kunnen op analoge wijze $F_r^{(8)}$ en $\Theta_r^{(8)}$, bepaald worden.

Correspondeert voor een bepaalden dag i het S uur 12 met het getijuur τ, dan komt de waterstand op een der S uren $2-5 \ldots 23$ of $12 - (3q + 1)$; $(q = -4 \ldots\ldots + 3)$ overeen met het juiste getijuur:

$$\tau - (3q + 1) + \alpha + (3q + 1)\beta = \tau' + \alpha + (3q + 1)\beta$$

en wordt in dat geval de

grootste positieve afwijking $+ (0.5 + 10\,\beta)$ getijuur
„ negatieve „ $- (0.5 + 11\,\beta)$ „

De gemiddelde waarde der functie

$$R \cos \{n\,\tau' + \alpha + (3\,q + 1)\,\beta - \varepsilon\}$$

wordt dan, over een groot aantal waarnemingen waarbij

α varieert van 0.5 tot — 0.5,
q de waarden — 4 . . . . + 3 aanneemt,
τ' constant blijft

$$\frac{1}{8}\sum_{q=-4}^{q=+3}\int_{\alpha=-0.5}^{\alpha=0.5} R \cos\left[n\,\{\tau' + (3\,q + 1)\,\beta + \alpha\} - \varepsilon\right] d\alpha$$

of

$$\frac{\sin\frac{n}{2}}{\frac{n}{2}}\;\frac{\sin 8 \times 3\,\frac{n\beta}{2}}{8 \sin 3\,\frac{n\beta}{2}}\;R \cos\left\{n\left(\tau' - \frac{\beta}{2}\right) - \varepsilon\right\},$$

zoodat de berekende amplitude moet vermenigvuldigd worden met

$$F_r^{(8)} = \frac{\frac{n}{2}}{\sin\frac{n}{2}}\;\frac{8 \sin 3\,\frac{n\beta}{2}}{\sin 8 \times 3\,\frac{n\beta}{2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

en men aan de gevonden phase eene correctie moet aanbrengen van:

$$\Theta_r^{(8)} = -\,\frac{n\beta}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Stelt men in (1), (2), (3) en (4) $n = 15^\circ p$, dan wordt

$$F_r^{(24)} = \frac{7^\circ.5\,p}{\sin 7^\circ.5\,p}\;\frac{24 \sin 7^\circ.5\,p\,\beta}{\sin 24 \times 7^\circ.5\,p\,\beta}$$

$$\Theta_r^{(24)} = +\,7^\circ.5\,p\,\beta$$

$$F_r^{(8)} = \frac{7^\circ.5\,p}{\sin 7^\circ.5\,p}\;\frac{8 \sin 3\,\beta \times 7^\circ.5\,p}{\sin 8 \times 3\,\beta \times 7^\circ.5\,p}$$

$$\Theta_r^{(8)} = -\,7^\circ.5\,p\,\beta.$$

In de onderstaande staat zijn voor eenige getijden berekend $\log F$ en Θ

	$log\ F_r^{(24)}$	$\Theta_r^{(24)}$	$log\ F_r^{(8)}$	$\Theta_r^{(8)}$
M_1	0.00205	+ 0.25	0.00204	— 0.25
M_2	0.00825	+ 0.51	0.00821	— 0.51
M_3	0.01860	+ 0.76	0.01849	— 0.76
M_4	0.03320	+ 1.02	0.03301	— 1.02
M_6	0.07544	+ 1.52	0.07504	— 1.52
M_8	0.13617	+ 2.03	0.13544	— 2.03
N	0.01272	+ 0.78	0.01261	— 0.78
L	0.00568	+ 0.24	0.00567	— 0.24
ν	0.01200	+ 0.74	0.01190	— 0.74
λ	0.00591	+ 0.27	0.00590	— 0.27
O	0.00478	+ 0.53	0.00472	— 0.53
Q	0.00940	+ 0.80	0.00928	— 0.80
J	0.00232	— 0.29	0.00231	+ 0.29
MS	0.02330	+ 0.51	0.02326	— 0.51
$2\ MS$	0.01814	+ 1.02	0.01795	— 1.02
$2\ SM$	0.00824	— 0.51	0.00820	+ 0.51

De op deze wijze gevonden reductiefactoren verschillen weinig van die door DARWIN (Sc. P. I. p. 240) opgegeven en bepaald na constructie eener frequentie kromme der afwijkingen van τ' door de uitdrukking:

$$F = \frac{7^\circ.5\, p}{sin\ 7^\circ.5\, p} \frac{24 \times p \times 7^\circ.5\, \beta}{sin\ 24 \times p \times 7^\circ.5\, \beta}$$

Deze factor kan ook verkregen worden door de evaluatie van den dubbelintegraal:

$$\frac{1}{24\beta} \int_{r=-11\frac{1}{2}\beta}^{r=12\frac{1}{2}\beta} \int_{\alpha=-0.5}^{\alpha=+0.5} R\, cos\, \{n\,(\tau + \alpha + r) - \varepsilon\}\, d\alpha\, dr$$

na substitutie van $15^\circ p = n$.

23. De getijden van lange periode Mm Mf zijn bepaald geheel overeenkomstig de methode beschreven in DARWIN Sc. P. I. p. 244. Bovendien is op deze wijze eveneens het getij MSf berekend.

Met de dagelijksche sommen van de waterstanden te 2—5...23 uur, kan gemakkelijk nagegaan worden, dat de invloed op de gemiddelde som onder het uur τ van eene groepeering, bedraagt voor:

$$Mm: \quad \frac{(Mm)}{F_m} \cos \{6^\circ.8 - \varepsilon_m + 15^\circ \tau\} \quad \log F_m = 0.00098$$

$$Mf: \quad \frac{(Mf)}{F_f} \cos (13^\circ.7 - \varepsilon_{mf} + 30^\circ \tau) \quad \log F_f + 0.00375$$

$$MSf: \frac{(MSf)}{F_s} \cos (12^\circ.7 - \varepsilon_{msf} + 30^\circ \tau) \quad \log F_s = 0.00324$$

terwijl de invloed van M_2 op de gemiddelde som der voor MSf gerangschikte waterstanden onder het uur τ kan gesteld worden:

$$- 0.0384 f_{m2} M_2 \cos (- 2^\circ.3 + \varepsilon_{m2} + 30^\circ \tau)$$

waaruit volgt de correctie voor:

A: $\quad \Delta A = 0.0384 f_{m2} M_2 \cos (- 2^\circ.3 + \varepsilon_{m2})$

($\log coeff = 8.58394$)

B: $\quad \Delta B = - 0.0384 f_{m2} M_2 \sin (- 2^\circ.3 + \varepsilon_{m2})$

(M_2 = constante
f_{m2} = factor voor de reductie tot Amplitude
ε_{m2} = phase op den 1[en] dag der waarnemingen te 0 uur)

---

24. In de nummers 2—23 zijn aangegeven de wijzen waarop de componenten A en B der verschillende getijden kunnen bepaald worden.

De amplituden R en de phasen ε op den 1[en] dag te 0 uur vindt men dan door:

$$R = \sqrt{A^2 + B^2}$$

$$\varepsilon = bg.\ tg.\ \frac{B}{A}$$

en ten slotte de constanten H en k uit:

$$H = \frac{R}{f_\tau}$$

$$k = V_0 + u + \varepsilon$$

Voor de berekening van $\frac{1}{f_r}$ en $V_0 + u$ zie de tabellen van Börgen. Die harm. An. der Gez. pp. 61—67 en 35—37.

De constanten k, zooals reeds in 1 is vermeld, hebben betrekking op den Amsterdamschen tijd, zoodat alleen voor Ostende en Hellevoetsluis reductie's noodig waren. Om de juiste verachteringen van de golven met de fictieve sterren te verkrijgen, moeten de volgende correctie's aan de hierachter volgende k's aangebracht worden

$$+ \Delta t_1 \sigma$$

waarin Δt_1 voor alle plaatsen behalve Ostende en Hellevoetsluis de waarde van Δt in 1 heeft; terwijl voor

$$\begin{array}{ll} \text{Ostende} & \Delta t_1 = -0.13 \\ \text{Hellevoetsluis} & = -0.05 \end{array}$$

en σ de „spoed" van de fictieve ster. (zie de opgave in 1).

's-Gravenhage, Mei 1909.

GETIJCONSTANTEN

berekend uit de

24 (0—1....23) en

8 (2—5—8....23) uurwaterstanden per dag

voor het jaar

1900.

OPMERKINGEN.

In de laatste kolom van Hansweert is de M serie berekend volgens n°. 10, en de getijden MS, $2\,MS$, $2\,SM$, volgens n°. 15.

Voor MSf zijn de op de wijze van n°. 15 verkregen uitkomsten onder die volgens de methode van Darwin bepaald, geplaatst.

1900.

Plaats:	Hansweert		Hansweert		Hansweert	
Aantal waarnemingen per dag:	24		8			
GETIJ.	H (c.M.)	k (gr.)	H (c.M.)	k (gr.)	H (c.M.)	k (gr.)
A_0	—6.13		—6.20			
S_1	1.50	323.2	1.55	325.4		
S_2	48.14	128.8	48.29	128.7		
P	3.57	9.6	3.06	8.4		
K_1	7.38	23.7	7.38	25.8		
K_2	14.54	124.4	14.30	125.7		
T	3.08	101.3	3.21	96.5		
R	0.44	251.5	0.66	236.3		
Sa_1	4.96	206.1	4.94	207.5		
Sa_2	2.10	203.9	2.17	206.8		
Sa_3	2.18	204.0	2.18	200.8		
Sa_4	2.54	67.3	2.47	64.0		
M_1	0.72	35.4	0.74	38.4	0.67	16.8
M_2	186.89	69.4	187.04	69.3	187.24	69.3
M_3	0.57	98.3	0.49	101.0	0.73	162.9
M_4	5.05	133.0	4.48	133.8	4.76	124.1
M_6	6.05	167.5	6.68	169.3	6.77	166.6
M_8	3.56	130.5	2.99	114.4	2.96	152.7
N	31.84	42.0	31.68	41.4		
L	20.95	86.3	20.85	86.1		
ν	14.84	40.1	14.50	39.8		
λ	7.05	81.6	7.09	78.4		
O	10.65	200.8	10.37	201.3		
Q	4.04	153.2	4.17	154.4		
J	1.19	144.0	1.14	156.4		
MS	2.97	193.0	2.89	193.0	3.28	201.1
μ of 2 MS	17.40	169.7	16.73	171.5	17.68	170.2
2 SM	6.04	26.8			5.15	359.2
MK	1.11	13.4				
2 MK	2.41	234.8				
MN	2.34	85.4				
Mm	2.23	214.6	2.04	216.9		
Mf	1.48	119.3	1.51	117.1		
MSf	3.64	72.1	{ 3.89 { 2.75	70.2 89.3		

1900.

Plaats:	Brouwershaven		Brouwershaven	
Aantal waarnemingen per dag:	24		8	
GETIJ.	H (c.M.)	k (gr.)	H (c.M.)	k (gr.)
A_0	—15.93		—15.92[5]	
S_1	1.12	318.9	1.02	302.3
S_2	27.42	122.1	27.40	121.6
P	3.46	349.2	3.39	350.3
K_1	7.20	4.9	7.47	6.2
K_2	7.50	119.5	7.37	119.8
T	0.98	113.6	1.13	109.2
R	0.48	196.5	0.67	200.2
Sa_1	7.70	211.7	7.74	212.5
Sa_2	2.86	165.9	2.88	165.6
Sa_3	2.25	234.8	2.17	237.0
Sa_4	2.54	90.0	2.48	90.0
M_1	0.91	34.3	1.18	56.9
M_2	110.99	66.7	110.72	66.6
M_3	0.54	6.7	0.34	178.8
M_4	12.49	129.0	12.92	127.8
M_6	5.55	113.2	6.32	113.0
M_8	1.16	159.4	2.01	210.2
N	18.72	37.8	18.98	37.5
L	13.02	88.8	13.26	87.4
ν	8.70	40.9	8.49	42.0
λ	4.18	88.9	3.95	87.6
O	10.57	189.5	10.79	188.3
Q	4.58	144.9	4.71	144.4
J	1.02	112.5	1.22	143.6
MS	8.05	184.1	8.02	181.7
μ of $2MS$	9.26	189.8	10.33	184.3
$2SM$	4.83	30.1	3.57	16.2
Mm	2.89	210.4	2.74	207.9
Mf	1.15	163.9	1.18	169.7
MSf	0.70	145.2	0.63 1.45	127.0 148.8

1900.

Plaats:	Delfzijl		Delfzijl	
Aantal waarnemingen per dag:	24		8	
GETIJ.	H (c.M.)	k (gr.)	H (c.M.)	k (gr.)
A_0	— 9.90		—10.13	
S_1	1.60	53.4	1.49	53.4
S_2	29.85	26.4	29.07	26.3
P	2.23	28.9	2.31	29.6
K_1	7.29	33.7	7.33	33.3
K_2	8.40	20.3	8.22	22.2
T	0.95	10.4	0.91	8.9
R	0.38	38.5	0.73	20.7
Sa_1	11.82	204.0	12.54	204.5
Sa_2	1.92	102.4	1.88	105.6
Sa_3	3.84	155.9	4.03	156.1
Sa_4	6.41	46.3	6.59	47.3
M_1	0.60	85.8	0.55	75.3
M_2	122.71	317.4	122.31	317.2
M_3	0.03	216.1	0.40	241.6
M_4	14.51	114.9	14.21	114.2
M_6	6.37	303.3	6.20	305.9
M_8	1.57	140.7	1.26	194.4
N	20.19	288.0	20.08	287.8
L	13.48	336.4	13.43	336.4
ν	9.41	289.5	9.21	289.8
λ	5.90	313.6	5.66	315.4
O	9.63	237.1	8.90	237.4
Q	3.51	184.8	3.39	183.8
J	0.33	349.0	0.40	355.9
MS	7.81	187.4	7.93	189.3
μ of 2 MS	12.63	41.2	12.75	46.1
2 SM	4.60	225.3	2.40	181.2
Mm	4.80	215.6	4.84	206.6
Mf	2.37	169.7	2.11	169.0
MSf	1.17	134.5	{ 2.29	132.1
			{ 2.55	164.0

GETIJCONSTANTEN

berekend uit de

Waterstanden te *2—8—14—20* uur

van

1906.

1906.

Plaats:	Ostende		Wielingen		Neuzen		Hansweert	
GETIJ.	H (c.M.)	k (gr.)	H (c.M.)	k (gr.)	H (c.M.)	k (gr.)	H (c.M.)	k (gr.)
A_0	317.78		— 17.47		— 11.52		— 1.16	
S_1	0.37	257.7	0.67	270.6	1.54	22.2	0.92	3.3
$S_2 \cos (k s_2 - 60°)$	+ 54.63		+ 38.40		+ 28.03		+ 18.79	
P	2.17	318.4	2.53	350.0	2.82	0.3	3.26	16.7
K_1	6.62	0.1	5.84	358.7	6.60	19.5	7.36	25.2
K_2	18.51	62.8	12.93	75.8	14.77	111.2	12.30	125.3
T	2.72	79.1	3.81	70.3	3.81	88.7	3.29	75.0
Sa_1	7.83	230.7	6.64	255.7	7.32	264.8	8.99	270.3
Sa_2	4.15	193.2	3.35	188.9	4.31	192.8	5.81	223.4
Sa_3	1.20	245.9	0.98	290.3	1.80	341.7	1.13	272.6
Sa_4	1.88	225.8	2.63	230.5	1.52	277.4	2.05	233.7
M_2	176.55	16.5	160.49	36.1	179.59	59.1	189.79	68.8
M_4	10.60	342.4	14.99	51.4	9.29	94.4	5.62	108.2
M_6	6.32	335.6	13.14	22.1	9.76	84.2	5.45	130.3

1906.

Plaats:	Vlissingen		Veere		Wemeldinge		Zierikzee	
GETIJ.	H (c.M.)	k (gr.)	H (c.M.)	k (gr.)	H (c.M.)	k (gr.)	H (c.M.)	k (gr.)
A_0	— 17.84		— 14.04		— 8.27		— 14.12	
S_1	0.78	314.8	0.64	325.0	0.60	323.0	0.88	326.6
$S_2 \cos (ks_2 - 60^c)$	+ 36.40		+ 20.89		+ 10.71		+ 16.01	
P	2.41	352.3	2.55	356.2	3.67	12.8	2.83	3.3
K_1	6.73	6.4	6.68	359.2	7.37	22.5	7.21	15.5
K_2	12.66	93.9	9.90	108.0	10.22	140.6	8.58	127.4
T	3.75	68.8	3.51	88.1	2.65	115.8	3.27	100.5
Sa_1	7.29	249.0	7.35	218.9	7.68	241.7	8.02	244.0
Sa_2	4.04	189.6	4.23	169.1	3.85	198.2	4.40	201.0
Sa_3	1.08	350.4	2.00	13.7	2.30	10.0	2.37	353.3
Sa_4	2.28	226.9	2.02	261.0	2.24	284.3	1.59	258.6
M_2	168.73	43.8	132.97	56.8	150.59	76.2	136.66	66.4
M_4	12.76	63.0	10.16	96.7	5.98	199.8	5.44	126.0
M_6	12.47	46.6	9.31	69.7	3.18	146.2	5.83	81.9

1906.

Plaats:	Brouwershaven		Bruinisse		Steenbergsche Vliet		Willemstad	
GETIJ.	H (c.M.)	k (gr.)	H (c.M.)	k (gr.)	H (c.M.)	k (gr.)	H (c.M.)	k (gr.)
A_0	— 14.47		— 10.02		— 9.49		+ 6.69	
S_1	0.50	320.9	0.60	317.1	1.38	7.4	0.43	53.9
$S_2 \cos (ks_2 - 60°)$	+ 11.55		+ 7.33		— 0.46		— 9.52	
P	3.20	0.6	3.55	10.4	3.51	17.5	3.11	18.5
K_1	7.23	9.5	7.73	19.8	7.20	24.7	6.68	29.0
K_2	6.66	135.5	9.17	133.6	5.24	165.9	4.81	192.6
T	3.14	70.7	2.69	128.6	1.37	176.6	1.84	168.2
Sa_1	8.06	248.6	7.39	258.5	7.92	258.6	7.86	288.3
Sa_2	4.68	198.4	4.54	211.9	0.83	183.3	6.48	228.6
Sa_3	2.14	3.4	2.47	0.0	2.22	11.4	3.40	5.3
Sa_4	2.13	284.2	2.07	286.8	2.77	305.3	3.84	322.7
M_2	113.31	69.9	135.50	79.9	123.87	92.4	95.38	115.5
M_4	12.25	126.2	8.06	193.5	10.00	202.1	12.43	190.1
M_6	7.79	97.3	5.20	147.7	2.96	208.2	2.31	204.7

1906.

Plaats:	Moerdijk		Mond der Donge		Willemsdorp		Hellevoetsluis	
GETIJ.	H (c.M.)	k (gr.)	H (c.M.)	k (gr.)	H (c.M.)	k (gr.)	H (c.M.)	k (gr.)
A_0	+ 21.42		+ 22.40		+ 15.34		— 4.60	
S_1	0.80	69.2	0.76	84.7	0.60	36.3	0.86	321.6
$S_2 \cos(ks_2—60°)$	— 14.89		— 13.79		— 15.20		+ 2.36	
P	2.56	29.0	2.24	67.2	3.00	32.0	2.89	5.6
K_1	6.60	34.9	4.67	56.0	6.06	41.8	6.72	12.5
K_2	5.86	208.3	5.71	259.3	5.95	210.9	5.84	149.3
T	1.54	189.4	1.02	347.5	1.80	190.9	1.65	115.3
Sa_1	8.03	303.2	25.42	44.7	9.65	317.2	7.54	259.4
Sa_2	7.44	230.9	10.90	47.9	8.17	237.1	5.43	209.9
Sa_3	4.10	4.2	4.68	234.4	3.61	353.0	2.39	357.8
Sa_4	5.29	325.6	8.07	291.9	5.00	325.0	2.67	303.8
M_2	91.27	133.3	59.67	178.9	89.08	136.6	88.25	89.4
M_4	10.94	218.6	6.49	307.6	11.52	220.5	14.07	145.2
M_6	0.85	153.7	2.42	131.8	1.12	171.2	4.63	104.9

1906.

Plaats:	Spijkenisse		Puttershoek		's-Gravendeel		Dordrecht	
GETIJ.	H (c.M.)	k (gr.)	H (c.M.)	k (gr.)	H (c.M.)	k (gr.)	H (c M.)	k (gr.)
A_0	+ 11.67		+ 11.31[5]		+ 25.22		+ 35.63	
S_1	0.35	257.9	0.93	20.0	1.61	233.8	0.76	84.7
$S_2 \cos(ks_2 - 60°)$	— 7.43		— 14.41		— 15.68		— 14.98	
P	2.75	24.2	2.75	41.8	2.87	219.7	2.72	46.6
K_1	5.67	23.0	5.68	37.8	4.87	219.7	5.34	42.2
K_2	4.57	190.5	5.37	215.6	6.07	216.8	5.47	229.5
T	1.93	162.3	1.88	196.6	1.13	184.0	1.77	214.3
Sa_1	7.83	297.2	9.27	321.2	12.64	324.4	12.92	336.7
Sa_2	7.29	229.2	9.16	234.7	9.07	241.1	10.50	241.2
Sa_3	3.33	356.7	3.97	356.0	3.43	358.5	4.64	350.5
Sa_4	4.23	323.9	6.02	330.4	6.68	333.0	7.62	338.3
M_2	67.83	120.4	74.07	144.8	77.64	147.0	69.84	155.0
M_4	10.66	191.1	11.56	233.5	12.22	213.1	12.61	251.4
M_6	3.28	164.1	1.36	186.1	0.65	188.0	0.84	193.9

1906.

Plaats:	Alblasserdam		Vreeswijk		Schoonhoven		Streefkerk	
GETIJ.	H (c.M.)	k (gr.)	H (c.M.)	k (gr.)	H (c.M.)	k (gr.)	H (c.M.)	k (gr.)
A_0	+ 32.95		+ 190.94		+ 73.43		+ 45.94	
S_1	0.41	41.8	0.58	13.8	0.24	38.3	0.51	42.4
$S_2\ cos(ks_2—60°)$	— 13.90		— 0.46		— 8.33		— 11.40	
P	2.71	44.7	0.64	108.2	1.93	58.7	2.63	51.4
K_1	5.47	39.4	1.89	82.6	4.14	44.2	5.07	43.5
K_2	4.94	226.6	1.27	327.5	3.85	259.2	4.42	240.0
T	1.44	207.6	0.99	280.9	1.36	284.5	1.15	220.8
Sa_1	11.97	333.5	75.81	13.0	30.10	0.8	17.39	346.6
Sa_2	10.30	240.0	37.58	248.7	16.52	251.8	6.99	259.7
Sa_3	4.91	349.0	13.67	351.2	8.95	347.2	6.48	348.8
Sa_4	7.34	223.2	35.32	201.9	16.27	353.4	10.59	345.2
M_2	65.20	157.3	8.10	268.5	39.88	188.5	51.40	171.8
M_4	11.28	251.9	2.17	41.4	10.55	298.0	11.68	272.3
M_6	1.46	208.7	0.79	223.3	1.41	50.4	0.44	317.6

1906.

Plaats:	Krimpen		Rotterdam		Vlaardingen		Maassluis	
GETIJ.	H (c.M.)	k (gr.)	H (c.M.)	k (gr.)	H (c.M.)	k (gr.)	H (c.M.)	k (gr.)
A_0	+ 34.19		+ 19.45		+ 11.25		+ 0.89	
S_1	0.56	38.3	0.43	60.1	0.40	331.3	0.52	304.8
$S_2 \cos(ks_2—60°)$	— 12.73		— 9.80		— 5.61		— 1.00	
P	2.63	41.8	2.74	27.3	2.68	19.4	2.42	9.2
K_1	5.29	35.1	5.71	23.7	5.68	15.6	6.03	6.3
K_2	4.57	224.6	4.61	198.7	4.79	188.3	4.38	171.8
T	1.67	203.8	2.01	178.7	1.99	157.2	1.58	140.0
Sa_1	6.30	344.9	9.02	314.5	7.68	298.1	7.72	281.2
Sa_2	10.61	240.7	8.56	233.0	7.37	229.9	6.54	221.9
Sa_3	4.51	348.8	3.76	354.6	3.53	352.0	3.22	354.9
Sa_4	7.35	337.5	5.35	330.7	4.26	325.2	3.44	314.2
M_2	61.61	153.5	64.53	133.4	65.07	115.2	68.05	98.1
M_4	10.50	245.9	9.93	212.8	10.67	181.5	12.71	162.7
M_6	1.70	208.4	3.30	185.1	3.80	152.2	4.51	122.4

1906.

Plaats:	Hoek van Holland		Scheveningen		Katwijk		IJmuiden	
GETIJ.	H (c.M.)	k (gr.)	H (c.M.)	k (gr.)	H (c.M.)	k (gr.)	H (c.M.)	k (gr.)
A_0	— 11.59		— 6.40		— 6.65		— 16.57	
S_1	1.07	311.1	1.13	309.6	0.99	321.7	0.84	312.7
$S_2 \cos(k_{S_2}—60°)$	+ 6.74		+ 3.04		— 0.66		8.29	
P	2.56	337.9	2.43	338.4	2.29	332.3	3.30	338.7
K_1	7.54	355.7	7.22	354.8	7.10	347.4	7.81	355.5
K_2	4.46	132.2	3.97	143.1	3.53	166.1	4.25	198.1
T	1.09	87.1	0.45	81.0	0.41	189.2	1.31	174.5
Sa_1	8.78	252.7	10.50	268.5	10.57	264.6	11.46	251.8
Sa_2	5.51	204.8	4.86	208.7	5.26	203.3	6.95	222.0
Sa_3	2.07	353.1	2.18	355.8	4.28	359.8	1.64	298.8
Sa_4	2.27	275.8	2.76	282.7	7.24	351.3	2.11	225.2
M_2	77.51	73.0	72.48	82.4	66.54	92.8	66.35	115.0
M_4	17.26	128.9	19.27	131.7	19.21	147.5	17.18	156.4
M_6	5.62	70.6	3.29	94.2	2.23	163.7	3.33	262.2

1906.

Plaats:	Helder		Vlieland		Enkhuizen		Oranjesluizen	
GETIJ.	H (c.M.)	k (gr.)	H (c.M.	k (gr.)	H (c.M.)	k (gr.)	H (c.M.)	k (gr.)
A_0	— 11.12		— 12.73		— 5.69		— 1.75	
S_1	1.05	306.6	1.02	307.0	0.34	265.9	1.44	164.6
$S_2 \cos (ks_2 - 60°)$	— 15.03		— 10.19		+ 1.19		+ 2.16	
P	2.70	345.0	2.37	355.8	0.84	87.4	1.45	119.2
K_1	5.55	6.0	5.40	14.8	1.77	99.7	2.16	140.5
K_2	5.27	233.4	3.84	288.4	1.02	18.7	1.57	101.4
T	1.82	215.4	2.57	231.1	0.29	321.8	0.46	125.7
Sa_1	11.96	255.9	12.41	249.3	6.55	262.4	1.42	241.1
Sa_2	5.52	225.0	4.79	215.6	5.48	233.4	2.98	257.6
Sa_3	2.97	345.6	2.78	338.3	3.45	355.2	4.02	0.9
Sa_4	2.26	250.6	1.26	241.9	5.26	320.6	8.33	338.1
M_2	53.01	169.6	63.99	233.7	11.47	294.1	13.88	13.5
M_4	9.40	190.1	3.19	325.9	2.21	338.9	1.61	271.1
M_6	5.34	299.2	2.72	29.9	1.05	227.0	0.06	53.5

1906.

Plaats:	Nijkerk		Elburg		Kraggenburg		Schokland	
GETIJ.	*H* (c.M.)	*k* (gr.)	*H* (c.M.)	*k* (gr.)	*H* (c.M.)	*k* (gr.)	*H* (c.M.)	*k* (gr.)
A_0	— 2.27		+ 1.20		+ 6.12		+ 5.45	
S_1	1.21	117.6	0.65	75.1	0.68	350.0	0.43	302.2
$S_2 \cos(ks_2—60°)$	+ 2.49		+ 2.12[5]		+ 1.69		+ 1.25	
P	1.22	98.6	0.78	88.6	0.69	76.7	0.66	109.3
K_1	2.95	116.4	2.69	122.6	2.57	118.5	2.20	132.1
K_2	1.15	63.0	0.78	48.0	0.91	36.2	0.59	40.5
T	0.30	97.3	0.07	197.9	0.28	308.3	0.11	97.6
Sa_1	1.76	30.5	2.61	300.7	5.74	272.4	5.99	271.7
Sa_2	4.50	224.2	6.60	238.1	7.30	239.6	6.59	234.4
Sa_3	4.78	9.7	5.43	5.6	5.64	3.8	5.35	5.9
Sa_4	9.98	333.7	8.71	321.4	7.76	312.0	7.25	316.5
M_2	13.40	24.7	10.21	28.3	6.35	5.7	5.99	28.7
M_4	1.39	260.2	1.01	286.3	0.70	294.1	0.78	305.2
M_6	0.85	53.3	1.38	12.4	1.88	8.3	1.12	2.2

1906.

Plaats:	Urk		Lemmer		Stavoren		Hindeloopen	
GETIJ.	H (c.M.)	k (gr.)	H (c.M.)	k (gr.)	H (c.M.)	k (gr.)	H (c.M.)	k (gr.)
A_0	+ 3.07		+ 5.97		+ 0.66		+ 2.00	
S_1	0.29	282.0	0.68	303.0	0.60	320.6	0.85	322.2
$S_2 \cos(k s_2 - 60^\circ)$	+ 1.47		+ 1.17		— 1.63		— 1.72	
P	0.69	103.9	0.69	97.2	1.00	34.6	1.18	29.0
K_1	2.13	120.7	1.74	123.3	2.09	50.6	2.71	42.4
K_2	0.70	28.8	0.68	21.2	1.33	319.9	1.95	307.8
T	0.28	315.5	0.49	318.8	0.55	279.7	0.72	287.5
Sa_1	6.44	264.9	8.82	269.0	8.96	266.1	10.22	266.1
Sa_2	6.02	230.7	7.10	237.7	5.97	230.1	6.42	233.5
Sa_3	4.94	8.2	4.48	354.4	2.09	359.8	3.88	351.9
Sa_4	6.55	316.9	5.53	303.5	4.76	303.2	4.49	297.4
M_2	6.74	346.4	4.34	342.0	21.14	245.4	26.41	246.9
M_4	1.18	315.2	1.34	318.7	3.45	325.5	2.87	327.3
M_6	0.74	315.3	0.76	335.1	1.28	142.7	1.50	146.1

1906.

Plaats:	Harlingen		Roptazijl		Zoutkamp		Delfzijl	
GETIJ.	H (c.M.)	k (gr.)	H (c.M.)	k (gr.)	H (c.M.)	k (gr.)	H (c.M.)	k (gr.)
A_0	—4.23		—1.85		+0.01[5]		—10.68	
S_1	1.08	314.3	1.02	327.0	2.73	2.3	1.66	22.3
$S_2\ cos\,(ks_2—60°)$	—0.61		+0.07		+10.39		+26.23	
P	2.10	14.5	1.95	11.6	2.76	36.4	3.31	38.1
K_1	4.81	31.4	4.78	28.1	6.22	37.7	6.62	41.2
K_2	3.63	331.4	3.65	336.5	8.01	16.4	10.31	35.3
T	0.75	285.7	1.16	278.8	2.06	313.4	2.27	344.3
Sa_1	11.59	259.2	11.93	261.8	14.82	278.2	10.75	240.9
Sa_2	5.71	228.1	6.26	231.2	9.24	233.8	6.24	244.9
Sa_3	3.70	349.3	3.93	353.5	4.54	324.7	6.87	348.3
Sa_4	3.85	285.6	4.00	282.7	5.97	294.6	6.20	306.3
M_2	56.32	261.6	61.50	264.4	96.09	291.2	119.95	324.1
M_4	3.82	17.0	4.08	57.9	9.79	44.3	12.59	137.9
M_6	3.49	183.4	3.25	194.1	3.06	212.3	9.54	312.4

1906.

Plaats:	Nieuw-Statenzijl	
GETIJ	H (c.M.)	k (gr.)
A_0	+ 18.59	
S_1	4.58	357.4
$S_2 \cos(ks_2 - 60°)$	+ 24.89	
P	3.31	56.6
K_1	8.21	78.4
K_2	13.71	69.6
T	4.20	262.1
Sa_1	14.53	296.2
Sa_2	16.29	251.9
Sa_3	2.55	1.6
Sa_4	10.74	319.5
M_2	105.87	354.0
M_4	7.49	214.1
M_6	2.41	350.0

GETIJCONSTANTEN

berekend uit de

Waterstanden te *5—11—17—23* uur

van

1906.

1906.

Plaats:	Ostende		Neuzen		Hansweert		Vlissingen	
GETIJ	H (c.M.)	k (gr.)	H (c.M.)	k (gr.)	H (c.M.)	k (gr.)	H (c.M.)	k (gr.)
A_0	+ 317.38		— 10.68		— 0.79		— 17.20	
S_1	1.11	3.8	3.53	302.3	1.39	333.0	0.84	307.4
$S_2 \sin(ks_2—60°)$	— 1.55		+ 37.68		+ 43.77		+ 27.81	
P	3.20	348.7	2.76	341.4	2.92	0.4	3.31	337.7
K_1	6.07	356.2	8.11	15.9	7.12	22.3	6.83	5.5
K_2	15.18	73.1	14.68	112.9	15.15	129.3	13.66	103.6
T	3.36	75.5	3.16	93.2	3.38	132.4	3.10	91.3
Sa_1	8.94	231.2	7.10	264.5	8.58	283.9	7.72	251.8
Sa_2	5.40	197.1	3.89	210.4	4.61	231.2	4.12	209.6
Sa_3	0.75	355.2	1.60	321.7	0.87	334.1	1.30	349.3
Sa_4	0.66	218.0	1.97	255.5	1.47	211.2	1.03	193.6
M_2	181.86	17.5	174.82	56.0	183.38	65.9	167.13	41.2
M_4	11.43	357.0	8.44	122.2	5.67	155.5	10.43	89.0
M_6	6.06	316.1	6.64	131.5	9.61	189.2	6.03	64.2

1906.

Plaats:	Wemeldinge		Zierikzee		Brouwershaven		Willemstad	
GETIJ	H (c.M.)	k (gr.)	H (c.M.)	k (gr.)	H (c.M.)	k (gr.)	H (c.M.)	k (gr.)
A_0	— 7.48		— 13.53		— 13.16		+ 7.04	
S_1	1.24	341.9	1.20	331.6	1.35	290.6	0.78	39.3
$S_2 \sin (ks_2—60°)$	+ 34.90		+ 29.37		+ 24.88		+ 19.39	
P	2.89	346.3	2.83	343.2	3.33	353.0	2.51	22.8
K_1	7.59	21.2	7.31	12.5	7.19	7.3	5.56	6.1
K_2	12.33	131.8	9.93	128.3	8.96	126.6	6.71	172.6
T	3.71	112.9	3.34	99.7	2.80	109.1	1.51	141.3
Sa_1	7.27	248.2	7.57	248.6	7.74	255.3	9.00	290.7
Sa_2	3.56	192.9	3.32	200.7	3.78	208.5	5.86	220.1
Sa_3	1.61	12.1	2.76	352.3	2.33	343.0	1.48	2.6
Sa_4	2.37	287.1	2.44	276.7	1.88	280.2	2.92	335.5
M_2	143.17	74.1	130.66	63.7	106.71	67.7	92.20	117.7
M_4	8.37	198.9	6.71	155.0	13.17	135.4	14.00	183.3
M_6	7.28	217.1	3.26	183.9	4.08	139.7	6.50	249.8

1906.

Plaats:	Hellevoetsluis		Rotterdam		Hoek van Holland		IJmuiden	
GETIJ	H (c.M.)	k (gr.)	H (c.M.)	k (gr.)	H (c.M.)	k (gr.)	H (c.M.)	k (gr.)
A_0	— 3.86		+ 19.51		— 10.70		— 15.26	
S_1	1.00	20.1	0.48	31.7	1.32	292.4	1.17	306.3
$S_2 \sin(ks_2—60°)$	+ 20.17		+ 10.38		+ 17.20		+ 14.02	
P	2.75	0.0	2.23	28.1	3.41	338.9	3.29	341.0
K_1	6.80	12.5	5.63	20.8	7.82	357.0	7.74	353.5
K_2	6.70	143.8	3.95	198.6	6.53	131.1	4.97	170.5
T	2.83	111.7	1.38	170.1	2.43	120.5	1.89	127.6
Sa_1	8.04	264.1	8.78	314.4	8.19	254.9	10.00	255.8
Sa_2	5.09	211.8	7.28	230.2	4.73	217.6	5.40	224.7
Sa_3	2.51	350.0	3.61	342.2	2.03	345.7	1.84	289.9
Sa_4	2.45	291.8	4.56	324.8	2.00	263.9	3.58	218.9
M_2	82.63	89.6	64.43	136.2	72.86	71.7	63.86	116.7
M_4	15.28	149.6	10.67	200.3	16.74	132.5	19.18	152.7
M_6	1.34	127.4	3.52	226.9	1.56	74.0	5.48	248.8

1906.

Plaats:	Helder		Vlieland		Harlingen		Delfzijl	
GETIJ	H (c.M.)	k (gr.)	H (c.M.)	k (gr.)	H (c.M.)	k (gr.)	H (c.M.)	k (gr.)
A_0	— 10.92		— 12.47		— 3.99		— 10.80	
S_1	0.96	312.6	0.62	333.1	1.07	348.1	1.42	74.5
$S_2 \sin(ks_2—60°)$	+ 1.28		— 13.11		— 14.35		— 13.87	
P	2.67	346.0	2.36	358.7	2.16	24.5	3.09	15.3
K_1	5.46	359.3	5.15	8.9	4.98	31.0	6.75	39.3
K_2	3.91	242.0	5.69	298.7	5.68	320.8	8.11	28.7
T	0.70	185.9	1.45	272.1	1.83	295.5	1.07	329.2
Sa_1	11.63	255.5	13.23	251.0	12.06	262.4	10.55	241.7
Sa_2	4.95	213.6	5.45	214.5	6.22	231.1	5.95	244.4
Sa_3	2.43	336.1	2.76	334.5	3.72	349.4	6.98	350.3
Sa_4	3.13	247.7	1.26	244.6	3.41	293.4	5.53	291.3
M_2	56.10	170.4	65.30	229.9	55.27	261.7	121.98	325.7
M_4	13.35	183.3	1.97	278.0	4.58	77.1	14.03	137.8
M_6	7.47	292.1	5.93	33.6	2.00	148.4	2.80	21.4

GETIJCONSTANTEN

berekend uit de

Waterstanden te *2—5—8—11—14—17—20—23* uur

van

1906.

OPMERKINGEN.

De constanten van Hansweert van de *S* serie zijn berekend na schatting van de ontbrekende gemiddelden. Zie n°. 1. De overige, uitgezonderd het getij ν, waarvoor eene correctie wegens *N*, en *MSf*, waarvoor de bekende correctie wegens M_2 is aangebracht, zijn zonder correctie's voor storende getijden berekend. Evenwel met inachtneming van de voorschriften voor den aanvang en het einde van een bepaalde periode (zie DARWIN Sc. P. I p. 243).

Bij *MSf* zijn de uitkomsten volgens n°. 15 onder die, volgens de methode van DARWIN verkregen, geplaatst.

1906.

Plaats:	Ostende		Neuzen		Hansweert		Vlissingen	
GETIJ	H (c.M.)	k (gr.)	H (c.M.)	k (gr.)	H (c.M.)	k (gr.)	H (c.M.)	k (gr.)
A_0	+ 317.58		— 11.10		— 1.16		— 17.52	
S_1	0.72	327.1	1.59	305.3	1.00	339.2	0.79	308.0
S_2	54.65	68.4	46.96	113.4	47.63	126.8	45.80	97.4
$S_4 \cos(ks_4—120°)$	+ 0.20		— 0.42		— 0.37		— 0.32	
P	2.89	333.2	2.56	351.3	2.88	10.4	2.70	339.8
K_1	6.26	359.3	7.24	14.6	7.16	24.4	6.77	5.7
K_2	16.81	67.4	14.76	111.7	13.72	127.5	13.14	99.1
T	3.03	77.1	3.44	90.6	2.92	104.0	3.35	78.8
R	0.34	88.8	0.33	237.8	1.60	287.6	0.74	284.8
Sa_1	8.39	230.5	7.21	264.6	8.72	277.0	7.51	250.6
Sa_2	4.77	189.7	4.05	201.1	5.20	226.8	4.02	199.7
Sa_3	0.53	248.7	1.67	332.3	0.86	299.8	1.19	349.7
Sa_4	1.27	223.7	1.71	265.0	1.73	224.3	1.60	216.6
M_1	1.02	69.0	1.58	70.8	1.05	92.8	1.33	61.9
M_2	179.60	17.4	177.14	57.6	186.52	67.4	167.89	42.5
M_3	1.46	85.4	1.70	170.3	2.68	156.5	1.15	124.9
M_4	10.93	350.7	8.61	107.6	5.17	132.0	11.38	77.2
M_6	6.11	327.2	7.54	103.0	6.63	168.5	9.16	52.3
M_8	2.97	262.0	2.95	72.4	3.98	148.3	3.20	26.8
N	30.80	357.6	29.11	41.9	29.70	44.5	27.78	23.5
L	10.74	23.5	14.58	78.8	16.87	70.9	12.15	46.3
ν	9.92	321.0	9.85	354.5	12.75	31.2	9.05	350.3
λ	5.24	55.4	4.54	90.2			4.60	72.1
O	9.89	183.3	10.70	201.7	12.69	197.6	10.60	191.0
Q	4.57	123.6	4.54	138.2	5.74	140.8	5.18	132.3
J	0.08	359.0	0.87	152.1	0.86	187.3	0.33	228.6
MS	7.49	229.8	5.23	173.9	2.09	206.0	6.89	142.4
μ of 2 MS	10.73	252.4	13.76	167.4	17.63	167.7	11.09	153.5
2 SM	3.07	272.6	5.30	359.2	5.67	356.8	3.92	359.2
Mm	4.72	246.9	4.66	272.0	5.65	259.1	4.77	260.1
Mf	5.13	62.8	4.91	55.0	2.11	19.8	5.20	51.2
MSf	{ 1.68	63.8	4.50	31.2	4.79	19.4	3.59	53.3
	{ 1.98	21.4	5.09	22.7	5.37	7.3	3.79	23.4

1906.

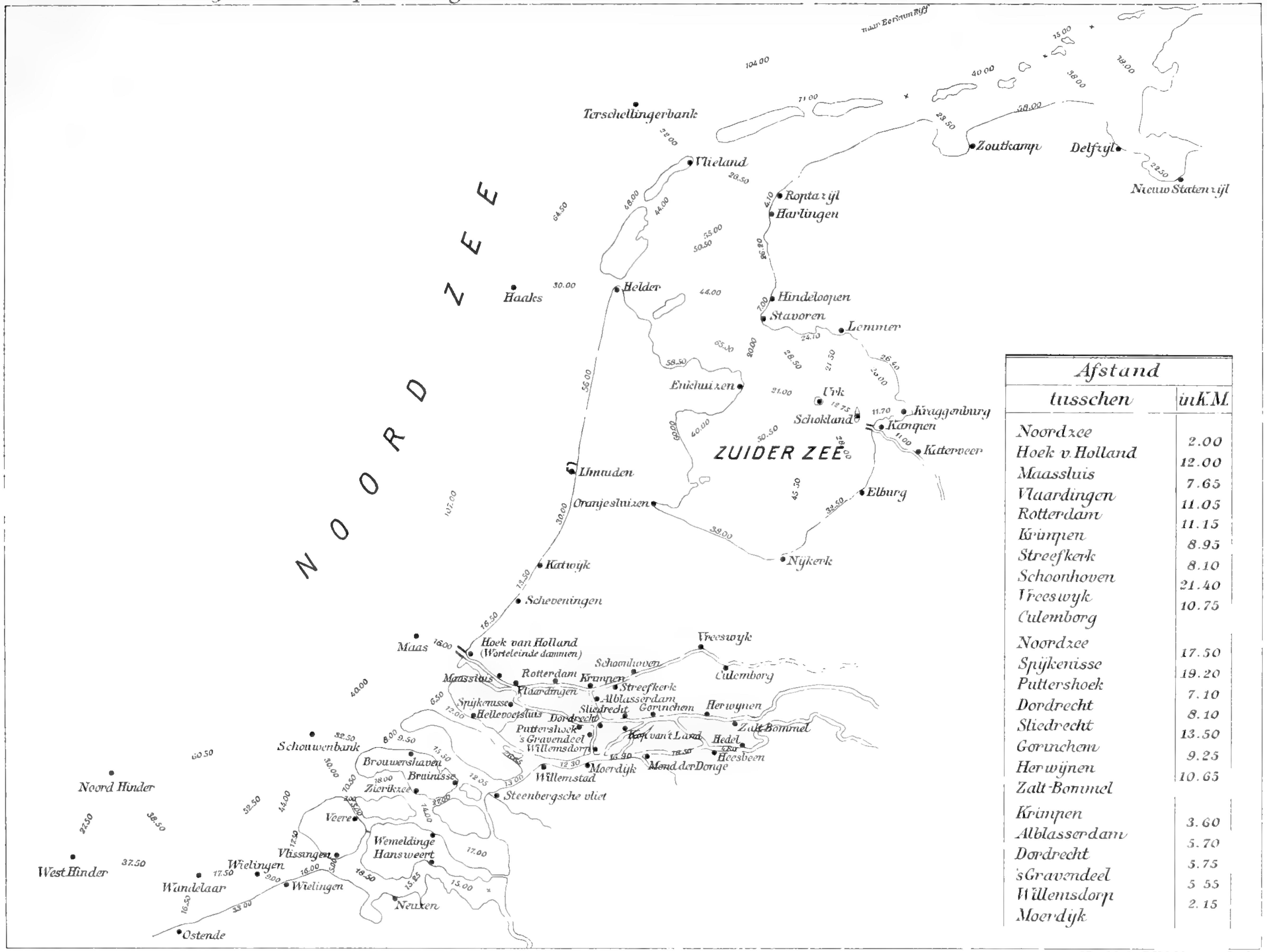
Plaats:	Wemeldinge		Zierikzee		Brouwershaven		Willemstad	
GETIJ.	H (c.M.)	k (gr.)	H (c.M.)	k (gr.)	H (c.M.)	k (gr.)	H (c.M.)	k (gr.)
A_0	— 7.88		-13.82^5		— 13.82		+ 6.87	
S_1	0.98	327.8	1.06	325.9	0.91	301.4	0.57	37.4
S_2	36.51	132.9	33.46	121.4	27.43	125.1	21.03	176.3
$S_4 \cos (ks_4—120°)$	—0.39		—0.30		—0.66		— 0.17	
P	2.93	3.9	2.58	353.3	3.17	356.2	2.85	23.0
K_1	7.44	21.5	7.18	13.9	7.15	8.5	6.29	22.5
K_2	11.24	135.8	9.25	127.9	7.79	130.4	5.67	180.9
T	3.17	114.1	3.29	100.0	2 78	88.9	1.63	156.1
R	0.54	13.5	0.05	45.9	0.98	291.6	0.42	77.9
Sa_1	7.47	244.8	7.79	247.2	7.89	251.9	8.43	289.6
Sa_2	3.70	195.7	3.86	200.9	4.21	202.9	6.15	224.5
Sa_3	1.96	10.9	2.57	352.7	2.20	352.8	2.44	4.5
Sa_4	2.30	285.7	1.98	269.5	2.01	282.3	3.36	328.2
M_1	1.43	83.1	1.20	79.0	1.39	75.5	1.00	90.1
M_2	146.85	75.2	133.62	65.1	109.99	68.8	93.78	116.8
M_3	1.02	186.1	0.74	174.6	0.58	163.3	0.77	229.6
M_4	7.16	199.3	5.88	142.1	12.67	131.0	13.19	186.5
M_6	4.42	197.2	3.02	102.6	5.58	111.5	4.48	238.4
M_8	1.51	101.7	2.64	95.6	0.41	185.5	1.42	31.1
N	23.97	58.0	21.65	46.7	16.79	49.5	14.03	98.4
L	12.94	78.7	11.34	71.8	9.56	79.0	8.89	124.6
ν	9.49	17.2	7.65	8.4	5.96	20.2	6.28	63.3
λ	5.06	111.9	3.91	104.2	3.21	101.8	3.12	163.3
O	10.55	202.6	10.52	196.8	9.22	195.6	9.15	212.7
Q	5.13	148.4	5.04	143.1	5.05	137.6	4.54	162.8
J	0.45	195.1	0.27	128.5	0.32	121.4	0.47	316.8
MS	4.32	239.3	3.71	196.4	7.81	185.4	7.77	244.2
μ of $2\,MS$	12.54	184.1	10.75	180.0	9.64	187.3	8.99	227.9
$2\,SM$	4.53	10.5	4.31	9.0	3.92	18.6	2.19	46.6
Mm	4.58	268.8	5.53	267.4	5.20	261.0	5.53	275.9
Mf	4.94	48.4	4.95	50.0	4.57	50.0	5.67	46.7
MSf	3.52	42.8	3.31	34.1	3.18	38.5	4.51	42.8
	3.78	23.2	3.89	14.9	3.38	15.1	4.53	28.0

1906.

Plaats:	Hellevoetsluis		Rotterdam		Hoek van Holland		IJmuiden	
GETIJ.	H (c.M.)	k (gr.)	H (c.M.)	k (gr.)	H (c.M.)	k (gr.)	H (c.M.)	k (gr.)
A_0	—4.23		+19.48		—11.18		—15.91	
S_1	0.99	351.4	0.39	43.5	1.10	297.6	1.01	301.0
S_2	20.30	144.8	14.28	193.5	18.48	128.6	16.29	180.6
$S_4 \cos(ks_4—120^\circ)$	—0.37		—0.03		—0.47		—0.65	
P	2.76	3.5	2.49	30.1	3.00	335.1	3.32	339.9
K_1	6.75	12.2	5.61	22.4	7.72	355.9	7.72	354.7
K_2	6.26	146.3	4.29	198.6	5.50	131.5	4.47	183.2
T	2.23	112.7	1.69	175.4	1.70	110.4	1.47	146.6
R	0.61	14.5	0.34	101.9	0.82	338.0	0.69	36.4
Sa_1	7.78	261.8	8.90	314.5	8.49	253.7	10.73	253.7
Sa_2	5.26	210.8	8.08	231.7	5.09	210.7	6.18	223.2
Sa_3	2.44	353.8	3.66	348.5	2.05	349.4	1.73	294.1
Sa_4	2.55	298.0	4.95	328.0	2.12	280.2	2.84	221.3
M_1	1.01	74.6	1.08	96.6	1.35	67.6	1.38	65.3
M_2	85.44	89.5	64.37	134.8	75.18	72.4	65.10	115.8
M_3	0.75	199.5	0.40	219.6	0.48	230.7	0.49	271.0
M_4	14.67	147.5	10.24	206.3	16.99	130.7	18.18	154.5
M_6	2.94	109.9	3.19	206.7	3.57	71.1	4.37	253.9
M_8	1.22	163.5	0.43	189.6	1.40	172.1	2.66	265.0
N	13.29	71.3	9.46	118.4	10.84	52.8	9.15	101.6
L	8.23	98.7	9.74	146.0	7.15	82.4	7.75	112.7
ν	5.24	39.2	4.32	82.3	4.46	21.8	4.37	65.2
λ	2.75	131.1	1.46	141.9	2.60	118.3	2.26	148.9
O	10.25	198.6	8.39	211.7	11.19	187.0	11.42	187.2
Q	4.86	146.8	3.98	160.8	4.94	132.7	4.89	134.5
J	0.88	341.2	0.36	5.9	0.31	127.6	0.46	166.6
MS	8.99	201.2	6.85	261.1	10.21	182.9	10.10	211.2
μ of 2 MS	8.32	207.8	7.76	253.9	7.68	201.3	8.06	219.7
2 SM	2.82	37.5	1.53	73.3	2.48	27.7	1.59	42.6
Mm	5.47	267.8	6.49	277.8	5.66	262.0	6.20	257.4
Mf	5.30	65.2	4.73	45.9	4.15	47.3	3.79	52.5
MSf	2.86	23.1	4.45	40.2	2.93	39.9	3.01	40.2
	3.73	31.2	4.13	40.2	3.08	18.3	3.13	13.4

1906.

Plaats:	Helder		Vlieland		Harlingen		Delfzijl	
GETIJ.	H (c.M.)	k (gr.)	H (c.M.)	k (gr.)	H (c.M.)	k (gr.)	H (c.M.)	k (gr.)
A_0	— 11.02		— 12.60		— 4.11		— 10.74	
S_1	1.03	310.5	0.88	322.4	1.16	331.2	1.66	48.1
S_2	15.08	235.1	16.60	292.1	14.36	327.6	29.67	32.1
$S_4 \cos (ks_4 - 120°)$	— 0.10		— 0.14		— 0.12		+ 0.06	
P	2.70	345.6	2.39	357.3	2.20	19.3	2.88	28.0
K_1	5.36	2.9	5.15	12.5	4.89	30.6	6.64	40.0
K_2	4.58	237.1	4.75	294.5	4.61	325.0	9.20	32.4
T	1.23	207.4	1.89	245.7	1.29	292.6	1.66	339.5
R	0.63	62.8	0.88	101.6	0.55	177.9	0.63	302.9
Sa_1	11.80	255.7	12.84	250.2	11.82	260.6	10.65	241.3
Sa_2	5.20	219.6	5.11	215.0	5.97	229.5	6.10	244.7
Sa_3	2.70	341.3	2.77	336.4	3.70	349.3	6.92	349.2
Sa_4	2.69	248.9	1.26	243.3	3.62	289.2	5.82	299.3
M_1	1.01	80.3	0.91	101.7	0.93	169.4	0.76	136.0
M_2	54.56	170.0	64.64	231.8	55.79	261.7	120.95	324.9
M_3	0.54	249.3	0.44	353.9	0.20	54.5	0.69	153.2
M_4	11.36	187.1	2.37	308.0	3.64	50.0	13.31	137.8
M_6	6.39	295.1	4.32	32.5	2.62	170.9	5.42	326.0
M_8	1.34	340.6	1.41	296.0	3.07	315.5	0.21	152.3
N	8.79	158.3	10.27	219.0	9.58	254.0	18.99	307.8
L	5.18	158.9	5.20	224.9	5.48	253.5	9.09	323.8
ν	3.42	106.8	3.79	173.0	3.93	204.2	7.23	269.6
λ	2.31	186.6	2.05	257.3	2.01	246.5	4.50	353.2
O	8.27	196.4	6.35	210.9	6.72	227.1	9.08	241.4
Q	3.73	142.5	3.04	156.0	2.70	170.8	3.39	178.0
J	0.41	205.7	0.45	191.9	0.34	189.7	0.70	252.4
MS	6.43	247.4	1.82	44.0	2.93	123.4	7.48	213.6
μ of $2MS$	5.80	262.0	7.25	344.2	6.29	358.1	12.56	57.9
$2SM$	1.58	117.4	2.24	213.2	0.56	218.7	1.96	276.2
Mm	6.51	261.6	6.23	264.3	7.18	271.0	5.94	279.3
Mf	4.60	59.9	4.15	49.5	5.76	59.5	5.22	37.8
MSf	1.98	47.6	2.79	54.5	5.22	55.2	2.69	108.4
	2.44	8.1	2.39	28.2	5.17	37.6	1.77	51.1



Afstand tusschen	in K.M.
Noordzee	
Hoek v. Holland	2.00
Maassluis	12.00
Vlaardingen	7.65
Rotterdam	11.05
Krimpen	11.15
Streefkerk	8.95
Schoonhoven	8.10
Vreeswyk	21.40
Culemborg	10.75
Noordzee	
Spijkenisse	17.50
Puttershoek	19.20
Dordrecht	7.10
Sliedrecht	8.10
Gorinchem	13.50
Herwijnen	9.25
Zalt-Bommel	10.65
Krimpen	
Alblasserdam	3.60
Dordrecht	5.70
's Gravendeel	5.75
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Analytical treatment of the polytopes regularly derived from the regular polytopes.

(Section I: The simplex).

BY

P. H. SCHOUTE.

Verhandelingen der Koninklijke Akademie van Wetenschappen te Amsterdam.

(EERSTE SECTIE).

DEEL XI. N°. 3.

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AMSTERDAM,

JOHANNES MÜLLER.

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INTRODUCTION.

In a memoir recently published by this Academy (*Verhandelingen*, vol. XI, n°. 1) M[rs]. A. BOOLE STOTT has given the geometrical treatment of the polytopes regularly derived from the regular polytopes of polydimensional space. In these pages I wish to complete her beautiful considerations by giving the analytical counterpart [1]).

The basis of this analytical counterpart is the fact that the coordinates of the vertices of the tetrahedron may be represented by one of the symbols $(1, 0, 0, 0)$ and $\frac{1}{2}[1, 1, 1]$, while those of the vertices of cube, octahedron and icosahedron can be put in the forms $[1, 1, 1]$, $[1, 0, 0]$ and $[1+\sqrt{5}, 2, 0] : 2$ respectively. The meaning of these symbols will be explained later on.

This paper is divided into five sections. In the first, concerned with the offspring of the regular simplex, we will meet chiefly the amplifications of the symbol $(1, 0, 0, 0)$ of the tetrahedron. The second and the third, dealing in the same manner with the measure polytope and the cross polytope, will bring us chiefly amplifications of the symbols $[1, 1, 1]$ and $[1, 0, 0]$ of cube and octahedron. The fourth will deal with the half measure polytopes and allied forms represented by amplifications of the symbol $\frac{1}{2}[1, 1, 1]$. Finally in the fifth section about the extra regular polyhedra and polytopes we will have to use the symbol of the icosahedron.

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[1]) I had the great advantage of reading the original manuscript to M[rs]. STOTT; the ensuing discussion — I acknowledge this with thankfulness — has led to a simplification of the proofs of several of the theorems.

Section I: Polytopes deduced from the simplex.

A. *The symbol of coordinates.*

1. In a preceding paper (*Nieuw Archief voor Wiskunde*, vol. IX, p. 133) I found that the distance r between two points P, P', the barycentric coordinates of which — with respect to a regular simplex $S(n+1)$ of space S_n — are $\mu_1, \mu_2, \ldots, \mu_{n+1}$ and $\mu'_1, \mu'_2, \ldots, \mu'_{n+1}$, is represented, with the length of the edge of the simplex of coordinates as unit, by the simple formula

$$r^2 = \tfrac{1}{2}\sum_{i=1}^{n+1}(\mu_i - \mu'_i)^2 \ldots\ldots\ldots\ldots\ldots \quad 1).$$

We insert here a much simpler deduction of this formula; of this deduction fig. 1 gives a geometrical represensation for the particular case $n = 2$ of the plane.

Let $\sum\limits_{i=1}^{n+1} x_i = 1$ represent the space S_n determining in a space of operation S_{n+1}, on the axes of a given system of rectangular coordinates with origin O, the points $A_1, A_2, \ldots, A_{n+1}$ at positive distances unity from O.

Let P and P' with the orthogonal coordinates $m_1, m_2, \ldots, m_{n+1}$ and $m'_1, m'_2, \ldots, m'_{n+1}$ be any two points of this S_n; then, according to the expression for the distance in rectangular coordinates, we have $\overline{PP'}^2 = \sum\limits_{i=1}^{n+1}(m_i - m'_i)^2$ and, as the points lie in S_n, the conditions $\sum\limits_{i=1}^{n+1} m_i = 1$, $\sum\limits_{i=1}^{n+1} m'_i = 1$ hold.

Now let us consider the normal distance coordinates $\overline{\mu}_i$ and $\overline{\mu}'_i$ $(i = 1, 2, \ldots, n+1)$ of P and P' with respect to the regular simplex $S(n+1)$ with the vertices $A_1, A_2, \ldots, A_{n+1}$ in S_n; then from similar rectangular triangles we deduce immediately the relation

$$\frac{\overline{\mu}_i}{m_i} = \frac{\overline{\mu}'_i}{m'_i} = \frac{h}{1},$$

where h is the height of the regular simplex. But, as the barycentric coordinates are normal distance coordinates measured by the corresponding height of the simplex and all these heights are equal in the regular simplex, we find for the barycentric coordinates μ_i and μ'_i

$$\mu_i = \frac{\overline{\mu}_i}{h} = m_i, \quad \mu'_i = \frac{\overline{\mu}'_i}{h} = m'_i,$$

and therefore

$$\overline{PP'}^2 = \sum_{i=1}^{n+1} (\mu_i - \mu'_i)^2.$$

But here PP' is expressed in OA_i as unit. By taking the edge $A_i A_k = OA_i \sqrt{2}$ as unit we find as above

$$r^2 = \tfrac{1}{2} \sum_{i=1}^{n+1} (\mu_i - \mu'_i)^2.$$

The formula 1) enables us to find an answer to the following question, now forming our starting point:

"Under what circumstances will the series of points obtained by giving to the set of barycentric coordinates $x_1, x_2, \ldots, x_{n+1}$ a determinate set of values taken in all possible permutations form the vertices of a polytope all the edges of which have the same length, say the length of the edge of the simplex of coordinates?"

The very simple answer is given by the theorem:

THEOREM I. "If the $n+1$ values $a_1, a_2, \ldots, a_{n+1}$ satisfying the relation $\sum_{i=1}^{n+1} a_i = 1$ are arranged in decreasing order, so that we have

$$a_1 \geqq a_2 \geqq \ldots \geqq a_k \geqq a_{k+1} \geqq \ldots \geqq a_{n+1},$$

the difference $a_k - a_{k+1}$ of any two adjacent values must be either one or zero."

Proof. According to 1) the square of the distance PQ between any two vertices P, Q of the set is a sum of squares; from this it is evident that in order to make the distance a minimum we have to select two points P, Q which are transformed into each other by interchanging only one pair of coordinate values, say a_k and $a_{k'}$. But then the square of PQ is $\frac{1}{2}\left[(a_k - a_{k'})^2 + (a_{k'} - a_k)^2\right] = (a_k - a_{k'})^2$, and therefore PQ itself is $a_k - a_{k'}$. Now this difference becomes a minimum, if a_k and $a_{k'}$ are unequal adjacent values. As this minimum distance must be an edge, the condition that all the edges are to have the length unity implies that the difference between any two different adjacent values must be one.

2. As the condition stated in theorem I depends upon the *differences* of the corresponding barycentric coordinates x_i and x'_i we may drop the conditions $\sum x_i = 1$, $\sum x'_i = 1$ by allowing these coordinates either to increase or to diminish *all* of them by the *same* amount, so as to make e. g. either the smallest or the greatest of the $n+1$ values equal to zero. So in order to avoid fractions

we will indicate the system of points $(2\frac{1}{5}, 1\frac{1}{5}, \frac{1}{5}, -\frac{4}{5}, -1\frac{4}{5})$ of S_4 for which the difference of any pair adjacent values is unity by $(4, 3, 2, 1, 0)$. Indeed, it is easy to return from $(4, 3, 2, 1, 0)$, where the sum of the values is 10 and therefore too large by 9, to the real coordinate values by subtracting $\frac{9}{5}$ from each; so this important simplification — of a temporary character — can do no harm.

We indicate the entire system of points obtained by taking the values $4, 3, 2, 1, 0$ in all possible orders of succession by putting these values in decreasing order between round brackets; this symbol $(4\ 3\ 2\ 1\ 0)$ will be called the "zero symbol" in distinction with the symbol of true coordinates.

3. The simplification alluded to above allows us to indicate in a very transparent manner the sets of values furnishing the polytopes with one kind of vertex and one length of edge (equal to the edge of the simplex of coordinates) found by M^rs^. Stott. So we have for $n = 2, 3, 4, 5$ successivily in the symbols explained in the memoir of M^rs^. Stott:

$$n = 2$$

$(100) = p_3$, $(110) = -p_3$, $*(210) = p_6$

$$n = 3$$

$(1000) = T$, $(1110) = -T$, $*(2110) = CO$

$*(1100) = O$, $(2100) = tT$, $*(3210) = tO$

$$n = 4$$

$(10000) = S(5)$, $(21000) = e_1 S(5)$, $(32100) = e_1 e_2 S(5)$

$(11000) = c e_1 S(5)$, $(21100) = e_2 S(5)$, $(32110) = e_1 e_3 S(5) = - e_2 e_3 S(5)$

$(11100) = c e_2 S(5) = - c e_1 S(5)$, $*(21110) = e_3 S(5)$, $(33210) = c e_1 e_2 e_3 S(5) = - e_1 e_2 S(5)$

$(11110) = c e_3 S(5) = - S(5)$, $*(22100) = c e_1 e_2 S(5)$, $*(43210) = e_1 e_2 e_3 S(5)$

$(22110) = c e_1 e_3 S(5)$
$= c e_2 e_3 S(5) = - e_2 S(5)$,

$$n = 5$$

$(100000) = S(6)$, $(221000) = c e_1 e_2 S(6)$, $*(332100) = c e_1 e_2 e_3 S(6)$

$(110000) = c e_1 S(6)$, $*(221100) = c e_1 e_3 S(6)$, $(432100) = e_1 e_2 e_3 S(6)$

$*(111000) = c e_2 S(6)$, $(321000) = e_1 e_2 S(6)$, $(432110) = e_1 e_2 e_4 S(6)$

$(210000) = e_1 S(6)$, $(321100) = e_1 e_3 S(6)$, $*(432210) = e_1 e_3 e_4 S(6)$

$(211000) = e_2 S(6)$, $(321110) = e_1 e_4 S(6) = - e_3 e_4 S(6)$, $(433210) = e_2 e_3 e_4 S(6)$

$(211100) = e_3 S(6)$, $(322100) = e_2 e_3 S(6)$, $*(543210) = e_1 e_2 e_3 e_4 S(6)$

$*(211110) = e_4 S(6)$, $*(322110) = e_2 e_4 S(6)$,

In general a form obtained in this way may present itself in two different positions with respect to the simplex $S(n+1)$ of coordinates. So we may write e. g. for (21100) also $(0-1-1-2-2)$, or, if we invert the sign of all the coordinates and indicate that we have done so by putting the sign minus before the brackets, $-(01122)$, i.e. $-(22110)$; so $(21100)=-(22110)$ and likewise $(22110)=-(21100)$. Really the symbols with the values satisfying the condition $\Sigma x_i = 1$ corresponding to (21100) and (01122) are $(\frac{7}{5}\,\frac{2}{5}\,\frac{2}{5} -\frac{3}{5} -\frac{3}{5})$ and (-10011) representing, if we omit the brackets, two points P, P' situated symmetrically to each other with respect to the centre of gravity $(\frac{1}{5}\,\frac{1}{5}\,\frac{1}{5}\,\frac{1}{5}\,\frac{1}{5})$ of the simplex $S(5)$ of coordinates. So (22110) is the form (21100) in opposite orientation; in the equation $(22110)=-(21100)$ the difference in orientation is indicated by the sign minus.

The forms the symbols of which are not affected by the "inversion" mentioned are marked by an asterisk; as they do not alter as a whole when they are put into the opposite position they possess central symmetry [1]).

4. The results obtained show that the geometrical method followed by M^rs^. Stott and the analytical method developed here cover exactly the same ground, i. e. that they lead up to the same system of forms. Nevertheless we should jump to a wrong conclusion, if we deduced from this coincidence of results that by either of the methods all the possible forms with one kind of vertex and one length of edge have been found. We show this by remarking that the combination of the two zero symbols (1000) and (1110), or of the proper values (1000) and $(\frac{1}{2}\,\frac{1}{2}\,\frac{1}{2} -\frac{1}{2})$, of T and $-T$ gives us the vertices of the cube, which implies that all the forms deduced from the cube by Mrs. Stott can be represented by couples of symbols in barycentric coordinates as derived from the tetrahedron [2]).

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[1]) For the deduction of the e and c symbols from the symbols of coordinates and reversely compare the part D of this section.

The first of the tables added at the end of this memoir is destined to put on record for $n=3$, 4, 5 the different polyhedra and polytopes deduced from the simplex with their principal properties. Of this table the first column contains Mrs. Stott's symbol, the second the symbol of coordinates and the third the value by which the coordinates have to be diminished in order to find the true coordinate values for which $\Sigma x_i = 1$. The following columns will be explained farther on.

[2]) So the most complicated form $e_1 e_2 C = tCO$ can be represented by

$$\left|\begin{matrix}\left(\frac{7+3'}{4}, \frac{3-1'}{4}, \frac{-1-1'}{4}, \frac{-5-1'}{4}\right) \\ \left(\frac{7-3'}{4}, \frac{3+1'}{4}, \frac{-1+1'}{4}, \frac{-5+1'}{4}\right)\end{matrix}\right|,$$

if $1'$ and $3'$ stand for $\sqrt{2}$ and $3\sqrt{2}$.

5. We finish the first part of this section by mentioning a theorem already proved in the paper quoted in art. 1, as this theorem will be very useful in future. It is:

"Any two spaces S_{k-1}, S_{n-k} containing together the vertices of a regular simplex $S(n+1)$ of S_n are perfectly normal to each other."

This theorem is an immediate consequence of the property that any two edges without common end point determine a regular tetrahedron and are therefore at right angles to each other. For this implies that, if S_{k-1} contains the vertices $A_1, A_2, \ldots, A_k$ and S_{n-k} the vertices $A_{k+1}, A_{k+2}, \ldots, A_{n+1}$, each of the $k-1$ independant lines $A_1 A_l$ $(l = 2, 3, \ldots, k)$ of S_{k-1} is normal to each of the $n-k$ independent lines $A_{k+1} A_{k+m}$ $(m = 2, 3, \ldots, n-k+1)$ of S_{n-k} [1]).

B. *The characteristic numbers.*

6. We will now explain how the characteristic numbers of the vertices, edges, faces, limiting bodies, etc. can be deduced from the symbol of coordinates.

The larger n is, the more elaborate the process becomes. So, in order to divide the difficulties, we will begin by treating the cases $n = 4$ and $n = 5$ at full length by means of an easy method, working from two different sides, the vertex side and the limiting polyhedron $(n = 4)$ or limiting polytope $(n = 5)$ side, of the series vertex, edge, face, etc. Afterwards we will show a more direct way leading immediately to the knowledge of the forms and the numbers of the different kinds of limits $(l)_p$ of p dimensions.

In the cases $n = 4$ and $n = 5$ of the four and the five characteristic numbers we determine for itself the first two and the last two, using the law of Euler for $n = 4$ as a check and for $n = 5$ as a means of finding the lacking middle number of the faces.

The number of vertices is easily found in all possible cases. If all the $n+1$ digits of the symbol of coordinates of the polytope in S_n are different it is $(n+1)!$ This number $(n+1)!$ must be divided by $2!$ for any two, by $3!$ for any three, by $4!$ for any four digits being equal, etc.

The number of edges can be calculated as soon as we know how many edges pass through each vertex. For the product of this latter number by the number of the vertices indicates how many times

[1]) Compare the theorem of art. 30 (on p. 42 of the first volume of my textbook, "Mehrdimensionale Geometrie") where the "two groups of lines through O" may be replaced by "one group of lines through O (here A_1) and an other group of lines through O' (here A_{k+1})."

an edge passes through a vertex; so the total number of edges is half this product. Now the number of edges passing through a vertex is equal to the number of vertices lying at distance unity from the chosen vertex, and this number is easily determined, as will be shown by examples for $n = 4$ and $n = 5$ separately.

In order to be able to find the number of the limiting bodies ($n = 4$) and that of the limiting polytopes ($n = 5$) we prove in general the following theorem:

THEOREM II. „The non vanishing coefficients c_i of the coordinates x_i in the equation $c_1 x_1 + c_2 x_2 + \ldots = p$ of a limiting space S_{n-1} of the polytope deduced from the regular simplex $S(n+1)$ of S_n must all be equal to each other".

Proof. The linear equation $c_1 x_1 + c_2 x_2 + \ldots = p$ represents, as far as the vertices of the polytope are concerned, more than one equation, if the coefficients c_i are different. We show this by a simple example. If in the case (32110) of S_4 we start from the equation $2x_1 + x_2 = p$ and we try to determine the vertices of the polytope for which the expression $2x_1 + x_2$ becomes either a maximum or a minimum we find the maximum 8 for $x_1 = 3$, $x_2 = 2$ and the minimum 1 for $x_1 = 0$, $x_2 = 1$. So for values of p situated *between* 8 and 1 the space $2x_1 + x_2 = p$ *intersects* the polytope, while it contains a limiting face only — and not a limiting body — for the extreme values 8 and 1 of p, as each of the *couples* of equations $x_1 = 3$, $x_2 = 2$ and $x_1 = 0$, $x_2 = 1$ determines a plane; of these planes the first contains the triangle $x_1 = 3$, $x_2 = 2$ and $x_3, x_4, x_5 = (110)$, the second the hexagon $x_1 = 0$, $x_2 = 1$ and $x_3, x_4, x_5 = (321)$.

From this example can be deduced generally that the equation $c_1 x_1 + c_2 x_2 + \ldots = p$ represents k different equations, as far as the vertices of the polytope are concerned, if the non vanishing coefficients c_i admit together k different values.

The theorem is not reversible, i. e. not every linear equation with equal coefficients c_i represents for the maximum or the minimum value of p a limiting space S_{n-1} of the polytope in S_n. So in the case of the simplex (10000) of S_4 only the five spaces $x_i = 0$ bear limiting tetrahedra of $S(5)$, while the ten spaces $x_i + x_k = 0$ bear faces (100), the ten spaces $x_i + x_k + x_l = 0$ bear edges (10), etc.

In order to find the number of the faces ($n = 4$) and that of the limiting bodies ($n = 5$) we determine the *form* of the limiting bodies ($n = 4$) and that of the limiting polytopes ($n = 5$). For the number of faces ($n = 4$) is half the sum of the numbers of the faces of all the limiting bodies, and the number of limiting bodies

($n = 5$) is half the sum of the numbers of the limiting bodies of the limiting polytopes.

7. We now treat at full length the example **(32110)** mentioned above.

The number of vertices is 5! divided by 2! i. e. $120 : 2 = 60$. The number of edges passing through each vertex is five, for in

$$3\ 2\ 1\ 1\ 0$$

each of the five brackets indicates two coordinates with difference unity the interchanging of which furnishes a new vertex joined by an edge to the "pattern vertex", i. e. to the point with the coordinates 3, 2, 1, 1, 0. So the number of edges is $\frac{60 \times 5}{2} = 150$.

In the case of this polytope we have to consider successively the equations.

$$\begin{array}{llcl}
a) & \ldots\ x_1 + x_2 + x_3 + x_4 = 7 & \text{or} & x_5 = 0, \\
b) & \ldots\ x_1 + x_2 + x_3 = 6 & \text{or} & x_4 + x_5 = 1, \\
c) & \ldots\ x_1 + x_2 = 5 & \text{or} & x_3 + x_4 + x_5 = 2, \\
d) & \ldots\ x_1 = 3 & \text{or} & x_2 + x_3 + x_4 + x_5 = 4.
\end{array}$$

Each of the two equations placed on the same line is a consequence of the other; for in treating the polytope (32110) we have to suppose that the true coordinate values of any point have been increased by such a common amount as to make the sum of the coordinates equal to seven.

a). Both the equations $x_1 + x_2 + x_3 + x_4 = 7$ and $x_5 = 0$ give $x_1, x_2, x_3, x_4 = (3211)$, i. e. for the coordinates x_1, x_2, x_3, x_4 we can take any permutation of 3, 2, 1, 1. But if we subtract a unit from all the coordinates and write $x_5 = -1$ and $x_1, x_2, x_3, x_4 = (2100)$, whereby the constant sum 7 of the coordinates is changed into 2, we see (compare the table in art. 3) that the obtained form is a tT[1]). This tT presents itself five times, as the subscript 5 in $x_5 = 0$ may be any of the five numbers 1, 2, 3, 4, 5.

b). In the case $x_1 + x_2 + x_3 = 6$ which implies $x_4 + x_5 = 1$ we have to combine the two systems $x_1, x_2, x_3 = (321)$ and $x_4, x_5 = (10)$; i. e. we have to combine the system $x_1, x_2, x_3 = (321)$ with each of the two possibilities $x_4 = 1, x_5 = 0$ and $x_4 = 0, x_5 = 1$ giving

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[1]) That (2100) is a polyhedron with 12 vertices, 18 edges, 8 faces limited by 4 regular hexagons and 4 equilateral triangles is immediately found by treating the symbol (2100) in the manner taught here.

two regular hexagons in parallel planes, or we have to combine the system $x_4, x_5 = (10)$ with each of the six possibilities 321, 312, 231, 213, 132, 123 for x_1, x_2, x_3 contained in $x_1, x_2, x_3 = (321)$ giving six parallel edges of the same length. So in order to prove that the result is a hexagonal prism P_6 we have only to show yet that the planes of the two hexagons are normal to the six edges. But this follows from the theorem in art. 5, as the planes of the hexagons are parallel to the plane $x_4 = 0, x_5 = 0$, i. e. to the face $A_1\, A_2\, A_3$ of the simplex of coordinates, while the six edges are parallel to the line $x_1 = 0, x_2 = 0, x_3 = 0$, i. e. to the opposite edge $A_4\, A_5$ of that simplex.

The hexagonal prism P_6 obtained in this manner occurs ten times, as for the subscripts 4 and 5 in the equation $x_4 + x_5 = 1$ we can take any combination of the five numbers 1, 2, 3, 4, 5 by two.

c). For $x_1 + x_2 = 5$ we have either $x_1 = 3, x_2 = 2$ or $x_1 = 2, x_2 = 3$ and in both cases $x_3, x_4, x_5 = (110)$. So we find here ten prisms P_3.

d). Finally $x_1 = 3$ gives $x_2, x_3, x_4, x_5 = (2110)$; so we find here five CO.

All in all we have got the limiting polyhedra

$$5\ tT,\quad 10\ P_6,\quad 10\ P_3,\quad 5\ CO;$$

so their number is 30.

The number of the faces is easily found. As the numbers of faces of tT, P_6, P_3, CO are respectively 8, 8, 5, 14 we get

$$\tfrac{1}{2}(5 \times 8 + 10 \times 8 + 10 \times 5 + 5 \times 14) = 120.$$

So the final result (e, k, f, r) [1]) is (60, 150, 120, 30), in accordance with the law of Euler.

Remark. In the case of the simplex (1000) of S_4 we would have to consider the equations

$$\begin{array}{llll}
a) \ldots & x_1 + x_2 + x_3 + x_4 = 1 & \text{or} & x_5 = 0, \\
b) \ldots & x_1 + x_2 + x_3 = 1 & \text{or} & x_4 + x_5 = 0, \\
c) \ldots & x_1 + x_2 = 1 & \text{or} & x_3 + x_4 + x_5 = 0, \\
d) \ldots & x_1 = 1 & \text{or} & x_2 + x_3 + x_4 + x_5 = 0,
\end{array}$$

containing — as we remarked already in art. 6 — respectively a limiting tetrahedron, a face, an edge, a vertex of the simplex $S(5)$. Therefore in the expressive language of M^rs^. STOTT the limiting polyhedra of (32110) are distinguished, as to their orientation, as

[1]). This is the general symbol I used for S_4 in my textbook; here e, k, f, r stand for "ECKE, KANTE, FLÄCHE, RAUM," i. e. for vertex, edge, face, limiting body.

5 tT of *body import*,
10 P_6 „ *face* „ ,
10 P_3 „ *edge* „ ,
5 CO „ *vertex* „ .

8. We add an other example, this time of a polytope in S_5, and choose **(432110)**, showing all possible particularities. [1])

The number of vertices is $6!$ divided by $2!$ i. e. $720:2 = 360$.

The number of edges passing through each vertex is six, for in

$$4\ 3\ 2\ 1\ 1\ 0$$

each of the six brackets indicates two coordinates with difference unity. So the number of edges is $\dfrac{360 \times 6}{2} = 1080$.

Here we have to consider the equations

$$\begin{array}{llll}
a) & \ldots\ x_1 + x_2 + x_3 + x_4 + x_5 = 11 & \text{or} & x_6 = 0, \\
b) & \ldots\ x_1 + x_2 + x_3 + x_4 = 10 & \text{or} & x_5 + x_6 = 1, \\
c) & \ldots\ x_1 + x_2 + x_3 = 9 & \text{or} & x_4 + x_5 + x_6 = 2, \\
d) & \ldots\ x_1 + x_2 = 7 & \text{or} & x_3 + x_4 + x_5 + x_6 = 4, \\
e) & \ldots\ x_1 = 4 & \text{or} & x_2 + x_3 + x_4 + x_5 + x_6 = 7.
\end{array}$$

a). The equation $x_6 = 0$ gives $x_1, x_2, x_3, x_4, x_5 = (43211)$, or — if we diminish all the coordinates by one — we find in $x_6 = -1$ the polytope $x_1, x_2, x_3, x_4, x_5 = (32100)$, i. e. an $e_1\,e_2\,S(5)$, occurring six times [2]).

b). For $x_5 + x_6 = 1$ we have the two possibilities $x_5 = 1, x_6 = 0$ and $x_5 = 0, x_6 = 1$, combined with $x_1, x_2, x_3, x_4 = (4321)$, which may be reduced to $x_1, x_2, x_3, x_4 = (3210)$ by subtracting unity from all the coordinates. So we find a rectangular fourdimensional prism P_{tO} with tO as base, occurring fifteen times.

c. Here we have to combine the two systems $x_1, x_2, x_3 = (432)$, and $x_4, x_5, x_6 = (110)$. So we get a polytope with $6 \times 3 = 18$ vertices arranged in six equilateral triangles in planes parallel to the plane $x_1 = 0, x_2 = 0, x_3 = 0$ containing the face $A_4\,A_5\,A_6$ of

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[1]) The fourth and the sixth columns of Table I contain the characteristic numbers and the limiting elements of the highest number of dimensions of the new polytopes. The meaning of the small subscripts in column four and of the fractions in column five will be explained in part G of this section.

[2]) The characteristic numbers of this form — compare Table I — can be deduced in the manner indicated in art. 7; see farther under a').

In the memoir quoted of Mrs. Stott the regular simplex of space S_4 is indicated by the symbol C_5; we prefer to use here $S(5)$, as this allows us to discriminate between the regular simplex $S(8)$ of space S_7 and the measure polytope C_8 of S_4, etc.

the simplex of coordinates and in three regular hexagons in planes parallel to the plane $x_4 = 0, x_5 = 0, x_6 = 0$ containing the opposite face $A_1 A_2 A_3$ of that simplex. As these faces are perpendicular to each other according to the theorem of art 5, we find a regular prismotope (3 ; 6), occurring twenty times.

d). For $x_1 + x_2 = 7$ we have to combine each of the possibilities $x_1 = 4, x_2 = 3$ and $x_1 = 3, x_2 = 4$ with $x_3, x_4, x_5, x_6 = (2110)$. So we find fifteen prisms P_{CO}.

e). The equation $x_1 = 4$ gives $x_2, x_3, x_4, x_5, x_6 = (32110)$. So here we find six $e_1\, e_3\, S(5)$.

So the limiting polytopes are

$$6\, e_1\, e_2\, S(5),\ 15\, P_{tO},\ 20\,(3\,;6),\ 15\, P_{CO},\ 6\, e_1\, e_3\, S(5);$$

their number is 62. Of these the $6\, e_1\, e_2\, S(5)$ are of polytope import, the $15\, P_{tO}$ of polyhedron import, the 20 (3 ; 6) of face import, the $15\, P_{CO}$ of edge import and the $6\, e_1\, e_3\, S(5)$ of vertex import.

In order to find the number of the limiting polyhedra we enumerate the limiting polyhedra of the limiting polytopes.

a'). The determination of the limiting polyhedra of the six polytopes (32100) runs parallel to the investigation of (32110) in the preceding article, as we remarked already in the last footnote. So we find in the same order of succession and with the import with respect to the simplex $S(5)$ of its space: $5\,tO$ of polyhedron import, $10\, P_3$ of edge import and $5\, tT$ of vertex import, i. e. *twenty* limiting polyhedra.

b'). The prism P_{tO} is limited by two tO and fourteen prisms, viz. six P_4 (or cubes) and eight P_6, i. e. by *sixteen* polyhedra.

c'). The prismotope (3 ; 6) is limited by *nine* prisms, six P_3 and three P_6.

d'). The prism P_{CO} is limited by two CO and fourteen prisms, viz. six P_4 (or cubes) and eight P_3, i. e. by *sixteen* polyhedra.

e'). The polytope (32110) of the preceding article is limited by *thirty* polyhedra.

So the number of polyhedra is

$$\tfrac{1}{2}\,(6 \times 20 + 15 \times 16 + 20 \times 9 + 15 \times 16 + 6 \times 30) = 480.$$

According to the law of Euler the number of faces is

$$1080 + 480 + 2 - (360 + 62) = 1140.$$

So the resulting symbol of characteristic numbers is (360, 1080, 1140, 480, 62).

9. The direct method alluded to in the beginning of art. 6 rests

on the division of the limiting elements $(l)_p$ of p dimensions according to their symbol into different groups; as introduction we explain what this means by considering the edges of the polytope **(432110)** treated in the preceding article.

The edges are obtained by joining two points which pass into each other by interchanging in the symbol of coordinates two digits with difference unity. So we have edges with four different *symbols*, viz. (43), (32), (21), (10), if by the symbol (p, q) we indicate any edge the end points of which pass into each other by interchanging two coordinates with the numerical values p and q.

It is easy to find their numbers. To that end we calculate first the numbers of edges of different symbol passing through a determinate vertex, e.g. through the pattern vertex 4, 3, 2, 1, 1, 0, which point will be indicated by P. Through P passes only one edge (43) and one edge (32), as the symbol of coordinates contains only one 4, one 3 and one 2; but two edges (21) and two edges (10) concur in P, as the symbol of coordinates contains two digits 1. So we have

$$1 \text{ edge } (43) + 1 \text{ edge } (32) + 2 \text{ edges } (21) + 2 \text{ edges } (10)$$

through P. Now the number of vertices of the polytope being 360, the numbers indicating how many times each of the four edges of different symbol passes through any vertex would be 360, 360, 720, 720; as each edge bears two vertices we find

$$180 \text{ edges } (43) + 180 \text{ edges } (32) + 360 \text{ edges } (21) + 360 \text{ edes } (10),$$

i. e. once more altogether 1080 edges.

In order to be able to extend the method of deduction of edges with different symbols to limits $(l)_p$ of p dimensions we are obliged to introduce some new terms. So we call (43), (32), (21), (10) the *unextended* symbols of the four groups of edges found above and designate as the *extended* symbols of these groups respectively

$$(43)(2)(1)(1)(0), \quad (4)(32)(1)(1)(0), \quad (4)(3)(21)(1)(0),$$
$$(4)(3)(2)(1)(10),$$

containing also the remaining four digits, each placed in brackets. Moreover we call the constituents (43), (2), (1), (1), (0) of the extended symbol (43) (2) (1) (1) (0) the *syllables* of that symbol and exclude from our considerations the "petrified" syllables with two or more equal digits as (22), (111), etc., where the influence of the interchange of the digits is nihil.

10. We now prove the general theorem:

Theorem III. "We obtain all the groups of d-dimensional limiting polytopes $(P)_d$ with different symbols of any given n-dimensional polytope $(P)_n$ derived from the simplex $S(n+1)$ of S_n, if we split up the $n+1$ digits of the pattern vertex in all possible ways into $n-d+1$ groups of adjacent digits and consider these groups, each of them placed in brackets, as the syllables of the extended symbol."

We remark that the extended symbols of the four groups of edges of the preceding article satisfy the conditions of the theorem.

We represent the $n-d+1$ different syllables by $(\ldots)^{k1}, (\ldots)^{k2}, \ldots, (\ldots)^{k_{n-d+1}}$, where $k_1, k_2, \ldots, k_{n-d+1}$ indicate the numbers of the digits, so that we have $k_1 + k_2 + \ldots + k_{n-d+1} = n+1$; moreover in order to fix the ideas we suppose that the coordinate values of $(\ldots)^{k1}$ correspond to the coordinates $x_1, x_2, \ldots x_k$, those of $(\ldots)^{k2}$ to the coordinates $x_{k_1+1}, x_{k_1+2}, \ldots x_{k_1+k_2}$, etc.

Proof. If we put it short the three moments of the proof are:

a) As petrified syllables have been excluded we obtain by proceeding according to the indications of the theorem a d-dimensional polytope P_d, the vertices of which are vertices of P_n.

b) As the digits of the syllables are adjacent digits of the symbol of $(P)_n$, the $(P)_d$ obtained is a limiting polytope of $(P)_n$.

c) As the system of equations representing any limiting polytope $(P)_d$ of $(P)_n$ occurs under all the systems of equations corresponding to the limiting $(P)_d$ of $(P)_n$ furnished by the theorem, we obtain by means of the theorem all the limiting polytopes $(P)_d$ of $(P)_n$.

We consider each of these three parts for itself.

a) *The polytope obtained is a* $(P)_d$.

By the exclusion of petrified syllables we are sure that any syllable $(\ldots)^k$ with k digits allows the vertex, the coordinates of which are the $n+1$ digits of the symbol of $(P)_n$, to coincide successively with all the vertices of a determinate $k-1$-dimensional polytope $(P)_{k-1}$ situated in a space S_{k-1} parallel to a limiting space S_{k-1} of the simplex $S(n+1)$ of coordinates. So in the case of the $n-d+1$ syllables $(\ldots)^{k1}, (\ldots)^{k2}, \ldots, (\ldots)^{k_{n-d+1}}$ under consideration the polytope obtained will be a prismotope, the constituents of which are polytopes $(P)_{k-1}$, where k is successively $k_1, k_2, \ldots, k_{n-d+1}$, situated in spaces parallel to the limiting spaces $S_{k_1-1} = (A_1 A_2 \ldots, A_{k_1})$, $S_{k_2-1} = (A_{k_1+1} A_{k_1+2} \ldots, A_{k_1+k_2})$, etc. of the simplex of coordinates, which spaces are by two normal to one another according to art. 5. This prismotope which may be represented by the symbol $(P_{k1-1}; P_{k2-1}; \ldots; P_{k_{n-d+1}-1})$ is a polytope $(P)_d$. For its number of dimen-

sions is the sum of the numbers $k_1 - 1, k_2 - 1, \ldots, k_{n-d+1} - 1$ of dimensions of the constituents, i. e. the sum of the numbers $k_1, k_2, \ldots, k_{n-d+1}$ diminished by the number of the constituents, i. e. $n+1$ diminished by $n-d+1$, i. e. d.

We pass from the extended symbol of a $(P)_d$ formed according to the prescriptions of the theorem to the unextended symbol by omitting the syllables containing only one digit. So the unextended symbol contains only syllables with two and more than two digits. If all the syllables of the unextended symbol bear two digits, the polytope $(P)_d$ is a measure polytope; if this symbol contains only one syllable with more than two digits, the polytope $(P)_d$ is a prism, may be of higher rank; if the symbol contains at least two syllables of more than two digits the polytope $(P)_d$ is a prismotope, may be of higher rank, in the restricted sense of the word. This explains how we have to interprete the result found above that all the limits of $(P)_n$ are prismotopes. In the particular case of the limits $(l)_{n-1}$ of the highest number of dimensions, where we have to split up the digits of the pattern vertex of $(P)_n$ into two groups, we find, if $n+1$ is split up into k and $n-k+1$, the result $(P_{k-1}; P_{n-k})$, which is a non specialized [1]) polytope $(P)_{n-1}$ for $k=1$ and $k=n$, a prism on a non specialized polytope $(P)_{n-2}$ as base for $k=2$ and $k=n-1$, and a prismotope in the narrower sense for all the intermediate values; i. o. w. of the limits $(l)_{n-1}$ represented elsewhere (*Proceedings* of the Academy of Amsterdam, vol. XIII, p. 484) in connexion with the notion of *import* by $g_0, g_1, \ldots, g_{n-1}$ the forms g_0 and g_{n-1} are non specialized polytopes, the forms g_1 and g_{n-2} are prisms and the forms $g_2, g_3, \ldots g_{n-3}$ are prismotopes.

b). *The* $(P)_d$ *obtained is a limiting body of* $(P)_n$.

A polytope $(P)_d$, the vertices of which are vertices of $(P)_n$, is a limiting body of $(P)_n$ — and not a *section* of it —, if we can indicate $n-d$ limiting spaces S_{n-1} of $(P)_n$ containing it. Now, according to the manner in which $(P)_d$ is obtained, the coordinates of its vertices satisfy the $n-d+1$ equations

$$x_1 + x_2 + \ldots + x_{k_1} = p_1,\ x_{k_1+1} + x_{k_1+2} + \ldots + x_{k_1+k_2} = p_2,$$
$$x_{k_1+k_2+1} + x_{k_1+k_2+2} + \ldots + x_{k_1+k_2+k_3} = p_3, \text{ etc.},$$

if p_1 is the sum of the first k_1 digits of the pattern vertex, p_2 the sum of the next k_2 digits, p_3 of the then next k_3 digits, etc. These $n-d+1$ equations, only connected by the relation holding for

[1]) Here "non specialized" means: according to the mode of generation neither prism nor prismotope. About this last form art. 13 will give more particulars.

all points of S_n, that the expression $\sum\limits_{i=1}^{n+1} x_i$ is equal to the sum of all the digits of $(P)_n$, form a system of $n-d$ mutually independent equations, representing therefore, in accordance with the result of the first part of the proof, a space S_d, bearing the $(P)_d$ found above. If we write this system of equations in the form:

$$\sum_{i=1}^{k_1} x_i = p_1, \sum_{i=1}^{k_1+k_2} x_i = p_1 + p_2, \ldots, \sum_{i=1}^{k_1+k_2+\ldots+k_{n-d}} x_i = p_1 + p_2 + \ldots + p_{n-d},$$

it is evident that each of the equations represents an $n-1$-dimensional limiting space of $(P)_n$, the constant of the right hand member being a maximum.

As we remarked already the crux of the proof of this part lies in the true interpretation of the expression "adjacent digits". It cannot be replaced by the condition that all the syllables should be formed according to the first theorem. We show this by means of two simple cases concerned with the determination of faces of threedimensional polyhedra. In the case of the $(2110) = CO$ the hexagon (210) (1) is no face but a section, likewise in the case of the $(1100) = O$ the square (10) (10) is no face but a section. In both cases the syllables satisfy the conditions of the first theorem; but the impossibility of putting the syllables behind each other so as to obtain the order of succession of the digits of the pattern vertex implies the impossibility of finding an equation where the constant that is equal to the sum of some of the coordinates is either a maximum or a minimum. Under the five polyhedra T, O, tT, CO, tO which can be represented by a symbol with four digits (compare the small table of art. 3 for $n = 3$) the O and CO are the only ones with sections p_4 and p_6 with sides equal to the edges; at the same time they are the only ones with four edges through each vertex.

c). By means of the theorem we obtain all the limits $(P)_d$ of $(P)_n$. It is always possible to represent any limit $(P)_d$ of $(P)_n$ by $n-d$ equations of spaces S_{n-1} containing $n-1$-dimensional limits $(l)_{n-1}$ of $(P)_n$; as the vertices of this $(P)_d$ are also vertices of $(P)_n$, this system of equations will be in accordance with the symbol of $(P)_n$, i. e. this system must be included into the set of systems of equations provided by the theorem.

11. We apply the theorem III to an other fivedimensional form **(321100)**, showing at the same time how we can determine the numbers of *all* the different limits.

Vertices. There is only one kind of vertex (3) (2) (1) (1) (0) (0). According to the rule given in art. 6 the number of vertices is 6! divided by 2^2, i. e. 180.

Edges. There are three groups of edges, represented in extended and in unextended [1]) symbols by

$$(32)(1)(1)(0)(0) = (32),\quad (3)(21)(1)(0)(0) = (21),$$
$$(3)(2)(1)(10)(0) = (10).$$

We indicate a new method of determining the numbers of these edge groups. In the case of (10) the coordinates corresponding to the two digits between the same brackets can be x_i, x_k where i, k is any combination of the subscripts 1, 2, 3, 4, 5, 6 by two, giving $(6)_2 = \frac{6.5}{1.2} = 15$ possibilities; these two coordinates having been chosen the four remaining ones can be assigned anyhow to the four digits (3), (2), (1), (0), giving $4! = 24$ possibilities. So the number of edges (10) is $(6)_2 . 4! = 360$. In the case of (21) the number 360 must be divided by 2 on account of the two equal syllables (0), (0), in the case of (32) this number must be divided by 2^2 on account of the two pairs of equal syllables (1), (1) and (0), (0). So we have

$$90 \text{ edges } (32) + 180 \text{ edges } (21) + 360 \text{ edges } (10)$$

i. e. altogether 630 edges.

Faces. There are six groups of faces, represented in extended and in unextended symbols by

$$\begin{array}{ll} (321)(1)(0)(0) = (321) = p_6, & (32)(1)(10)(0) = (32)(10) = p_4, \\ (3)(211)(0)(0) = (211) = p_3, & (3)(21)(10)(0) = (21)(10) = p_4, \\ (3)(2)(110)(0) = (110) = p_3, & (3)(2)(1)(100) = \quad (100) \quad = p_3. \end{array}$$

Taken in the order of succession of the rows the numbers of these polygons are

$$\begin{array}{lll} (6)_3 \,.\, 3!:2 = \ 60, & (6)_2 (4)_2 .\ 2! = 180, & (6)_3 \,.\, 3!:2 = \ 60, \\ (6)_2 (4)_2 .\ 2! = 180, & (6)_3 \,.\, 3! \quad = 120, & (6)_3 \,.\, 3! \quad = 120, \end{array}$$

i. e. we find

$$300\, p_3 + 360\, p_4 + 60\, p_6 = 720 \text{ faces.}$$

Limiting bodies. There are seven groups of limiting bodies, viz.:

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[1]) As we have seen in the preceding article the unextended symbols are deduced from the extended ones by omitting the syllables of one digit.

$$(3211)(0)(0) = (3211) = tT,\ (321)(10)(0) = (321)(10) = P_6,$$
$$(32)(110)(0) = (32)(110) = P_3,\ (32)(1)(100) = (32)(100) = P_3,$$
$$(3)(2110)(0) = (2110) = CO,\ (3)(21)(100) = (21)(100) = P_3,$$
$$(3)(2)(1100) = (1100) = O,$$

the numbers of which are respectively

$$(6)_4 \,.\, 2!:2 = 15\ ,\ (6)_3 \,.\, (3)_2 = 60\ ,\ (6)_3 \,.\, (3)_2 = 60\ ,$$
$$(6)_3 \,.\, (3)_2 = 60\ ,\ (6)_4 \,.\, 2! = 30\ ,\ (6)_3 \,.\, (3)_2 = 60\ ,$$
$$(6)_4 \,.\, 2! = 30\ ,$$

i e.

$$15\, tT + 30\,(O + CO) + 60\, P_6 + 180\, P_3 = 315 \text{ limiting bodies.}$$

Limiting polytopes. There are four groups of limiting polytopes, viz.:

$$(32110)(0) = (32110) = e_1\, e_3\, S_5\ ^{1}),\ (321)(100) = (6\,;\,3),$$
$$(32)(1100) = P_O,\ (3)(21100) = (21100) = e_2\, S_5,$$

the numbers of which are.

$$(6)_5 = 6\ ,\ (6)_3 = 20\ ,\ (6)_4 = 15\ ,\ (6)_5 = 6.$$

So we find

$$6\, e_1\, e_3\, S_5 + 20\,(6\,;\,3) + 15\, P_O + 6\, e_2\, S_5 = 47 \text{ limiting polytopes}$$

and the characteristic numbers are

$$(180,\ 630,\ 720,\ 315,\ 47),$$

in accordance with the law of Euler.

12. Though the introduction of the extended symbols has enabled us to simplify the theoretical considerations it cannot be denied that the unextended symbols are better fit for practical use. Therefore we insert here a corresponding version of theorem III, but to that end we have to enter first into a distinction of the digits of the syllables of the unextended symbols. We will distinguish the digits contained in any of these syllables into *end digits* and *middle digits*, the first and the last digits and the digits equal to these being the end digits, the remaining ones — if there are some — the middle digits. So in (3210) there are two middle digits 2 and 1, in (2110) there are two equal middle digits 1, while in (2210), (2100) there is only one middle digit and in (1000), (1100) none. Now we can repeat theorem III in the new form:

THEOREM III'. "We obtain a $(P)_d$ the vertices of which are vertices

[1]) Compare the small table unter art. 3.

of the given polytope $(P)_n$, if we fix either the values of $n - d$ coordinates and allow the remaining $d + 1$ to interchange their values, or the values of $n - d - 1$ coordinates and split up the remaining $d + 2$ into two groups of interchangeable ones, or the values of $n - d - 2$ coordinates and split up the remaining $d + 3$ into three groups of interchangeable ones, etc., this process winding up for $n < 2d$ in a symbol with $n - d + 1$ and for $n > 2(d - 1)$ in a symbol with d groups."

"This $(P)_d$ will be limiting polytope of $(P)_n$, if:

1°. each syllable of the unextended symbol with middle digits exhausts these digits of the symbol of $(P)_n$,

2°. no two syllables without middle digits have the same end digits."

Proof. The first part of the new theorem is a consequence of this that in the different cases communicated the corresponding extended symbol is always consisting of $n - d + 1$ syllables, i. e. of k syllables with more than one digit and $n - d - k + 1$ syllables with only one digit for $k = 1, 2, \ldots, d$; so it is equivalent to part *a*) of the proof of theorem III. The second part of the new theorem is equivalent to part *b*) of the proof of theorem III; for the only cases in which it is impossible to put the syllables of the extended symbol behind one another so as to obtain the order of succession of the pattern vertex are the two excluded by the two items 1° and 2°, i. e. 1° that a syllable with middle digits does not exhausts these digits and 2° that two syllables without middle digits do have the same end digits. Finally the part *c*) of the proof of theorem III can be repeated here.

By means of theorem III′ we find e. g. in the case of the $(P)_9$ represented by **(5443322210)** the following 58 different kinds of limiting $(P)_6$:

(5443322), — (544332) (21), (544332) (10), — (54433) (221), (54433) (210), — (5443) (3222), (5443) (322) (21), (5443) (322) (10), (5443) (32) (221), (5443) (32) (210), (5443) (2221), (5443) (2210), — (544) (33222), (544) (3322) (21), (544) (3322) (10), (544) (332) (221), (544) (332) (210), (544) (32221), (544) (3222) (10), (544) (322) (210), (544) (32) (2210), (544) (22210), — (54) (433222), (54) (43322) (21), (54) (43322) (10), (54) (4332) (221), (54) (4332) (210), (54) (433) (2221), (54) (433) (2210), (54) (43) (32221), (54) (43) (3222) (10), (54) (43) (322) (210), (54) (43) (32) (2210), (54) (43) (22210), (54) (332221), (54) (33222) (10), (54) (3322) (210), (54) (332) (2210), (54) (322210) — (4433222), — (443322) (21), (443322) (10), — (44332) (221), (44332) (210), — (4433) (2221), (4433) (2210), — (443) (32221), (443) (3222) (10), (443) (322)

(210), (443) (32) (2210), (443) (22210), — (4332221), — (433222) (10), — (43322) (210), — (4332) (2210), — (433) (22210), — (43) (322210), — (3322210).

13. We will insert a few remarks about the character of the limiting $(P)_6$ obtained.

In the case (5443322) of one syllable we find a non specialized form (3221100) which will prove to be an $e_2 e_4 S(7)$ in M[rs]. Stott's language.

In the cases (544332) (21) and (544332) (10) we find right prisms on (322110) $= e_2 e_4 S(6)$ as base.

In the case (54433) (221) we find a prismotope the constituents of which are a (21100) $= e_2 S(5)$ and a (110) $= p_3$. So this $(P)_6$ can be generated in the following way. Consider a space S_4 and a plane S_2 perfectly normal to each other. Take in S_4 an $e_2 S(5)$, in S_2 a p_3, and let P be a definite vertex of the former, Q a definite vertex of the latter. Now move either $e_2 S(5)$ parallel to itself in such a way that P coincides successively with all the points inside p_3, or p_3 parallel to itself in such a way that Q coincides successively with all the points inside $e_2 S(5)$. Then the $(P)_6$ can be considered as the locus either of the $e_2 S(6)$ in the first case or of the P_3 in the second; its vertices are given in the first case by the three positions of $e_2 S(5)$ in which P coincides with one of the vertices of p_3, in the second by the thirty positions of p_3 in which Q coincides with one of the vertices of $e_2 S(5)$. We represent it by the symbol $\{e_2 S(5); 3\}$.

In the case (54433) (210) we find an $\{e_2 S(5); 6\}$.

In the three cases (5443) (3222), (5443) (2221), (5443) (2210) we find successively $(CO; T)$, $(CO; T)$, $(CO; tT)$.

In the cases (5443)(322)(21), (5443)(322)(10), (5443)(32)(221) we have to deal with right prisms on a $(CO; 3)$ as base, whilst (5443) (32) (210) is a right prism on a $(CO; 6)$ as base. These prisms may also be represented by the symbols $(CO; 3; 2)$ and $(CO; 6; 2)$ as prismotopes of the second rank. But a prismotope proper of the second rank is the $(P)_6$ represented by (544)(322)(210), which may be represented as such by the symbol $(3; 3; 6)$. To generate it we have to start from three planes α_1, α_2, α_3 two by two perfectly normal to one another, and to place in α_1 and α_2 equilateral triangles and in α_3 a regular hexagon; then the $(P)_6$ is obtained by the parallel motion of the hexagon in such a way that a definite vertex of that hexagon coincides successively with all the points inside the fourdimensional prismotope $(3; 3)$ determined by the two triangles.

The (54) (43) (32) (2210) is a prism of the third rank on a tT as base; it may also be considered as a prismotope $(C; tT)$.

If in the case of $(P)_n$ we deduce the limits $(P)_{n-1}$ we find them in the order of succession $g_{n-1}, g_{n-2}, \ldots, g_0$ of polytope import to vertex import, when, in proceeding from left to right we take in the first syllable as many digits as possible and keep in it the first digit as long as possible. This principle has been followed throughout in the enumeration of the limiting $(P)_6$ of the given $(P)_9$, as well as in the sixth columm of Table I.

In the notation of art. 10 (page 17 in the middle) a limit $(P)_{n-1}$, represented as to its import by g_k, is a prismotope $(P_k; P_{n-k-1})$.

14. It is worth noticing that in space S_n the series of limiting elements may include the series of the measure polytope M, for n even up to the polytope $M_{\frac{n}{2}}$ of $S_{\frac{n}{2}}$, for n odd up to the polytope $M_{\frac{n+1}{2}}$ of $S_{\frac{n+1}{2}}$. So, for $n = 2m + 1$ the $(P)_{2m+1}$ represented by

$$e_1\, e_2\, e_3 \ldots e_m\, S(2m + 2) = (2m + 1, 2m, 2m - 1, \ldots, 3, 2, 1, 0)$$

admits as limiting element $(P)_{m+1}$ the M_{m+1} with the symbol

$$(2m + 1, 2m)\,(2m - 1, 2m - 2) \ldots (3, 2)\,(1, 0).$$

On the other hand, amongst the polytopes *themselves*, no measure polytope occurs and of the cross polytopes only the octahedron presents itself. We prove that this must be so, for each of the two series separately.

Measure polytopes. The number of vertices of (5443322210) is $\frac{10!}{2!\,2!\,3!}$, which can be written in the form $\frac{10!}{3!\,(2!)^2\,(1!)^3}$ so as to be able to generalize it for any $(P)_n$ as

$$\frac{(n+1)!}{a!\,b!\,c!\ldots k!},$$

where $a, b, c, \ldots k$ are arranged in decreasing order and their sum is $n + 1$. Now this form is a product of binomial coefficients

$$(n + 1)_a\,(n - a + 1)_b\,(n - a - b + 1)_c \ldots$$

and there is only one possibility under which this product contains no factors different from two and is therefore a power of two, i. e. in the case $n + 1 = 2^p$, $a = 2^p - 1$, $b=1$, giving

$$\frac{2^p!}{(2^p - 1)!\;1!} = 2^p.$$

But this case corresponds to the *simplex* $S(2^p)$ of space S_{2^p-1}.

Cross polytopes. The cross polytope is characterized by the property of having all its diagonals of the same length ($=\sqrt{2}$ times an edge) and passing through the same point. So in order to represent a cross polytope the symbol of coordinates of $(P)_n$ can contain end digits only, for the supposition of *three* different digits as in (210) leads inevitably to three different distances. Let us suppose the two end digits are 1 and 0. Then we have to take in at least two of each in order to create the possibility of interchanging two pairs of digits; this gives us the octahedron, the diagonals of which are the joins of the pairs of vertices represented by

$$\left.\begin{matrix}1100\\0011\end{matrix}\right\}, \qquad \left.\begin{matrix}1010\\0101\end{matrix}\right\}, \qquad \left.\begin{matrix}1001\\0110\end{matrix}\right\},$$

and pass therefore through the point $\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}$. Finally in the cases $(11\,\overline{00\ldots0}^{\,n-1})$ and $(\overline{11\ldots1}^{\,n-1}00)$ we have to deal also with polytopes admitting only diagonals $=\sqrt{2}$ times an edge, but here these diagonals do not pass through the same point (centre). For in the case of (11000) the centre is the point all the coordinates of which are $\frac{2}{5}$ and this point lies not on the diagonal joining the points 11000 and 00110, etc.

C. *Extension number and truncation integers and fractions.*

15. "How can all the new polytopes $(a\,b\,c\ldots)$ found analytically be deduced geometrically from the regular simplex?"

As we remarked in the introduction the new polytopes *have been discovered* geometrically by M[rs]. A. Boole Stott; we will consider her method thoroughly under *D*. Here we wish to indicate first that the answer to this question can also be given by the theorem:

Theorem IV. "The new polytopes, all with edges of length unity, can be found by means of a regular extension of the regular simplex of coordinates followed by a regular truncation, either at the vertices alone, or at the vertices and the edges, or at the vertices, edges and faces, etc."

Proof. This theorem is an immediate consequence of that given in art. 6 (theorem II) about the equality of the non vanishing coefficients c_i of the coordinates x_i in the equation $c_1 x_1 + c_2 x_2 + \ldots = p$ of a limiting space S_{n-1} of the new polytope deduced from the simplex $S(n+1)$ of S_n. So in treating in art. 7 the example (32110) we found that the limiting spaces

$$\begin{array}{lll} x_1 & = 3 & \text{containing a limiting } CO, \\ x_1 + x_2 & = 5 & \quad ,, \quad ,, \quad ,, \quad P_3, \\ x_1 + x_2 + x_3 & = 6 & \quad ,, \quad ,, \quad ,, \quad P_6, \\ x_1 + x_2 + x_3 + x_4 & = 7 & \quad ,, \quad ,, \quad ,, \quad tT \end{array}$$

are respectively parallel to the spaces

$$\begin{array}{lll} x_1 & = 1 & \text{containing a vertex}, \\ x_1 + x_2 & = 1 & \quad ,, \quad \text{an edge}, \\ x_1 + x_2 + x_3 & = 1 & \quad ,, \quad \text{a face}, \\ x_1 + x_2 + x_3 + x_4 & = 1 & \quad ,, \quad ,, \text{ limiting body} \end{array}$$

of the regular simplex, while they are normal to the line joining the centre O of the simplex to the centre of that limiting element. Moreover it is evident that all the spaces of the same group, say $x_k + x_l + x_m = 6$, have the same distance from the corresponding spaces $x_k + x_l + x_m = 1$, etc.

16. The meaning of the expression "extension number" is clear by itself: an extension to an amount ε transforms the simplex $S^{(1)}(n+1)$ with edge unity of S_n into a simplex $S^{(\varepsilon)}(n+1)$ of edge ε. But we have to define beforehand what we will understand e. g. by a truncation $\frac{1}{2}$. If we split up the $n+1$ vertices of the extended simplex $S^{(\varepsilon)}(n+1)$ into two groups of $k+1$ and $n-k$ points (see fig. 2, where the case $n=6$, $k=3$ is represented), forming the vertices of regular simplexes $S^{(\varepsilon)}(k+1)$, $S^{(\varepsilon)}(n-k)$ lying in spaces S_k, S_{n-k-1}, and we cut $S^{(\varepsilon)}(n+1)$ by any space S_{n-1} at the same time parallel to these spaces S_k, S_{n-k-1}, i. e. normal to the line joining the centres M, M' of $S^{(\varepsilon)}(k+1)$, $S^{(\varepsilon)}(n-k)$ in a certain point O, any edge PQ joining a vertex P of $S^{(\varepsilon)}(k+1)$ to a vertex Q of $S^{(\varepsilon)}(n-k)$ will be cut in a certain point R for which the ratio $\dfrac{PR}{PQ}$ is equal to $\dfrac{MO}{MM'}$, and therefore independent from the choice of the vertices P, Q. This ratio is the "truncation fraction" of $S^{(\varepsilon)}(n+1)$ at the limiting $S^{(\varepsilon)}(k+1)$ by the truncating space and its complement $\dfrac{QR}{QP}$ to unity is the "truncation fraction" of $S^{(\varepsilon)}(n+1)$ at the limiting $S^{(\varepsilon)}(n-k)$ opposite to $S^{(\varepsilon)}(k+1)$ by the same space.

But, if we like, we can use the term "truncation number" for the number of units contained in the segment PR or QR, according to the truncation being performed at the side of P or of Q. As the number of units of the denominator of the truncation fraction,

i. e. the number of units of the edge of the extended simplex, is the extension number ε, the truncation numbers τ, which — as ε itself — will prove to be always integer, are simply the numerators of the truncation fractions with the extension number ε as common denominator.

17. If we indicate the truncation numbers corresponding successively to a truncation at a vertex, an edge, a face,... by $\tau_0, \tau_1, \tau_2, \ldots$, the theorem holds:

Theorem V. "Let $(m_0, m_1, m_2, \ldots, 0)$ be the zero symbol of the polytope; then the sum $m = \Sigma m_i$ of the digits is the extension number ε, and the truncation numbers $\tau_0, \tau_1, \tau_2, \ldots$ are represented by the forms

$$\tau_0 = m - m_0,\ \tau_1 = m - m_0 - m_1,\ \tau_2 = m - m_0 - m_1 - m_2, \ldots \text{"}$$

Proof. By the extension ε the simplex of coordinates $S^{(1)}(n+1) = (1\,\overline{00\ldots0}^{\,n})$ of S_n is changed into the concentric simplex $S^{(\varepsilon)}(n+1) = (\varepsilon\,\overline{00\ldots0}^{\,n})$ with edges ε. Then the space S_{n-1} represented in the latter case by $x_1 = 0$ contains a limit $(l)_{n-1}$ of the considered polytope $(P)_n$ forming a part of the limiting simplex $S^{(\varepsilon)}(n) = (\varepsilon\,\overline{00\ldots0}^{\,n-1})$ of $(\varepsilon\,\overline{00\ldots0}^{\,n})$, at which limit of the *highest* order of dimensions this $S^{(\varepsilon)}(n+1)$ is not sliced off. If now we go back to true coordinate values the last digits in the two symbols of $(P)_n$ and $S^{(\varepsilon)}(n+1)$ must still be the same, which will be the case, if we have to subtract from nought in both cases the same amount, i. e. if $\dfrac{\Sigma m_i - 1}{n+1}$ and $\dfrac{\varepsilon - 1}{n+1}$ are equal, i. e. if we have $\varepsilon = \Sigma m_i = m$.

From what we remarked in the preceding article it follows that the truncation fraction of the extended simplex $S^{(m)}(n+1)$ at any limiting $S^{(m)}(k+1)$ can be derived from the mutual position of three parallel spaces S_{n-1} normal to the line joining the centre of $S^{(m)}(k+1)$ to the centre of the opposite $S^{(m)}(n-k)$; of these three spaces one passes through $S^{(m)}(k+1)$, an other through the opposite $S^{(m)}(n-k)$, whilst the third is the truncating space lying between these two. For, if as in the preceding article P is any vertex of $S^{(m)}(k+1)$, Q any vertex of $S^{(m)}(n-k)$ and R the point of intersection of PQ with the third of these three parallel spaces, which may be represented by $S^{(1)}{}_{n-1}$, $S^{(2)}{}_{n-1}$, $S^{(3)}{}_{n-1}$, according to definition

the truncation fraction τ_k is the ratio $\frac{PR}{PQ}$ and now we have the theorem:

"If a line cuts any three parallel spaces $S^{(1)}{}_{n-1}$, $S^{(2)}{}_{n-1}$, $S^{(3)}{}_{n-1}$ of S_n represented by the three equations $b_1 x_1 + b_2 x_2 + \ldots\ldots + b_q x_q = c_t$, $(t = 1, 2, 3)$ in the points P, Q, R, we have

$$\frac{PR}{PQ} = \frac{c_1 - c_3}{c_1 - c_2}\text{''}$$

For this is obviously true for the edge $A_1 A_{q+1}$ of the simplex of coordinates, the values of x_1 for the three points of intersection with this line being determined by the relations $b_1 x_1 = c_t$, $(t = 1, 2, 3)$; therefore it is true for any transversal, according to a well known theorem, already used implicitly in the preceding article, the ratio in question being the same for all possible transversals.

Now in the case under consideration the spaces $S^{(1)}{}_{n-1}$, $S^{(2)}{}_{n-1}$, $S^{(3)}{}_{n-1}$ may be represented by the equations $x_1 + x_2 + \ldots + x_{k+1} = c_t$, $(t = 1, 2, 3)$, where c_1 and c_2 are the maximum and minimum values of the left hand member with respect to the vertices of the extended simplex $(m\,\overline{00\ldots.0}^{\,n})$, while c_3 is the maximum value of the same expression with respect to the vertices of $(m_1, m_2, m_3, \ldots, 0)$. So we have

$$c_1 = m,\ c_2 = 0,\ c_3 = m_0 + m_1 + \ldots + m_k$$

giving for the truncation fraction the result $\frac{m - (m_0 + m_1 + \ldots + m_k)}{m}$ and therefore $\tau_k = m - (m_0 + m_1 + \ldots + m_k)$.

So we find in the case of the $(P)_9$ of art. 12 represented by (5443322210) $m = 26$, $\tau_0 = 21$, $\tau_1 = 17$, $\tau_2 = 13$, $\tau_3 = 10$, $\tau_4 = 7$, $\tau_5 = 5$, $\tau_6 = 3$, $\tau_7 = 1$.

For $n = 3, 4, 5$ the extension number and the truncation numbers are indicated in Table I, the seventh column containing the extension number ε, the eighth column giving what may be called the "truncation symbol". So in the case of $e_1 e_2 e_4 S(6)$ the extension number is 11, the truncation symbol is $7, 4, 2, 1$ where these numbers represent successively the values of $\tau_0, \tau_1, \tau_2, \tau_3$; so we find mentioned here a truncation $\frac{7}{11}$ at the vertices, $\frac{4}{11}$ at the edges, $\frac{2}{11}$ at the faces and $\frac{1}{11}$ at the limiting bodies.

D. *Expansion and contraction symbols.*

18. If we compare the symbols containing the operators e_i and c of expansion and contraction introduced by M[rs]. Stott for the offspring of the $T = S(4)$ in S_3 and the $S(5)$ in S_4 with the zero symbol of these polyhedra and polytopes, we remark that all these cases underlie certain general laws, up to now of an empirical character. By proving these laws we will promote them to theorems, the first of which can be stated as follows:

Theorem VI. "The expansion e_k, $(k = 1, 2, 3, \ldots, n-1)$ applied to the $S^{(1)}(n+1)$ of S_n changes the symbol of coordinates $(1\,\overline{00\ldots0}^{\,n})$ of that simplex into an other zero symbol which can be obtained by adding a unit to the first $k+1$ digits."

Indeed this gives (compare the small table $n = 4$ under art. 3):

$$e_1\, S(5) = (21000), \quad e_2\, S(5) = (21100), \quad e_3\, S(5) = (21110).$$

Proof. The operation of expansion e_k consists in moving the limiting $S(k+1)$ of $S^{(1)}(n+1)$ to equal distances away from the centre O of $S^{(1)}(n+1)$, each $S(k+1)$ moving in the direction of the line OM joining O to its centre M, these $S(k+1)$ "remaining parallel to their original position, retaining their original size and being moved over such a distance that the two new positions of any vertex which was common to two adjacent limits $(l)_k$ in the original $S^{(1)}(n+1)$ shall be separated by the length of an edge". [1])

Now let us consider (fig. 3) the plane through OM and any vertex A of the $S(k+1)$ of which M is the centre; then, on account of the regularity of $S^{(1)}(n+1)$, the angle AMO is a right one. This plane will also contain the new position $A'M'$ of AM. What we have to do now is this: We select from the symbol of coordinates $(1\,\overline{00\ldots0}^{\,n})$ the vertices of any limiting $S^{(1)}(k+1)$, calculate the coordinates of M, deduce from the coordinates of O and M those of M' on the supposition that $OM' : OM = \lambda$ is known. Then we have to determine the coordinates of A' by adding to the coordinates of the vertex A chosen arbitrarily among the $k+1$ vertices of the $S^{(1)}(k+1)$ the differences of the corresponding coordinates of M' and M. Finally we have to determine λ by the stated condition that two new positions of the same vertex A of $S^{(1)}(n+1)$ shall be separated by the length of an edge,

[1]) Compare p. 5 of the memoir quoted of M[rs]. Stott.

or — which comes to the same — by the condition that the coordinates of A' satisfy the law stated in theorem I, that the difference of any two different adjacent values must be unity.

We now set to work and select for the $k+1$ vertices the vertices $A_1, A_2, \ldots, A_{k+1}$ of $S^{(1)}(n+1)$ and for A of fig. 3 the vertex A_1. According to this choice the coordinates of M are

$$x_1 = x_2 = \ldots = x_{k+1} = \frac{1}{k+1}, \quad x_{k+2} = x_{k+3} = \ldots = x_{n+1} = 0.$$

So the coordinates of the three points O, A, M satisfy the equations

$$x_2 = x_3 = \ldots = x_{k+1}, \quad x_{k+2} = x_{k+3} = \ldots = x_{n+1};$$

but then these relations hold for any point of the plane OAM, as the $n-2$ equations represent a plane. As moreover

$$AA' = MM' = (\lambda - 1)\, OM,$$

or in the notation of vector analysis

$$A' - A \equiv M' - M \equiv (\lambda - 1)(M - O),$$

we have successively for the mentioned coordinates in true values and for the mentioned differences of coordinates:

	x_1	$x_2 = x_3 = \ldots = x_{k+1}$	$x_{k+2} = x_{k+3} = \ldots = x_{n+1}$
O	$\frac{1}{n+1}$	$\frac{1}{n+1}$	$\frac{1}{n+1}$
M	$\frac{1}{k+1}$	$\frac{1}{k+1}$	0
$M-O$	$\frac{1}{k+1} - \frac{1}{n+1}$	$\frac{1}{k+1} - \frac{1}{n+1}$	$-\frac{1}{n+1}$
$A'-A$	$(\lambda-1)\left(\frac{1}{k+1} - \frac{1}{n+1}\right)$	$(\lambda-1)\left(\frac{1}{k+1} - \frac{1}{n+1}\right)$	$-(\lambda-1)\frac{1}{n+1}$
A	1	0	0
A'	$1+(\lambda-1)\left(\frac{1}{k+1} - \frac{1}{n+1}\right)$	$(\lambda-1)\left(\frac{1}{k+1} - \frac{1}{n+1}\right)$	$-(\lambda-1)\frac{1}{n+1}$

i. e. the difference $\dfrac{\lambda-1}{k+1}$ of the coordinates x_{k+1} and x_{k+2} of A' is either unity or zero. But if we make it zero we get $\lambda = 1$, i. e. we find back the *original* $S^{(1)}(n+1)$. So for the *expanded* polytope we have to take $\dfrac{\lambda-1}{k+1} = 1$ or $\lambda = k+2$, giving for A' the coordinates

$$x_1 = 2\,,\; x_2 = \ldots = x_{k+1} = 1\,,\; x_{k+2} = \ldots = x_{n+1} = 0$$

i. e. the symbol $(2\,\overbrace{11\ldots1}^{k}\,\overbrace{00\ldots0}^{n-k})$, what was to be proved.

If by the operation e_k the limits $S^{(1)}(k+1)$ of $S^{(1)}(n+1)$ are moved away from the centre O to a distance equal to λ times the original distance, the extended simplex $S^{(x)}(n+1)$, the limits $S^{(x)}(k+1)$ of which will contain these $S^{(1)}(k+1)$ in their new positions, will be a simplex $S^{(\lambda)}(n+1)$, i. o. w. $\lambda = k+2$ is the extension number of the new polytope. This comes true, for according to theorem V the sum $k+2$ of the digits of the zero symbol $(2\,\overbrace{11\ldots1}^{k}\,\overbrace{00\ldots0}^{n-k})$ *is* the extension number. So we find by the way:

THEOREM VII. "In the expansion e_k the limits $S^{(1)}(k+1)$ are moved away from the centre to a distance equal to $k+2$ times the original distance."

Remark. We may express the influence of the operation e_k on the symbol $(1\,\overbrace{00\ldots0}^{n})$ of the simplex $S^{(1)}(n+1)$ presenting only one unit interval between the first and the second digit by saying that it creates a second unit interval between the $k+1^{st}$ and the $k+2^{nd}$ digit. This remark which holds also with respect to the symbol of true coordinates will be of use in the following articles.

19. THEOREM VIII. "The influence of any number of expansions $e_k, e_l, e_m, \ldots$ of $S^{(1)}(n+1)$ on its zero symbol $(1\,\overbrace{00\ldots0}^{n})$ is found by adding the influences of each of the expansions taken separately".

Indeed this gives (compare the small table $n=4$ under art. 3):

$$e_1\,e_2\,S(5) = (32100)\,,\; e_1\,e_3\,S(5) = (32110)\,,\; e_2\,e_3\,S_5 = (32210),$$
$$e_1\,e_2\,e_3\,S(5) = (43210).$$

Proof. We begin to prove the theorem for the case of two operations of expansion only.

It is stipulated expressly by M[rs]. STOTT that in the succession of two operations of expansion the subject of the second is to be what its original subject has become under the influence of the first. So in the case $e_2\,e_1\,T$ of the tetrahedron (fig. 4^a) the original triangular subject of e_2 is transformed by e_1 into a hexagon (fig. 4^b) and now the hexagon is moved out, in the case $e_1\,e_2\,T$ the linear subject of e_1 is transformed by e_2 into a square (fig. 4^c) and now the

square is moved out; in both cases the result (fig. 4^d) is the same, tO. In general, for $k > l$, in the case $e_k\, e_l\, S^{(1)}(n+1)$ the subject $S^{(1)}(k+1)$ of e_k is transformed by e_l into an $e_l\, S^{(1)}(k+1)$ and now this $e_l\, S^{(1)}(k+1)$ is moved away from the centre, while in the case $e_l\, e_k\, S^{(1)}(n+1)$ the subject $S^{(1)}(l+1)$ of e_l is transformed by e_k into an n—1-dimensional polytope of the import l corresponding to $S^{(1)}(l+1)$ which polytope is moved away from O as a whole. Now it is evident that the geometrical condition "that the two new positions of a vertex shall be separated by the length of an edge" makes the distance over which the second motion of any of these two pairs has to take place equal to the distance described in the first motion of the other pair; i. e. if $S^{(1)}(l+1)$ is a limiting element of $S^{(1)}(k+1)$ and A is a vertex of that $S^{(1)}(l+1)$, the segments described by A in transforming $S^{(1)}(n+1)$ into the two polytopes $e_k\, e_l\, S(n+1)$ and $e_l\, e_k\, S(n+1)$ are the two pairs of sides of a parallelogram leading from A to the opposite vertex A'. In other words: we find the true coordinates of A' by adding to the coordinates of A the variations corresponding to the motions due to each of the operations e_k and e_l taken separately.

Taking for $S^{(1)}(k+1)$ the simplex $A_1 A_2 \ldots A_{k+1}$, for $S^{(1)}(l+1)$ the simplex $A_1 A_2 \ldots A_{l+1}$ and for A the point A_1 we have to vary the coordinates $1, \overline{0, 0, \ldots 0}^{\,n}$ of A so as to admit two more unit intervals, one between the $k+1^{st}$ and the $k+2^{nd}$, an other between the $l+1^{st}$ and the $l+2^{nd}$ digit. If then afterwards we pass to the zero symbol we get $(3\, \overline{22\,..\,2}^{\,l}\, \overline{11\,..\,1}^{\,k-l}\, \overline{00\,..\,0}^{\,n-k})$, what was to be proved.

Now we have still to add that the proof for the composition of three and more operations of expansion runs entirely on the same lines. In the case of three operations we will have to compose three displacements according to the rule of the diagonal of the parallelopipedon, in the case of more we will have to use the extension of this rule to parallelotopes. To this geometrical composition of motions always corresponds the arithmetical addition of the symbol influences, where the order of succession is irrelevant; this arithmetical addition leads to the creation of new unit intervals independently. So the general rule is proved.

The preceding developments lead to a new theorem, viz:

THEOREM IX. "The operation e_k can still be applied to any polytope deduced from the simplex in the zero symbol of which the $k+1^{st}$ and the $k+2^{nd}$ digit are equal."

This theorem enables us to find immediately the expansion symbol of a polytope with given zero symbol. We show this by the example **(5443322210)** of art. 12.

In (5443322210) five unit intervals occur, viz. if we represent the p^{th} digit by d_p between (d_1, d_2), (d_3, d_4), (d_5, d_6), (d_8, d_9), (d_9, d_{10}). Of these the first corresponds to the original unit interval of the simplex $(1\,\overline{00\ldots0}^{\,9})$, whilst the others are introduced by the expansion operations e_2, e_4, e_7, e_8. So we find $e_2\, e_4\, e_7\, e_8\, S(10)$.

Reversely it is quite as easy to find back the zero symbol of $e_2\, e_4\, e_7\, e_8\, S(10)$. As there are to be four unit intervals more than the original one the zero symbol begins by 5 and 4, and has to show a unit interval behind the third, the fifth, the eighth, the ninth digit, etc.

20. It is obvious that the system of expansion operations cannot lead to a zero symbol with two or more equal largest digits. So the system of the expansion forms is not complete as to the total number of possible forms. But the scope of this incompleteness is not so large as we might think at first. For, if the zero symbol winds up in two or more zeros, the inversion indicated in art. 3 will bring about a new zero symbol with more than one largest digit. Nevertheless, after this extension of the system of expansion forms, still the forms with a zero symbol containing two or more largest digits and two or more zeros are lacking.

So it was desirable to have at hand a new geometrical operation leading to forms with a zero symbol containing more than one largest digit. This now is given us by M^rs^. Stott in the operation of *contraction*; but before we show this we may devote a single word to the introduction of different kinds of contraction.

The *subject* of the operation c of contraction of an expansion form in S_n is always a group of limiting elements of the same import and of the highest order of dimensions available; so we designate the contraction c as a c_0, a c_1, a c_2, etc. according to the subject elements being of vertex import, of edge import, of face import, etc. Moreover these limits of the same import can be subject of contraction, when and only when all their vertices form together exactly all the vertices of the expansion form, each vertex taken once; in this case any two of these limits are still separated from each other by the distance of an edge at least and now the operation of contraction consists merely in this that all these limits undergo a parallel displacement, of the same amount, towards the

centre O of the expansion form, by which any of these limits gets a vertex or some vertices in common with some of the other ones.

We illustrate this by the example of fig. 4. Here the results can be tabulated as follows:

$$\left.\begin{array}{r} c_0 e_1 T = \quad O \\ c_0 e_2 T = -T \\ c_0 e_1 e_2 T = -e_1 T \end{array}\right\}, \quad \left.\begin{array}{r} c_1 e_1 T = T \\ c_1 e_2 T \text{(impossible)} \\ c_1 e_1 e_2 T = e_2 T \end{array}\right\}, \quad \left.\begin{array}{r} c_2 e_1 T \text{(impossible)} \\ c_2 e_2 T = T \\ c_2 e_1 e_2 T = e_1 T \end{array}\right\}.$$

In this small table the negative sign indicates the inverse orientation; the impossibility of $c_1 e_2 T$ and $c_2 e_1 T$ is caused by the fact that the polygons, in the first case of edge and in the second of face import, forming the subject of contraction, have already a vertex or an edge in common.

But we can also account for the impossibility of $c_1 e_2 T$ and $c_2 e_1 T$— and for other similar results — by remarking that the contraction c_k undoes the expansion e_k and that it can be applied, when and only when the expansion form has been obtained by applying amongst the different expansions the operation e_k. So c_0 is the only contraction operation which we have to introduce in order to be able to deduce all the forms with a symbol satisfying the law of theorem I.

As we will use henceforth exclusively the operation c_0, the subscript of the c can be omitted.

21. We now prove the general theorem:

Theorem X. "By applying the contraction c to any expansion form the largest digit of the zero symbol of this form is diminished by one".

Proof. The groups of polytopes of vertex import of the expansion form represented by the zero symbol $(a+1, a, b, c, .., 0)$, where $a \geqq b \geqq c \ldots$, is found by putting $x_i = a+1$, leaving $(a, b, c, .., 0)$ for the other coordinates. By diminishing $a+1$ by one we get an other form with the zero symbol $(a, a, b, c, .., 0)$ possessing also polytopes of vertex import represented by $(a, b, c, .., 0)$. So the polytopes g_0 of vertex import of the second form are congruent to and equally orientated with the corresponding polytopes of the first, but they lie in spaces $x_i = a$ nearer to the centre than $x_i = a+1$. For, if p is the extension number of the original form $(a+1, a, b, c, .., 0)$ of S_n, and therefore $p-1$ that of the new form $(a, a, b, c, .., 0)$, the true coordinate values of x_i corresponding to the values $a+1$ of the first and a of the second zero symbol are $a+1-\dfrac{p-1}{n+1}$

and $a - \frac{p-2}{n+1}$; as the true coordinates of O are $\frac{1}{n+1}$ in both cases, the distance to O is diminished by

$$a + 1 - \frac{p-1}{n+1} - \frac{1}{n+1} - \left(a - \frac{p-2}{n+1} - \frac{1}{n+1}\right) =$$
$$= 1 - \frac{p-1}{n+1} + \frac{p-2}{n+1} = \frac{n}{n+1}.$$

Moreover it is evident that any two of these polytopes g_0 of the first form, e. g. those lying in the spaces $x_1 = a + 1$, $x_2 = a + 1$ are separated by the right prism with the base polytopes

$$x_1 = a + 1\,, \quad x_2 = a \quad , \quad x_3, x_4, \ldots \quad = (b, c, \ldots, 0)\,,$$
$$x_1 = a \quad , \quad x_2 = a + 1\,, \quad x_3, x_4, \ldots \quad = (b, c, \ldots, 0)\,,$$

while the corresponding two g_0 of the second form are in contact with each other by the $n-2$-dimensional polytope

$$x_1 = a\,, \quad x_2 = a\,, \quad x_3, x_4, \ldots \quad = (b, c, \ldots, 0).$$

By combining the theorems IX and X we can find the symbol in operators c and e_k of any contraction form, i. e. of any form the zero symbol of which contains two or more largest digits. To that end we have.

1°. to pass to the corresponding expansion form by adding one to the first digit,

2°. to treat the zero symbol of this expansion from according to the rule deduced from theorem IX,

3°. to put c before the obtained result.

In the following we give some examples of the deduction of c and e_k symbols from zero symbols, in connexion with the three possibilities which may present themselves, if we consider the two different zero symbols of a form without central symmetry, according to the appearance of the contraction symbol; they are

(4332210) = —(4322110), i. e. $e_2 e_4 e_5 S^{(1)}(7) = - e_1 e_3 e_5 S^{(1)}(7)$,
(4432210) = —(4322100), „ $c e_1 e_2 e_4 e_5 S^{(1)}(7) = - e_1 e_3 e_4 S^{(1)}(7)$,
(3332100) = —(3321000), „ $c e_2 e_3 e_4 S^{(1)}(7) = - c e_1 e_2 e_3 S^{(1)}(7)$.

Remark. According to the developments of the preceding article the contraction c_k always cancels the expansion e_k; so we can deduce from the theorems VI and IX that the operation c_k can only be applied to expansion forms in the zero symbol of which the $k+1^{st}$ and the $k+2^{nd}$ digit are unequal and that the zero symbol of the new form is found by subtraction of a unit from the first

$k+1$ digits of the zero symbol of the given form. Of this general result theorem X considers the special case k zero.

Now if we apply the contraction $c_0 = c$ to the simplex of coordinates $(\overline{100\ldots0}^{\,n})$ itself we find the point with the zero symbol $(\overline{000\ldots0}^{\,n+1})$ i. e. the centre O. This result is geometrically evident: if we bring the vertices nearer to the centre so as to annihilate the separating edges the result is a single point. In this point of view the inverse operation e_0 can be considered as corresponding to the generation of the simplex starting from a point.

Remark. By introducing the operation e_0 the contraction symbol c can be shunted out. So, if $S_0^{(1)}(n+1)$ represents the point which is to become the $S^{(1)}(n+1)$ by applying the operation e_0, we can replace $c\,e_k\,e_l\,S^{(1)}(n+1)$ by $e_k\,e_l\,S_0^{(1)}(n+1)$, but this implies that we write $e_0\,e_k\,e_l\,S_0^{(1)}(n+1)$ for $e_k\,e_l\,S^{(1)}(n+1)$.

This new notation will prove to be preferable in the case of the nets (see under E the art. 30 at the end of page 57).

E. *Nets of polytopes.*

22. As to recent literature about space fillings or nets we may mention A. ANDREINI's "Sulle reti di poliedri regolari e semiregolari e sulle corrispondenti reti correlative" (Roma, 1905), two papers of mine ("Fourdimensional nets and their sections by spaces" and "The sections of the net of measure polytopes M_n of space Sp_n with a space Sp_{n-1} normal to a diagonal", *Proceedings* of Amsterdam, vol. X, pp. 536, 688) and the memoir of M^rs^. STOTT quoted several times.

We exclude what may be called a prismatic net, i. e. a net in S_n obtained by prismatizing a net of S_{n-1} in a new direction, and divide the remaining *uniform* nets derived from the simplex into two groups [1]): 1°. *pure* nets with only one (central symmetric) constituent and 2°. *mixed* nets either with one non central symmetric constituent in two opposite positions or with constituents of different kind. If we restrict ourselves to the plane the first group consists of the hexagon net only, while the second is represented e. g. by the triangle net and the net of hexagons and triangles; if we proceed to ordinary space the first group contains the tO net only, while the second is represented e. g. by the net of T and O.

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[1]) This division — of no fundamental importance in itself — is introduced here, merely in order to smooth the way leading to the analytical representation of the nets.

It is our aim to unearth in the following articles *all* the nets of simplex extraction possible in space S_n from $n = 2$ to $n = 5$ included. This task, concerned with new material, breaks up into several parts. First we will have to deduce general characteristic properties of the analytical symbols which are to represent the nets. Secondly we will derive a simple rule solving the question under what circumstances the symbols obtained *do* represents possible nets. Thirdly application of this simple rule will lead to the knowledge of all the possible nets and to a tabularization of them. Finally we will pass in review the tabulated nets and devote some words to an other method by which at least a part of these results can be obtained.

23. *Theoretically* speaking a net can be determined analytically in two different ways, either as a whole or decomposed into its constituent polytopes. So we will try to find either *one* symbol of coordinates, representing all the vertices of the net at a time, or in the case of pure nets *one pair*, in the case of mixed nets *several pairs* of symbols, each pair consisting of a symbol representing all the vertices of any constituent and an other symbol from which can be deduced all the centres of the repetitions of that constituent in the same orientation occurring in the net.

In order to blow life into this theoretical skeleton — forming as it were a kind of working hypothesis — we consider the generally known and simple case of the *net of triangles* in the plane.

If we start (fig. 5) from a triangle $A_1 A_2 A_3 = p_3^{(1)}$, i. e. with sides unity, and complete the three sides produced to three systems of equidistant parallel lines, the distance of any two adjacent parallel lines being the height of triangle $p_3^{(1)}$, we get the net $N(p_3)$.

From this generation it is at once evident that with respect to the original $p^{(1)}_3$ as triangle of coordinates all the vertices of the net can be represented by the coordinate symbol (a_1, a_2, a_3), where $a_i, (i = 1, 2, 3)$ are any three integers for which $\Sigma a_i = 1$. So (a_1, a_2, a_3), $\Sigma a_i = 1$ is the *net symbol* of $N(p_3)$, under the condition stated that a_i are three integers. In this ever so simple case the round brackets may be omitted, for the faculty of taking for a_1, a_2, a_3 any set of three integers with a sum unity includes that of interchanging the three digits.

The net $N(p_3)$ consists of two sets of triangles, triangles $p^{(1)}_3$ corresponding in orientation with $A_1 A_2 A_3$ and triangles $p^{(1)}_3$ of opposite orientation. If we consider only one of these two sets of triangles and of these triangles only one of the three sets of homologous

vertices we get all the vertices of the net and each vertex once. In other words: the system of the centres of either of the two sets of triangles is equipollent to the system of vertices of the net, i. e. if we move all the vertices of the net in the direction $A_1 O$ over that distance it passes into the system of the centres of the triangles corresponding in orientation with $A_1 A_2 A_3$, whilst we get the system of the centres of the other set of triangles by a motion over the same distance in opposite direction. So, as the three coördinates of any vertex of the net are found by adding to the coordinates 1, 0, 0 of A_1 three integers with a sum zero, and the true coordinates of the centres O and O_1 are $\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$ and $-\frac{1}{3}, \frac{2}{3}, \frac{2}{3}$ the symbols $(b_1 + \frac{1}{3}, b_2 + \frac{1}{3}, b_3 + \frac{1}{3})$ and $(b_1 - \frac{1}{3}, b_2 + \frac{2}{3}, b_3 + \frac{2}{3})$ represent the centres of the two sets of triangles under the condition that the three b_i are integers with sum zero. In both cases the three integers b_i with sum zero indicate what is to be added to the coordinates of any centre of each set in order to obtain the whole set; as we call the two sets of centres the "frames" of the two kinds of triangles, we call the system of *differences* b_1, b_2, b_3 the "frame coordinates" and $(b_1, b_2, b_3), \Sigma b_i = 0$ the "frame symbol" of both sets of triangles.

Recapitulating we find the following result for $N(p_3)$:

	Net symbol	$(a_1, a_2, a_3), \Sigma a = 1.$
Set of triangles $A_1 A_2 A_3$	Symbol of constituent $A_1 A_2 A_3$	$(1, 0, 0),$
	Frame	$(b_1 + \frac{1}{3}, b_2 + \frac{1}{3}, b_3 + \frac{1}{3}), \Sigma b = 0.$
Other set of triangles	Symbol of $O_1 O_2 O_3$	$(-\frac{1}{3}, \frac{2}{3}, \frac{2}{3}),$
	Frame	$(b_1 - \frac{1}{3}, b_2 + \frac{2}{3}, b_3 + \frac{2}{3}), \Sigma b = 0.$
	Frame symbol	$(b_1, b_2, b_3), \Sigma b = 0.$

Here $O_1 O_2 O_3$ represents a central triangle oppositely orientated to $A_1 A_2 A_3$. We may still remark that the second frame may be written in the more symmetrical form $(b_1 + \frac{2}{3}, b_2 + \frac{2}{3}, b_3 + \frac{2}{3})$, $\Sigma b_i = -1$, or if one likes $(b_1 - \frac{1}{3}, b_2 - \frac{1}{3}, b_3 - \frac{1}{3}), \Sigma b_i = 2$.

But it is much simpler here to decompose the net into the repetitions of the two triangular constituents by introducing a new symbol still, the symbol $(b_1 + \mathbf{1}, b_2 + \mathbf{0}, b_3 + \mathbf{0})$, obtained by addition of the corresponding digits of the frame symbol and the symbol of the constituent $A_1 A_2 A_3$, the heavy round brackets meaning that only the parts of the digits written in heavy type are to interchange places, whilst the arbitrary integers b_i satisfy the condition $\Sigma b_i = 0$. For each system of values of the b_i satisfying the condition stated the symbol represents a definite triangular constituent of the set to which $A_1 A_2 A_3$ belongs; so by this symbol

the net $N(p_3)$ is decomposed into the different constituents of the set of triangle $A_1\ A_2\ A_3$. In the same way the symbol $(b_1+\mathbf{1},\ b_2+\mathbf{1},\ b_3+\mathbf{0})$ characterizes the other set of triangles under the condition $\Sigma\, b_i = -1$.[1])

24. In the first place we remark that the net of triangles admits a net symbol with only integer digits and we examine now to what extent this property is a general one.

If we choose for simplex of coordinates $S^{(1)}(n+1)$ a simplex with respect to which a definite polytope of the net — let us call it the central polytope $(P)^{\circ}$ of the net — can be represented by its zero symbol and we restrict ourselves to the cases in which the constituents of the net are exclusively forms derived from the simplex, we can easily prove that the coordinates of all the vertices of the net must be integer. To that end we call any polytope of the net "orientated" with respect to the simplex of coordinates, if a translational motion of the polytope which brings its centre into coincidence with the centre of that simplex gives it a position in which it is represented by its zero symbol or by the reverse; this definition enables us to state the following lemma:

"If two polytopes of the net in S_n have a limiting $n-1$-dimensional polytope in common, they are both orientated with respect to the simplex of coordinates, as soon as this is the case with one of them".

We remark — in order to prove this lemma first — that, if two polytopes derived from $S^{(1)}(n+1)$ have a limiting $n-1$-dimensional polytope in common, this limit has *either* with respect to both the *same* import *or* its import with respect to the one is *complementary* to that with respect to the other. For, according to the last two lines of art. 13, any two limits $(l)_{n-1}$, represente das to their imports by g_k and $g_{k'}$, are prismotopes $(P_k;\ P_{n-k-1})$ and $(P_{k'};\ P_{n-k'-1})$, and these prismotopes cannot coincide, unless we have either $k'=k$ or $k'=n-k-1$.

This remark leads to a proof of the lemma in the following way. Let $(P)_n^a$ and $(P)_n^b$ be the two given polytopes and $(P)_{n-1}$ their common $n-1$-dimensional limit lying in the space $S_{n-1}^{a,\,b}$. Let the $S^{(1)}(n+1)$, from which $(P)_n^a$ can be derived by means of the operations e_k and c, be our simplex of coordinates; then $P_n^{(a)}$ is not only orientated with respect to that simplex but also concentric with it.

[1]) As soon as the idea of splitting up the digits of the symbol into two parts, an unmovable part and a permutable one, had presented itself, the analytical deduction of the nets of simplex extraction was within grasp.

Let $(P')^b_n$ and $(P'')^b_n$ represent farthermore the two [1]) polytopes congruent to $(P)^b_n$ and concentric to $(P)^a_n$ which admit of a zero symbol with respect to the simplex of coordinates. Then we have only to prove that either $(P')^b_n$ or $(P'')^b_n$ is equipollent to $(P)^b_n$. Now from the fact that $(P)^a_n$ and $(P)^b_n$ have $(P)_{n-1}$ in common it follows that $(P')^b_n$ and $(P'')^b_n$ must admit a set of limits congruent and therefore of the same or of complementary import with $(P)_{n-1}$ of $(P)^a_n$; so one of these limits of $(P')^b_n$ — say $(P')_{n-1}$ — and one of these limits of $(P'')^b_n$ — say $(P'')_{n-1}$ — must lie in spaces S'_{n-1} and S''_{n-1} parallel to $S^{a,\,b}_{n-1}$ and on both sides at the same distance from the centre O of $(P)^a_n$. Of these spaces S'_{n-1} and S''_{n-1} let S'_{n-1} be that one which lies on opposite sides with respect to O with $S^{a,b}_{n-1}$. Then it will be possible to bring $(P')^b_n$ into coincidence with $(P)^b_n$ by means of a translational motion; for, if by such a motion the limit $(P')_{n-1}$ is brought into coincidence with $(P)_{n-1}$, the polytopes will coincide, as this is the case not only with the limits mentioned but also with their centres. So $(P)^b_n$ is orientated with respect to the simplex of coordinates.

From the lemma to the theorem in view we have only to take one step more. The lemma immediately shows that, if the net in S_n consists exclusively of polytopes derived from the simplex, all the polytopes are orientated with respect to the simplex of coordinates, as soon as this is the case with one of them; for we can always consider any two polytopes of the net as the first and the last of a series of polytopes any two adjacent ones of which are in $n-1$-dimensional contact. So with respect to the simplex from which the central polytope $(P)^{\circ}$ has been derived all the polytopes of the net are orientated. But this includes that by passing from any vertex of the net to an adjacent one the coordinates change by integers and as we can reach any vertex of the net by means of a set of these motions — starting from a determinate vertex of $(P)^{\circ}$ — the coordinates of any vertex of the net must be integers.

So we have shown now that the property of admitting vertices with integer coordinates only belongs to all the nets, the polytopes of which are exclusively of simplex extraction. This very general result brings us in contact with the two following questions:

a). Can the result be expressed by saying that any net with the assigned property of its constituents admits a *net symbol* with integer coordinates only?

This question must be answered negatively. We cannot pass to

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[1]) In the particular case of a central symmetric $(P)^b_n$ these two positions coincide, etc.

this new version, unless we prove that each net of the assigned kind *does* admit a net symbol; as soon as such a net admits a net symbol we can choose our system of coordinates in such a manner that this net symbol contains integer coordinates only. We take position with respect to this point by supposing beforehand that each net of the assigned kind admits a net symbol, which brings us under the obligation to prove afterwards that this is so.

b). Are there simplex nets not satisfying the condition that all the constituents are of simplex extraction?

We dispose of this question by pointing to three plane nets, viz.

1°. $N(p_3, p_4, p_6)$ of triangles, squares and hexagons (fig. 6),

2°. $N(p_4, p_6, p_{12})$ of squares, hexagons and dodecagons (fig. 7),

3°. $N(p_3, p_{12})$ of triangles and dodecagons (fig. 8),

which must undeniably be considered as simplex nets, as they can be derived from the three generally known plane nets $N(p_3)$, $N(p_6)$, $N(p_3, p_6)$ by means of the *e*-operations. If $N(p_x; p_y; p_z)$ represents a net with the polygonic constituents p_x, p_y, p_z of which p_x is of face, p_y of edge, p_z of vertex import, these deductions are indicated by the equations

$$e_2\, N(p_3) = N(p_3; p_4; p_6); \qquad e_2\, N(p_6) = N(p_6; p_4; p_3);$$
$$e_1\, e_2\, N(p_3) = N(p_6; p_4; p_{12}); \quad e_1\, e_2\, N(p_6) = N(p_{12}; p_4; p_6);$$
$$c\, e_1\, e_2\, N(p_3) = N(p_3; —; p_{12}).$$

As these three nets contain constituents not deducable from the simplex of the plane, the triangle, by means of the operations e_k and c, they *must* form exception to the general rule about the net symbol with integer coordinates only; for, in the coordinates with respect to the simplex, only the polytopes derived from the simplex can be represented by a symbol with integer coordinates only.

On account of the property of the three plane nets mentioned — to admit at the same time constituents derivable and constituents not derivable from the simplex — we call them "hybridous". In order to be able to deduce general results from the simple law of integers found above we discard provisionally the three hybribous plane nets and all the hybridous nets that space and hyperspace may contain, considering only the nets we call simplex nets "proper"; meanwhile we promise to come back to these exceptional cases, after having secured the general rule alluded to in art. 22 and the main results to which it leads (see art. 34).

25. In the second place we remark that the two sets of triangles of the net $N(p_3)$ admit the same frame symbol with integer coordinate

values only. We show that this property too is a general property i. e. that all the different sets of constituents of any simplex net proper admit the same frame symbol with integer coordinates.

In discussing the number of the regular polyhedra in ordinary space the plane nets $N(p_3)$, $N(p_4)$, $N(p_6)$ appear as polyhedra with an infinite number of faces, unyielding as to this that their faces remain in the same plane instead of bending round in three dimensions. Of these regular polyhedra with an infinite number of faces the centre is at infinity in the common direction of the normals to their plane in the space of three dimensions which is supposed to contain them and the anallagmatic [1]) rotations and reflections of the regular polyhedra proper pass into translations and reflections in the case of $N(p_3)$, $N(p_4)$, $N(p_6)$. In the same way each net of S_n may be considered as an $n+1$-dimensional polytope with an infinite number of limits $(l)_n$ which instead of bending round in S_{n+1} fill a space S_n. On account of this each net must be transformable in itself by a translational motion which brings a constituent polytope of the net into coincidence with any repetition of that constituent in the same orientation. By means of this property we prove now the following general theorem:

Theorem XI. "Any possible simplex net proper admits a net symbol and for all the different sets of constituents the same frame symbol. Moreover the frames of all the possible nets of S_n are similar to each other."

a) We show first that *all the different frames of a net are equipollent.*

Let (P) and (Q) with the centres C_p and C_q be any two polytopes of different kinds of a simplex net proper having at least one vertex V in common. Let (P') be any polytope of this net equipollent to (P) and let (Q'), V', C'_p, C'_q represent the new positions of (Q), V, C_p, C_q after a translational motion which brings (P) into coincidence with (P') and therefore the net with itself. Then (Q'), V', C'_p, C'_q are respectively a polytope of the net equipollent to (Q), a vertex of the net homologous to V, the centre of (P'), the centre of (Q'). From this we derive the equipollency of the three lines VV', $C_pC'_p$, $C_qC'_q$, i. e. $C_pC'_p$ and $C_qC'_q$ are mutually equipollent as they are both equipollent to VV'. So all the different frames of a net are equipollent, i. e. each of these frames can be brought into coincidence with any other of them by means of a translational motion.

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[1]) These rotations and reflections which transform a polytope in itself will be studied in part *G*.

b) In the second place we prove that *each net admits a frame symbol*, and that *the frames of all possible nets are similar.*

If a rectilinear translational motion of the net over a distance d brings any polytope (P) of it into coincidence with its repetition $(P)^{(1)}$ and therefore the net with itself, a rectilinear translational motion of the net over a p times larger distance pd in the same direction, p being any integer, will bring (P) into coincidence with an other of its repetitions $(P)^{(p)}$ and therefore also the net with itself. This is self evident if we consider the motion over pd as the result of p motions d in the same direction executed one after another. In other terms: the frame of any set of constituents of any simplex net proper must be characterized by the property of containing the point $C^{(p)}$ determined by the vector equation $CC^{(p)} \equiv p \,.\, CC^{(1)}$ as soon as it contains the centres C and $C^{(1)}$ and p is any integer. Now let $d_1, d_2, \ldots, d_{n+1}$ with the condition $\Sigma d_i = 0$ represent the frame coordinates of C with respect to any centre C_o of the frame, and let us consider $d_1, d_2, \ldots, d_n$, — i. e. all these integers, d_{n+1} alone excepted —, as the rectangular coordinates of a point V lying in an other space S'_n bearing the system of coordinates $O\,(X_1, X_2, \ldots, X_n)$. Then to each point C, $C^{(1)}$, $C^{(p)}$ of the frame correspond points V, $V^{(1)}$, $V^{(p)}$ of S'_n and the vector equation $CC^{(p)} \equiv p \,.\, CC^{(1)}$ includes the vector equation $VV^{(p)} \equiv p \,.\, VV^{(1)}$, i. e. there is a correspondence one to one between the centres C of the frame and the images V in S'_n, the points V in S'_n being characterized by the property of having integer coordinates $x_1, x_2, \ldots, x_n$. But *all* the points V with integer coordinates form evidently the vertices of a net of measure polytopes with edge unity; so the system of images V is *either* the total system of vertices of this net of measure polytopes *or* a portion of it, containing always the origin O corresponding to the centre C_o and partaking of the geometrical property of containing the point $V^{(p)}$, determined by the vector equation $VV^{(p)} \equiv p \,.\, VV'$ if it contains V, V' and p is integer. In this form it is immediately evident — in connection with the equivalence of the different coordinates — that the portion can only be a system of vertices, the coordinates of which are integers admitting a common factor r, i. e. the set of vertices of a net of measure polytopes with edge r. So we have shown now that the system of points V of S'_n must be $(rb_1, rb_2, \ldots, rb_n)$, where the n quantities b_i are arbitrary integers, whilst r is a definite integer. From this result it follows immediately that the frame of the centres C admits the frame symbol

$$(rb_1, rb_2, rb_3, \ldots, rb_n, rb_{n+1}), \ldots \ldots \ldots \ldots \quad F)$$

the arbitrary integers b_i satisfying the condition $\Sigma b_i = 0$. The

quantity r, which is the same for the different frames of the same net, may vary from net to net. We call it the *period* of the net.

All the simplex nets proper have similar frames, as their images of points V are similar. [1]) If, as in art. 1, our space S_n is the space $\sum_{i=1}^{n+1} x_i = 1$ lying in a space of operation S_{n+1} and determining on the axes of a given system $O(X_1, X_2, \ldots, X_{n+1})$ of coordinates equal intersepts OA_i, we can say that the nets of S_n, the vertices of which admit with respect to the simplex of coordinates $A_1 A_2 \ldots A_{n+1}$ integer coordinates only, always admit frames *projecting* themselves normally on any of the n- dimensional spaces S'_n of coordinates of S_{n+1} as sets of vertices of systems of measure polytopes of that S'_n.

c) In the third place we prove that *each net of S_n admits a net symbol.*

By combining the zero symbol $(q_1, q_2, \ldots, q_n, 0)$ of the central polytope with the frame symbol $(rb_1, rb_2, \ldots, rb_n, rb_{n+1})$, $\sum b_i = 0$ of the net we obtain the symbol

$$(rb_1 + q_1, rb_2 + q_2, \ldots\ldots, rb_n + q_n, rb_{n+1}), \ldots\ldots N)$$

where the q_i and r are given integers, whilst for the b_i we can take any system of integers with sum zero. As this symbol contains the coordinates of *all* [2]) the vertices of the net, it is the *net symbol.*

If we write this symbol in the form

$$(rb_1 + \mathbf{q}_1, rb_2 + \mathbf{q}_2, \ldots . rb_n + \mathbf{q}_n, rb_{n+1} + \mathbf{0})$$

we have got a symbol representing the net decomposed into the repetitions of the central polytope.

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[1]) We remark already here that later on cases will present themselves which are at variance with this simple result. We will treat these cases — and explain why they appear as exceptions — as soon as they turn up.

[2]) This is only true, if *each* vertex of the net is also vertex of a repetition of the central polytope in the same orientation. So, if the net contains a non central symmetric constituent in two opposite orientations and in each vertex only one of these two differently orientated constituents concurs, the net symbol corresponding to one of these constituents as central polytope would only contain half the number of vertices of the net and would have to be completed by a second symbol giving the other half. In that particular case the system of vertices breaks up into two equivalent parts P and Q with the property that the net is equipollent to itself for any two points of the same half as homologous but congruent with opposite orientation to itself for any two vertices of different halves as homologous. This particularity presents itself in the plane in the case of the net of triangles and dodecagons (fig. 8), already discarded above for an other reason. Here we exclude, also provisionally, all the eventually possible nets where this particularity of the division of the system of vertices into two equivalent systems might present itself.

At first sight it may seem that the introduction of the common factor r, by means of which the frame is only enlarged but not changed in form, is of no avail, as the *scale* of the diagrams is of no importance whatever. But then one overlooks the fact that the frame is enlarged, while the central polytope $(P)^{\circ}$ remains unaltered. So in the case of the central triangle $A_1 A_2 A_3$ in the plane: if we take $r = 1$ we have to deal with the net $N(p_3)$ of fig. 5, whilst the supposition $r = 2$ gives the vertices of the net $N(p_3, p_6)$ by means of the triangle $A_1 A_2 A_3$ and its equally orientated repetitions (fig. 9).

This simple example shows in the first place the influence of the period r. But on the other hand it gives a glimpse of the fact that with a given central polytope not all integer values of r lead to existing nets. So the supposition $r = 3$ brings already the central triangle $A_1 A_2 A_3$ and its repetitions too far apart. [1])

26. We pursue our investigation in the direction of the last sentence of the preceding article, entering into details about the relationship between the period r and the largest digit q_1 of the zero symbol $(q_1, q_2, \ldots, q_n, 0)$ of the central polytope $(P)^{\circ}$.

If we call any repetition (P) of the central polytope $(P)^{\circ}$ corresponding with it in orientation *adjacent* to it, if the distance between their centres C_0 and C is as small as possible [2]), i. e. if the coordinates of C can be deduced from the equal coordinates $x_i = \dfrac{\Sigma q_i}{n+1}$ of C_0 by altering only *one* pair of coordinates by addition and subtraction of only *one* time r, we find:

"The central polytope and one of its adjacent repetitions *overlap* for $r < q_1$, whilst for $r = q_1$ they are *in contact* and for $r > q_1$ *free* from each other".

Of these three cases of relationship between r and q_1 we consider first the case $r = q_1$, then the two cases $r \neq q_1$ at a time.

Case $r = q_1$. The two adjacent polytopes represented by

$$(q_1, q_2, q_3, \ldots, q_n, 0), \; (r + q_1, -r + q_2, q_3, \ldots, q_n, 0)$$

have all the vertices

[1]) Application of the case $r = 3$ to the triangle $A_1 A_2 A_3$ gives one of the two sets of triangles of the net of fig. 8, already discarded for two different reasons.

[2]) This is the case if the image V of C (compare the preceding article under b) lies on an axis OX_i at distance r from O.

$$x_1 = q_1,\ x_2 = 0,\ (x_3, x_4, \ldots, x_{n+1}) = (q_2, q_3, \ldots, q_n)$$

in common; this is immediately evident, if for x_1 and x_2 we take in the first symbol the digits q_1 and 0, in the second $r + 0$ and $-r + q_1 = -r + r$. In general these common vertices define a polytope of $n-2$ dimensions situated in the space S_{n-2} for which $x_1 = q_1$, $x_2 = 0$, i. e. the two polytopes are *in contact* with each other by a limit $(l)_{n-2}$; as any common vertex of the two polytopes lies at equal distances (radius of the circumscribed spherical space) from the centres C_0 and C, this common $(l)_{n-2}$ lies in the space S_{n-1} normally bisecting $C_0\,C$, i. e. this $(l)_{n-2}$ of contact has the midpoint M of $C_0\,C$ for centre, i. e. the contact by the $(l)_{n-2}$ is *external.* But in one exceptional case, in the case $q_1 = 2,\ q_2 = q_3 = \ldots = q_n = 1$ of the central symmetric polytope $(2\,\overbrace{11\ldots1}^{n-1}0)$, the common limit $(l)_{n-2}$ shrinks together into a single point, the midpoint of $C_0\,C$, as in that case $(q_2, q_3, \ldots, q_n)$ becomes a petrified syllable. At any rate, for $r = q_1$ the net is *mixed,* as the central polytope and one of it adjacent repetitions are *not* in contact by a limit $(l)_{n-1}$.

If for brevity we represent $\dfrac{\Sigma q_i}{n+1}$ by q the coordinates of C_0 and C are

$$\left.\begin{array}{lllll} C_0 \ldots\ldots & q & q, q, q, \ldots\ldots q \\ C \ldots\ldots & r+q, & -r+q, q, q, \ldots\ldots q \end{array}\right\}.$$

So, according to formula 1) of art. 1, the distance $C_0\,C$ is equal to the period r; this result will be useful in the treatment of the next case.

Case $r \neq q_1$. Let us start from the case $r = q_1$ treated above and vary r. As the relation $C_0\,C = r$ holds always, this variation of r implies a variation of $C_0\,C$, the effect of a translational motion of the repetition (P) of the central polytope $(P)^0$ in the direction $C_0\,C$ if r increases, in the opposite direction $C\,C_0$ if r decreases. In the first case when $C_0\,C$ is enlarged, the polytopes which were either in $(l)_{n-2}$-contact or in point contact, will become *free* from each other. In the second case when $C_0\,C$ diminishes the midpoint M of the new $C_0\,C$ will lie inside both polytopes, i. e. the polytopes will *overlap.* So the theorem is proved.

As we cannot use overlapping polytopes we have to discard all the cases $r < q_1$, i. e. we have to consider q_1 as an inferior limit of r. But if the net symbol — as we suppose — contains *all* the vertices, there is also a superior limit. For in the case $r = q_1 + k$

the distance of the two limits $(l)_{n-2}$ or of the two vertices, which coincided for $r=q$, has now become k and this distance may not surpass unity. So we have also to discard all the cases $r > q_1 + 1$. So the result is that we can only use the values $r = q_1$ and $r = q_1 + 1$, or inversely: the only values of the largest digit q_1 of the zero symbol of the central polytopes are r and $r-1$. Now as any polytope of the net can be promoted to central polytope we have in general:

THEOREM XII. "Any possible net with period r contains only constituents with zero symbols having for largest digit q_1 either $r-1$ or r. Two adjacent repetitions of a constituent for which $q_1 = r - 1$ are free from each other, whilst two adjacent repetitions of a constituent for which $q_1 = r$ are in contact, in general by a limit $(l)_{n-2}$, but in the particular case $(2\overline{11\ldots1}^{\,n-1}0)$ by a point."

27. But now unexpectedly a difficulty presents itself. In the case $q_1 = r$ any two adjacent repetitions of a definitely orientated constituent are in $(l)_{n-2}$-contact or in point contact, in the case $q_1 = r - 1$ these two repetitions are free from each other. In both cases we need other constituents to fill up gaps, in other words all the nets are *mixed*. But this result is at variance with the existence of the net $N(p_6)$ in the plane, of the net $N(tO)$ in space. So we have to look out for a way out of this difficulty. This way will present itself immediately, if we examine how to find the other constituents of a net, the central polytope and the period of which are given.

Let the zero symbol of the central polytope $(P)^0$ of a net with period r be represented once more by $(q_1, q_2, \ldots, q_n, 0)$, where we have either $q_1 = r$ or $q_1 = r - 1$. Then we can ask by what processes we can deduce from the symbol

$$(rb_1 + \mathbf{q}_1, rb_2 + \mathbf{q}_2, \ldots, rb_n + \mathbf{q}_n, rb_{n+1} + \mathbf{0}),$$

representing the net decomposed into the repetitions of the central polytope, other constituents. There are two of these processes completing each other in this sense that the first can be used in the case $q_n = 0$, the second in the case $q_n = 1$.

1°. In the case of the zero symbol $(q_1, q_2, \ldots, q_{n-1}, 0, 0)$ containing more than one zero we can write the decomposing symbol

$$(rb_1 + \mathbf{q}_1, rb_2 + \mathbf{q}_2, \ldots, rb_{n-1}, + \mathbf{q}_{n-1}, rb_n + \mathbf{0}, rb_{n+1} + \mathbf{0}), \Sigma b_i = 0$$

in the form

$$(rb_1+\boldsymbol{q}_1, rb_2+\boldsymbol{q}_2, \ldots, rb_{n-1}+\boldsymbol{q}_{n-1}, rb_n+\mathbf{0}, r(b_{n+1}-1)+\boldsymbol{r}), \Sigma b_i=0$$

by allowing r units to pass from the unmovable part rb_{n+1} of the digit $rb_{n+1}+\mathbf{0}$ to the permutable part; for by that variation we alter only the *grouping* of the vertices of the net to vertices of polytopes but not the total system of vertices of the net. If now we write b_0 for $b_{n+1}-1$ and put the permutable digit r foremost we get

$$(rb_0+\boldsymbol{r}, rb_1+\boldsymbol{q}_1, rb_2+\boldsymbol{q}_2, \ldots, rb_{n-1}+\boldsymbol{q}_{n-1}, rb_n+\mathbf{0}), \Sigma b_i=-1,$$

bringing to the fore the constituent with the zero symbol $(r, q_1, q_2, \ldots q_{n-1}, 0)$.

2°. In the case of the zero symbol $(q_1, q_2, \ldots q_{n-1}, 1, 0)$ containing only one zero an application of the same process leads from

$$(rb_1+\boldsymbol{q}_1, rb_2+\boldsymbol{q}_2, \ldots, rb_{n-1}+\boldsymbol{q}_{n-1}, rb_n+\mathbf{1}, rb_{n+1}+\mathbf{0}), \Sigma b_i=0$$

to

$$(rb_0+\boldsymbol{r}, rb_1+\boldsymbol{q}_1, rb_2+\boldsymbol{q}_2, \ldots, rb_{n-1}+\boldsymbol{q}_{n-1}, rb_n+\mathbf{1}), \Sigma b_i=-1$$

and therefore to the constituent $(r, q_1, q_2, \ldots, q_{n-1}, 1)$, the zero symbol of which is $(r-1, q_1-1, q_2-1, \ldots, q_{n-1}-1, 0)$. In order to obtain this zero symbol we can write the decomposing symbol in the form

$$(rb_0+1+\boldsymbol{r}-\mathbf{1}, rb_1+1+\boldsymbol{q}_1-\mathbf{1}, \ldots, rb_{n-1}+1 \\ +\boldsymbol{q}_{n-1}-\mathbf{1}, rb_n+1+\mathbf{0}), \Sigma b_i=-1$$

and pass to an other sum $\Sigma q_i-(n+1)$ of all the digits by omitting the unit of the unmovable part of the digits.

So, if we take notice only of the *zero symbols* of the constituents deduced by means of the two processes, we can word these processes as follows:

1°. "If the zero symbol of the given constituent contains more than one zero, we can replace one of these zeros by r".

2°. "If the zero symbol contains only one zero, we can replace this zero by r and diminish all the digits by unity afterwards".

We now come back to the difficulty about the pure nets stated above. To that end we have to ask under what circumstances one of the two processes leads back to the original constituent; therefore we repeat that:

the first deduces $(r, q_1, q_2, \ldots, q_{n-1}, 0)$ from $(q_1, q_2, \ldots q_{n-1}, 0, 0)$,
" second " $(r-1, q_1-1, q_2-1, \ldots, q_{n-1}-1, 0)$ " $(q_1, q_2, \ldots q_{n-1}, 1, 0)$.

As a polytope with a zero symbol with $k-1$ zeros cannot be a repetition of a polytope with a zero symbol with k zeros, the first process does not suit our aim; but the second may do so under the conditions

$$r-1=q_1, q_1-1=q_2, q_2-1=q_3, \ldots, q_{n-2}-1=q_{n-1}, q_{n-1}-1=1,$$

i. e. in the case of the central polytope $(r-1, r-2, \ldots, 2, 1, 0)$, i. e. if we have in S_n the case $(n, n-1, \ldots, 2, 1, 0)$ with $r=n+1$. It is indeed easy to prove that the particular case of the reappearance of the original constituent presents itself, when and only when we have $r=n+1$ and $q_1=n$. For, according to the law of theorem I, $q_1=n$ exacts that the zero symbol contains no two equal digits and under this circumstance the substitution of $n+1$ for zero followed by the diminution of all the digits by unity reproduces the original zero symbol. In art. 30 (page 57 at the top) it will be shown that the suppositions $r=n+1$, $q_1=n$ lead to the unique self space filler of S_n.

But now that the manner in which we have to account for the existing pure simplex nets is secured we have to revise our notion of "constituent of the same kind", if we will keep the analytical theory developed just now in touch with the geometrical facts. According to that theory we are obliged to say that the plane net $N(p_6)$ contains three different *groups* of hexagons, though geometrically all the hexagons are equipollent to each other and therefore of the same *kind*. For in the case $n=2$ the suppositions $r=n+1$, $q_1=n$ give rise to the net with the decomposing symbol $(3b_1+\mathbf{2},\ 3b_2+\mathbf{1},\ 3b_3+\mathbf{0})$, $\Sigma b_i=0$, corresponding (fig. 10) to the set of hexagons a with a heavy lined circuit, whilst the net contains two other groups of hexagons which admit alternately thick and thin sides, one group b where the horizontal thick side is below, an other group c where the horizontal thick side is above. So, though we keep saying that the hexagon is a self plane filler, we will consider $N(p_6)$ from an analytical point of view as admitting three different groups of hexagons, using here henceforward the more precise term of "group of constituents" in order to indicate a "set of equipollent polytopes, the vertices of which form together all the vertices of the net, *each vertex taken once*". Only under this extension of our former *kind* of constituents by our now introduced *group* of constituents the theorems XI and XII are *generally true*. If we follow the interpretation of the net $N(tO)$ as a net with one kind of

constituent only, we get a frame [1]) dissimilar to that of other nets, with any two adjacent repetitions of the unique constituent in contact by a limit $(l)_{n-1}$, i. e. a face here; these "exceptions" disappear, if we adhere to the analytical idea, according to which $N(tO)$ admits *four groups* of constituents. [2])

28. If we indicate by ρ_p the number of the digits of the zero symbol of the central polytope $(P)^{\circ}$ leaving the remainder p when divided by r, i. e. if ρ_p represents in general the number of the digits p of the zero symbol, but in the particular case of ρ_0 the sum of the numbers of the digits zero and the digits r (the latter being absent in the case $q_1 = r - 1$ of the original zero symbol), we have the:

Theorem XIII "The two operations stated above which may lead to new constituents of the same net do not affect the *circular* order of succession of the terms of the series $\rho_{r-1}, \rho_{r-2}, \ldots, \rho_1, \rho_0$. This series with the sum $n+1$ will be called "*partition cycle* of $n+1$, mod. r" of the net and be represented by $_r(\rho_{r-1}, \rho_{r-2}, \ldots, \rho_1, \rho_0)_n$".

This theorem is self evident. For the first of the two processes does not affect the series at all, whilst the second transforms it into $\rho_0, \rho_{r-1}, \rho_{r-2}, \ldots, \rho_1$.

We apply the two processes to an example in order to show the circular permutation of the partition and suppose to that end that in space S_6 there *is* a net with the period 4 admitting the constituent **(3222100)**. Then application of the two processes gives successively

			Partition cycle
(3222100)			1312
(4322210)			1312
(4432221)	=	(3321110)	2131
(4332111)	=	(3221000)	1213
(4322100)			1213
(4432210)			1213
(4443221)	=	(3332110)	3121
(4333211)	=	(3222100)	1312

Here every new symbol in the first column is derived by the first process from the one in the line immediately above it which con-

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[1]) In art. 39 the system of the centres of *all* the tO of $N(tO)$ will prove to form the vertices of a net of rhombic dodecahedra, which latter net is not of simplex extraction.

[2]) In order to avoid misunderstanding we stipulate expressly that it is not our intention to *replace* the notion of kind of constituent by that of group, but that we wish to stick to the notion of kind of constituent, *complemented* by that of group as soon as the partition cycle (see the next article) is a power cycle (see page 57).

tains a zero, whilst by the second process the symbols of the second column are deduced from those of the first column placed on the same line. In the third column we find the partition cycle 1312 proceeding one step to the right at every application of the second process.

So, *if* the case considered is that of an existing net — which question does not yet interest us here —, this net must admit seven different constituents, i. e. three pairs of oppositely orientated ones

$$\begin{array}{ccc} (3222100) & , & (3321110) \\ (3221000) & , & (3332110) \\ (4322100) & , & (4432210) \end{array}$$

and the central symmetric (4322210).

This example leads us to a general rule about the number of groups of constituents any net is to have; we state it in the form of:

THEOREM XIV. "In general the number of kinds of constituents of a net of S_n is $n+1$; in the case $r=1$ it is n."

The proof of the general case runs as follows. The zero symbol $(q_1, q_2, \ldots, q_{n-1}, q_n, 0)$ of the central polytope we start with passes either into $(r, q_1, q_2, \ldots, q_{n-1}, 0)$ or into $(r-1, q_1-1, q_2-1, \ldots, q_{n-1}-1, 0)$ according to q_n being either zero or one. So, in continuing the application of the two processes of art. 27, at *each* step the digit q_1 moves one place to the right and comes back to its original place after $n+1$ moves. Moreover it reappears there with its *original* value q_1. For the increase by r at the jump from the rear to the front is exactly counterbalanced by the loss of a unit every time when of two unequal adjacent digits the right hand one jumps to the fore, this loss occurring exactly r times; indeed, in the circular permutation of the digits from the left to the right — executed for simplicity for a moment without increasing or decreasing — the zero at the end has to be replaced successively by 1, by 2, ... by $r-1$ and finally in the case $q_1 = r-1$ by zero, in the case $q_1 = r$ by r. So after $n+1$ moves the original zero symbol recurs and the total process has come to a close. [1])

In the exceptional case $r=1$ we find only $q_1 = 1$, i. e. the zero symbol of any constituent can only contain units and zeros. So we can start with the simplex $(1\,\overline{00\ldots0}^{\,n})$ and find successively $(11\,\overline{00\ldots0}^{\,n-1})$, $(111\,\overline{00\ldots0}^{\,n-2})$, etc. But when we have to pass from $(\overline{11\ldots1}^{\,n}\,0)$

[1]) The process may come to a close sooner. Compare for this exception page 57.

to the next symbol we find by means of the second process $(\overline{00\ldots0}^{\,n+1})$, which falls out. So we only get n kinds of constituents for $r=1$.

29. We come now to the general rule about simplex nets proper; it can be stated in the following form:

THEOREM XV. "To every possible cyclical partition of $n+1$ corresponds a definite simplex net proper of S_n."

In order to prove the theorem for the general case with $n+1$ and the particular case with n groups of constituents we first of all determine a list containing these different groups of constituents, to be derived from the partition cycle. Then we select from this list a *definite* polytope $(P)_a$ of a definite group and show that this $(P)_a$ is in contact by *any* of its limits $(l)_n^{(a,b)}$ with one and only one other polytope $(P)_b$ of the list, whilst the list contains no polytope overlapping $(P)_a$.

Case $r>1$. We start from the partition cycle ${}_r(\rho_{r-1}, \rho_{r-2}, \ldots \rho_1, \rho_0)_n$ and deduce from it the net symbol

$$(\overline{ra_i+r-1}^{\,\rho_{r-1}},\ \overline{ra_i+r-2}^{\,\rho_{r-2}},\ \ldots\ldots,\ \overline{ra_i+1}^{\,\rho_1},\ \overline{ra_i+0}^{\,\rho_0})_{(a_1, a_2, \ldots\ldots, a_{n+1})}$$

i. e. the symbol with ρ_{r-1} digits congruent to $r-1$ mod. r, ρ_{r-2} digits congruent to $r-2$ mod. r, etc., the different quotients $a_1, a_2, \ldots, a_{n+1}$ of the division of these digits by r having a sum $\Sigma a_i = 0$, whilst the sum of the remainders $r-1, r-2, \ldots, 0$, i. e. $\sum\limits_{i=1}^{r-1} i\rho_i$ may be represented by k_0.

If we write this symbol in the form

$$(\overline{ra_i+\mathbf{r}-\mathbf{1}}^{\,\rho_{r-1}},\ \overline{ra_2+\mathbf{r}-\mathbf{2}}^{\,\rho_{r-2}}, \ldots,\ \overline{ra_i+\mathbf{1}}^{\,\rho_1},\ \overline{ra_i+\mathbf{0}}^{\,\rho_0})_{(a_1, a_2, \ldots\ldots a_{n+1})}$$

and permutate only the remainders $r-1$, $r-2, \ldots, 0$, the net is decomposed into the group of constituents to which the central polytope belongs; but we can have this rather complicated symbol in our mind quite as well if we simplify it by omission of the unmovable parts of the digits. So the first line of the following list repeats the group of constituents to which the central polytope belongs, while the other lines give all the other groups of constituents, deduced from the "central group" in the manner and order of succession of the preceding article.

	Σa_i	k
$\left(\frac{\rho_{r-1}}{r-1}, \frac{\rho_{r-2}}{r-2}, \ldots, \frac{\rho_1}{1}, \frac{\rho_0}{0} \right)$	0	k_0
$\left(\frac{1}{r}, \frac{\rho_{r-1}}{r-1}, \frac{\rho_{r-2}}{r-2}, \ldots, \frac{\rho_1}{1}, \frac{\rho_0-1}{0} \right)$	-1	$k_0 + r$
. . . .	. . . .	. . . . . .
$\left(\frac{\rho_0}{r}, \frac{\rho_{r-1}}{r-1}, \ldots, \frac{\rho_2}{2}, \frac{\rho_1}{1} \right)$	$-\rho_0$	$k_0 + \rho_0 r$
$\left(\frac{1}{r+1}, \frac{\rho_0}{r}, \frac{\rho_{r-1}}{r-1}, \ldots, \frac{\rho_2}{2}, \frac{\rho_1-1}{1} \right)$	$-(\rho_0+1)$	$k_0 + (\rho_0 + 1) r$
. . . .	. . . .	. . . . . .
. . . .	. . . .	. . . . . .
$\left(\frac{\rho_{r-2}}{2r-2}, \frac{\rho_{r-3}}{2r-3}, \ldots, \frac{\rho_0}{r}, \frac{\rho_{r-1}}{r-1} \right)$	$\rho_{r-1}-(n+1)$	$k_0 + (n+1-\rho_{r-1}) r$
$\left(\frac{1}{2r-1}, \frac{\rho_{r-2}}{2r-2}, \frac{\rho_{r-3}}{2r-3}, \ldots, \frac{\rho_0}{r}, \frac{\rho_{r-1}-1}{r-1} \right)$	$\rho_{r-1}-(n+2)$	$k_0 + (n+2-\rho_{r-1}) r$
. . . .	. . . .	. . . . . .
$\left(\frac{\rho_{r-1}-1}{2r-1}, \frac{\rho_{r-2}}{2r-2}, \frac{\rho_{r-3}}{2r-3}, \ldots, \frac{\rho_0}{r}, \frac{1}{r-1} \right)$	$-n$	$k_0 + nr$

For each of these groups of constituents have been indicated in a second column the value of Σa_i, in a third column the value of $k = \sum_{i=1}^{r-1} i\rho_i$. Moreover, in order to point out the regularity of the process by means of the variation of these two sums, the use of the zero symbol has been sacrificed for a moment, i. e. the diminution of the digits by unity every time as the last zero is replaced by r (exacted by the second of the two processes of the preceding article) is not executed here, which implies that a digit h jumping to the fore becomes $h+r$. Here at each step Σa diminishes by a unit and k increases by r. [1])

But in the selection of a definite polytope $(P)_a$ of the list we return to the zero symbol and suppose

1) that the cyclical permutation of the partition cycle from which $(P)_a$ has been derived begins by ρ_{l-1} and winds up in ρ_l,

2) that ρ_r of the ρ_l zeros have been replaced by r,

3) that the equation of the space S_{n-1} containing $(l)_{n-1}^{a,b}$ is

[1]) This relation also holds when we pass from the last group of constituents to the first, when we diminish all the a_i by unity at the transition. From this point of view we can introduce the notion of "cycle of constituents".

determined by making the sum $x_1 + x_2 + \ldots + x_\nu$ maximum, the ν digits which are to make that sum maximum consisting of ρ_r times r, ρ_{l-1} times $r-1$, etc. and ρ_μ of the ρ_m digits $r-l+m$.

Under these circumstances the polytope $(P)_a$ is represented by the symbol

$$\underset{(a_1,\, a_2\, \ldots\ldots\, a_{n+1})}{(\overset{\rho_r}{\overline{r}},\ \overset{\rho_{l-1}}{\overline{r-1}}, \ldots, \overset{\rho_m}{\overline{r-l+m}},\ \overset{\rho_{m-1}}{\overline{r-l+m-1}}, \ldots, \overset{\rho_0}{\overline{r-l}},\ \overset{\rho_{r-1}}{\overline{r-l-1}}, \ldots, \overset{\rho_{l+1}}{\overline{1}},\ \overset{\rho_l-\rho_r}{\overline{0}})},$$

while this symbol passes into that of $(l)^{a,b}_{n-1}$ by the introduction of intermediate brackets between the digits $a_\nu r + r - l + m$ and $a_{\nu+1} r + r - l + m$, i. e. $(l)^{a,b}_{n-1}$ is represented by

$$\underset{(a_1,\, a_2, \ldots\, a_\nu)}{(\overset{\rho_r}{\overline{r}}, \overset{\rho_{l-1}}{\overline{r-1}}, \ldots, \overset{\rho_\mu}{\overline{r-l+m}})}\ \underset{(a_{\nu+1},\, a_{\nu+2},\, \ldots\, a_{n+1})}{(\overset{\rho_m-\rho_\mu}{\overline{r-l+m}}, \overset{\rho_{m-1}}{\overline{r-l+m-1}}, \ldots, \overset{\rho_0}{\overline{r-l}}, \overset{\rho_{r-1}}{\overline{r-l-1}}, \ldots, \overset{\rho_{l+1}}{\overline{1}}, \overset{\rho_l-\rho_r}{\overline{0}})}$$

under the condition that of the two parts of this symbol the first refers to the coordinates $x_1, x_2, \ldots, x_\nu$ and the second to $x_{\nu+1}, x_{\nu+2}, \ldots, x_{n+1}$.

The determination of a second polytope $(P)_b$ of the list containing $(l)^{a,b}_{n-1}$ as limit must be guided by the remark that in *each* of the two parts of the symbol of $(l)^{a,b}_{n-1}$ considered for itself we may transfer the *same* amount from the unmovable parts of the digits to the permutable ones. But in order to obtain a symbol satisfying the law of theorem I, when the intermediate brackets are omitted, we have moreover to select these two amounts in such a way as to obtain a set of permutable parts containing $n+1$ integers distributed over r different ones, succeeding one another with differences unity. So we can *either* diminish all the digits $r, r-1, \ldots, r-l+m$ included between the first pair of brackets by r, counterbalancing this by increasing $a_1, a_2, \ldots, a_\nu$ by unity, *or* — which comes to the same — increase all the digits $r-l+m$, $r-l+m-1, \ldots, 0$ included between the second pair of brackets by r, counterbalancing this by diminishing $a_{\nu+1}, a_{\nu+2}, \ldots, a_{n+1}$ by unity; that these two results differ in form only can be shown by remarking that the first passes into the second if we increase all the digits $0, -1, \ldots, -l+m, r-l+m, r-l+m-1, \ldots, 0$ by r, counterbalancing it by diminishing $a_1+1, a_2+1, \ldots, a_\nu+1, a_{\nu+1}, a_{\nu+2}, \ldots, a_{n+1}$ by unity. So we find one and always one second polytope $(P)_b$ with $(l)^{a,b}_{n-1}$ as limit, represented by the symbol:

$$\underset{(a_1+1,\, a_2+1, \ldots, a_\nu+1,\ a_{\nu+1},\, a_{\nu+2},\, \ldots\, a_{n+1})}{(\overset{\rho_r}{\overline{0}}, \overset{\rho_{l-1}}{\overline{-1}}, \ldots, \overset{\rho_\mu}{\overline{-l+m}}, \overset{\rho_m-\rho_\mu}{\overline{r-l+m}}, \overset{\rho_{m-1}}{\overline{r-l+m-1}}, \ldots, \overset{\rho_0}{\overline{r-l}}, \overset{\rho_{r-1}}{\overline{r-l-1}}, \ldots, \overset{\rho_{l+1}}{\overline{1}}, \overset{\rho_l-\rho_r}{\overline{0}})},$$

We now pass to the determination of the centres G_a, G_b, G_l of $(P)_a$, $(P)_b$, $(l)_{n-1}^{a,b}$ which are collinear, as $G_l\,G_a$ and $G_l\,G_b$ are normal in the centre G_l to the space S_{n-1} bearing $(l)_{n-1}^{a,b}$, in order to prove that G_l lies between G_a and G_b, i. e. that $(P)_a$ and $(P)_b$ lie on different sides of that space S_{n-1}. If we consider the x_{n+1} of these three points we find for

$$G_a \ldots ra_{n+1} + \frac{1}{n+1}\Big\{ r\rho_r + (r-1)\rho_{l-1} + \ldots + \rho_m(r-l+m)$$
$$+ \ldots + \rho_0(r-l) + \rho_{r-1}(r-l-1) + \ldots + \rho_{l+1}\Big\},$$

$$G_b \ldots ra_{n+1} + \frac{1}{n+1}\Big\{ -\rho_{l-1} - 2\rho_{l-2} - \ldots - \rho_\mu(l-m)$$
$$+(\rho_m-\rho_\mu)(r-l+m)+\ldots+\rho_0(r-l)+\rho_{r-1}(r-l-1)+\ldots+\rho_{l+1}\Big\},$$

$$G_l \ldots ra_{n+1} + \frac{1}{n+1}\Big\{$$
$$(\rho_m-\rho_\mu)(r-l+m)+\ldots+\rho_0(r-l)+\rho_{r-1}(r-l-1)+\ldots+\rho_{l+1}\Big\}.$$

Now in comparing the three values $x^{(a)}$, $x^{(b)}$, $x^{(a,b)}$ of x_{n+1} we can omit the common part ra_{n+1}. But then if we write y_{n+1} for $x_{n+1} - ra_{n+1}$ it is evident that we have $y^{(b)} < y^{(a,b)} < y^{(a)}$. For $y^{(a,b)}$, $y^{(a)}$, $y^{(b)}$ are arithmetic means, $y^{(a,b)}$ of a series S of positive integers $1, 2, \ldots, r-l+m$, each of them taken a certain number of times, $y^{(a)}$ of an other series of integers consisting of S and of positive numbers $r-l+m$, $r-l+m+1$, $\ldots$, $r-1$, r equal to or larger than the largest of S, $y^{(b)}$ of a third series of integers consisting of S and negative numbers. So G_a and G_b lie on different sides of the space S_{n-1} bearing $(l)_{n-1}^{(a,b)}$, i. e. the system of polytopes contained in the list admits no holes, every limit $(l)_{n-1}$ of an arbitrarily chosen polytope P_a being covered by an other polytope P_b.

We have still to show that no two polytopes of the net can overlap. We do so by simply remarking that the vertices of the polytopes of *any* group of constituents form together the total system of vertices of the symbol derived from the partition cycle [1]) (see above at the beginning of the treatment of the case $r > 1$ under consideration), as each of the groups of constituents of the list has been deduced from that symbol according to the processes of art. 27. For — while overlapping of polytopes of the *same* group

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[1]) This fact can also be put on duty in the proof about the position of two polytopes with common $(l)_{n-1}$ on different sides of that limit.

is already excluded (art. 26) — this remark excludes overlapping of any two polytopes, as we can derive from it that not a single vertex can lie inside any polytope of any group of constituents.

Case $r=1$. In this case the enumeration of the zero symbols

$$(1\,\overline{00\,..\,0}^{\,n}),\ (11\,\overline{00\,..\,0}^{\,n-1}),\ \ldots,\ (\overline{11\,..\,1}^{\,p}\,\overline{00\,..\,0}^{\,n-p+1}),\ \ldots,\ (\overline{11\,..\,1}^{\,n}\,0)$$

of the n groups of constituents is much simpler. Moreover the polytopes of the first group and those of the last admit only one kind of limits $(l)_{n-1}$ viz. simplexes, those of any other group only two, limits $(l)_{n-1}$ with respect to $(1\,\overline{00\,..\,0}^{\,n})$ of the lowest and of the highest import.

Here overlapping is also excluded, as can be shown by means of the same remark used above. Here the polytope $(P)_a$ can be represented by

$$(a_1+\mathbf{1},\ a_2+\mathbf{1},\ \ldots\ldots,\ a_p+\mathbf{1},\ a_{p+1}+\mathbf{0},\ \ldots\ldots,\ a_{n+1}+\mathbf{0}),$$

its limit $(l)_{n-1}^{(a,\,b)}$ lying in the space S_{n-1} with the equation $x_1=a_1+1$ by

$$(a_1+1)\ (a_2+\mathbf{1},\ \ldots\ldots,\ a_p+\mathbf{1},\ a_{p+1}+\mathbf{0},\ \ldots\ldots,\ a_{n+1}+\mathbf{0}),$$

the second polytope $(P)_b$ of the list containing also this limit by

$$(a_1+1+\mathbf{0},\ a_2+\mathbf{1},\ \ldots\ldots,\ a_p+\mathbf{1},\ a_{p+1}+\mathbf{0},\ \ldots\ldots,\ a_{n+1}+\mathbf{0}).$$

For the rest the proof can be copied from that given above.

Now that theorem XV has been proved we go back to the polytopes $(P)_a$ and $(P)_b$ in contact with each other by a common limit $(l)_{n-1}$ in order to indicate a relation between the *import* of that common limit with respect to $(P)_a$ and $(P)_b$ on one hand and the *places* of the groups of constituents, to which $(P)_a$ and $(P)_b$ belong, in the list of polytopes of the general case $r>1$ on the other. To that end we indicate by $G_1, G_2, \ldots, G_n, G_{n+1}$ successively the kinds of polytopes represented by the first, the second, ... the last but one, the last line of the list of polytopes and — as on page 17 — by the symbols $g_0, g_1, g_2, \ldots, g_{n-1}$ in relation to any n-dimensional polytope limits $(l)_{n-1}$ of vertex import, edge import, face import, ..., the highest import of that polytope. Then we find:

THEOREM XVI. "If two polytopes of the net, $(P)_a$ of group G_k and $(P)_b$ of group $G_{k-\nu}$, are in $(l)_{n-1}$ contact, the common limit is a $g_{\nu-1}$ for $(P)_a$ and a $g_{n-\nu}$ for $(P)_b$".

The proof of this theorem lies in the remark that $(P)_b$, according to the subscript $(a_1+1, a_2+1, \ldots, a_\nu+1, a_{\nu+1}, a_{\nu+2}, \ldots, a_{n+1})$

of the symbol representing it, gives for Σa a value surpassing the corresponding sum for $(P)_a$ by ν, on account of the units added to the digits $a_1, a_2, \ldots a_\nu$. As Σa diminishes by a unit if we go down one line in the list, our $(P)_b$ belongs to group $G_{k-\nu}$ if $(P)_a$ belongs to group G_k. Now we know that the sum of the ν definite coordinates is maximum for $(P)_a$ and minimum for $(P)_b$ in the space bearing the common limit, which proves that this limit is a $g_{\nu-1}$ for $(P)_a$ and a $g_{n-\nu}$ for $(P)_b$.

By means of this theorem we can indicate the group to which belong the polytopes touching a given polytope along its limits of a given import; if $(P)_a$ belongs to group G_k and it has limits g_h, it is touched along these limits by polytopes $(P)_b$ belonging to group G_{n-h-1}.

The theorem also holds for the case $r = 1$, where the zero symbols of the successive groups $G_1, G_2, \ldots, G_p, \ldots, G_n$ are

$$(\overbrace{100\,..\,0}^{n}),\ (\overbrace{1100\,..\,0}^{n-1}),\ \ldots,\ (\overbrace{11\,..\,1}^{p}\overbrace{00\,..\,0}^{n-p+1}),\ \ldots,\ (\overbrace{11\,..\,10}^{n}).$$

There we can state it in this form: "Any polytope $(P)_a$ of a net for which $r = 1$ is touched along its limits of vertex import by polytopes of the preceding, along its limits of highest import by polytopes of the following group".

30. We now apply the theorem XV to the cases $n = 2, 3, 4, 5$ and put the results on record in the second table added at the end of this memoir.

First one word about the general plan of this table. Horizontally it is divided into four parts, corresponding successively to the cases $n = 2, 3, 4, 5$. Vertically it breaks up into seven columns with the first five of which we are concerned here. The first column, indicating the rank number of the net, enables us to individualize each net by a very short symbol, consisting of the value of n in italian figures, bearing at the right a roman rank index, 2_{III} indicating the net of hexagons in the plane. The second column gives the value of the period r from 1 to $n + 1$ upward. The third column contains the partition cycle, represented by that permutation in which the first digit is as small as possible. The fourth column brings the net symbol corresponding to that cyclical permutation of the partition cycle, whilst the fifth is concerned with the zero symbols of the different groups of constituents. With respect to these columns — the others will be explained in part G — we have to insert a few remarks.

In the cases 2_{III}, 3_{V}, 4_{VII}, 5_{XII}, where the partition cycle consists of $n+1$ units, we find back the self space fillers of simplex extraction, $p_6 = (210)$, $tO = (3210)$, etc. The net symbol of these self space fillers is characterized by the property that its $n+1$ digits, when divided by $n+1$, leave all possible remainders $n, n-1, \ldots, 1, 0$, each remainder once.

In the case of the partition 2,2 of the net 3_{III} an other particularity presents itself: in the process of formation of new zero symbols we fall back at the second step on the original symbol

$$(1100),\ (2110),\ (2211) = (1100).$$

This is due to the fact that the partition cycle consists of (two) equal parts. So this particularity repeats itself in the cases 5_{IV}, 5_{VII} and 5_{X} with the partition cycles $(3, 3)$, $(2, 2, 2)$ and $(1, 2, 1, 2)$, in general if we have $n+1 = uv$ and the partition cycle consists of the v digits $a_1, a_2, \ldots a_v$, this set of v digits being repeated in the same order of succession so as to have u sets. In the latter case where the partition cycle is said to be "a cycle of power v", we find only $\dfrac{n+1}{v} = u$ constituents of different form; it includes the self space fillers, which present themselves for $v = n+1$, $u = 1$ [1]).

We point out two other particularities occurring for the first time in S_5. The two partition cycles $1, 2, 3$ and $1, 3, 2$ of which the second written in the form $3, 2, 1$ is the inversion of the first, have been inscribed both as 5_{VI}, as these two nets, differing only in orientation with respect to the simplex of coordinates, are essentially the same. On the other hand the two nets 5_{IX} and 5_{X} are essentially different, though the four digits of the partition cycle are two times 2 and two times 1 for both.

The fifth column forms the principal part of the table. As to the number of different constituents of a net in S_n this column is subdivided into $n+1$ small ones. In the first of these $n+1$ small columns is placed the central polytope; on each horizontal line the polytope mentioned in a following small column is deduced by the two processes of art. 27 from that in the immediately preceding one. For brevity we have only inscribed the *geometrically* different forms, using from $n = 4$ upward the symbol e_o explained at the end of art. 21 and indicating the orientation by means of the signs.

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[1]) If we wish to indicate the number of constituents of different form and orientation we can complete theorem XIV by saying that this number is n for $r = 1$ and $\dfrac{n+1}{v}$ if the partition cycle is a cycle of power v.

We now come back to the particularity of the nets $5_{VI}{}^a$, $5_{VI}{}^b$ hinted at above. We see now at a glance that these two nets are one and the same, the polytope of the p^{th} small column of the one being equal but oppositely orientated to the polytope of the $7 - p^{th}$ small column of the other ($p = 1, 2, \ldots, 6$). So in each the six constituents present themselves in only one of the two possible orientations, which implies that none of them can be central symmetric, as in the range of the $n + 1$ different constituents of a net of S_n in the order of succession obtained by a regular application of the processes of art. 27 *adjacent* polytopes of a central symmetric one differ in orientation only. Or otherwise: two opposite limits $(l)_{n-1}$ of a central symmetric constituent are covered by two congruent but oppositely orientated polytopes, i.e. if we project on the line CC' joining the centre C of any polytope of the net to the centre C' of any limit $(l)_{n-1}$ of this polytope all the polytopes of the net the centres of which lie on that line, the projection of any central symmetric polytope with its centre on CC' acts as a "turn table" with respect to that projection.

31. The simple rule of theorem XV enables us to extend the list of nets to any value of n we like. So we would find for $n = 6$ and $n = 7$ respectively the 17 and the 29 cases represented as to their roman rank index, their partition cycle and the character of their constituents in the following small tables, where the three subdivisions of each last column give successively the number of central symmetric constituents, the number of the asymmetric constituents occurring in pairs and the number of asymmetric constituents occurring in one orientation only.

$n = 6.$

I	7		6	
II	16	1	6	
III	25	1	6	
IV	34	1	6	
V	115	1	6	
VI	124			7

VII	133	1	6	
VIII	223	1	6	
IX	1114	1	6	
X	1123			7
XI	1213	1	6	
XII	1222	1	6	

XIII	11113	1	6	
XIV	11122	1	6	
XV	11212	1	6	
XVI	111112	1	6	
XVII	1111111	1		

$n = 7.$

I	8	1	6		XI	1115		8		XXI	11213			8
II	17		8		XII	1124			8	XXII	11222	2	6	
III	26	2	6		XIII	1214	2	6		XXIII	12122	2	6	
IV	35		8		XIV	1133	2	6		XXIV	111113		8	
V	44	2	2		XV	1313		4		XXV	111122	2	6	
VI	116	2	6		XVI	1223		8		XXVI	111212		8	
VII	125			8	XVII	1232		8		XXVII	112112	2	2	
VIII	134			8	XVIII	2222	2			XXVIII	1111112	2	6	
IX	224	2	6		XIX	11114	2	6		XXIX	11111111	1		
X	233	2	6		XX	11123			8					

Under $n = 6$ no cases of a power partition cycle (except 6_{XVII}, the self space filler) present themselves, as $n+1$ is prime here. For $n = 7$ we find besides 7_{XXIX} still 7_V, 7_{XV}, 7_{XXVII} with $v = 2$ and 7_{XVIII} with $v = 4$.

Instead of pushing this general investigation any further we will give here the generalizations of the three nets of the plane to space S_n.

THEOREM XVII. "In space S_n the central symmetric polytope with the zero symbol $(n, n-1, n-2, \ldots, 1, 0)$, represented also by the expansion symbol $e_1 e_2 e_3 \ldots e_{n-2} e_{n-1} S(n+1)$, is the only self space filler of simplex extraction. This unique geometric constituent of the net presents itself in $n+1$ different groups with the property that the vertices of the constituents of each group form the vertices of the net, each vertex taken once, in other words: that no two constituents of the same group have a vertex in common. In this "cycle of constituents" (compare the footnote of art. 29) formed by these groups $G_0, G_1, \ldots, G_n$ any polytope of the group G_k is touched along its limits g_0 of vertex import by $(n+1)_1$ polytopes of group G_{k-1}, along its limits g_1 of edge import by $(n+1)_2$ polytopes of group G_{k-2}, etc. So, in order to perform the task of colouring the polytopes of this net in such a way that any two polytopes bearing the same colour are free from each other (a polydimensional bud of the renowned shrub "map colouring") it will be necessary and sufficient to have at hand $n+1$ different paints, one for the polytopes of each group".

"The n-dimensional angle of the self space filler of S_n is $\frac{2^n}{n+1}$ right ones".

What is said above about contact of constituents of different groups by a limit $(l)_{n-1}$ is a mere application [1]) of theorem XVI.

As the net of measure polytopes M_n of S_n shows, the n-dimensional space round any point contains 2^n right angles, if the n-dimensional angle of M_n is called a right one. As all the n-dimensional angles of the self space filler are equal and $n+1$ of these polytopes concur in a vertex of the net, the n-dimensional angle of the self space filler is $\frac{2^n}{n+1}$ right angles.

THEOREM XVIII. "The net of S_n with the period unity admits the n constituents $(\overline{11\,..\,1}^{\,p}\ \overline{00\,..\,0}^{\,n-p+1})$, $(p = 1, 2, \ldots, n)$ consisting of $\frac{n}{2}$ constituents in both orientations for n even and of $\frac{n-1}{2}$ constituents in both orientations and one central symmetric constituent for n odd".

THEOREM XIX. "The net of S_n with the partition cycle $1, n$ admits the $n+1$ constituents $(1\ \overline{00\,..\,0}^{\,n})$, $(\overline{22\,..\,2}^{\,p}\ 1\ \overline{00\,..\,0}^{\,n-p})$ for $p = 1, 2, \ldots, n-1$ and $(\overline{11\,..\,1}^{\,n}0)$ consisting of $\frac{n}{2}$ constituents in both orientations and one central symmetric constituent for n even and of $\frac{n+1}{2}$ constituents in both orientations for n odd".

These theorems immediately follow by specializing the general results. We give them here expressis verbis as we will indicate later on an other deduction of them. [2])

32. A survey of the results for $n = 2, 3, \ldots, 7$ suggests one or two general remarks.

The first can be stated in the form of:

THEOREM XX. "Every simplex polytope partakes in the formation of two nets. This is true without any reserve for the central symmetric constituents, it is also true for *each* of the two different positions of an asymmetric constituent."

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[1]) It is an easy task to demonstrate theorem XVII by itself by showing that the image points of the centre of the central polytope with respect to the spaces S_{n-1} bearing the limits $(l)_{n-1}$ as mirrors form the centres of the polytopes in $n-1$-dimensional contact with the central polytope. We consider this verification as a useful exercise, even in the special case $n = 3$ of ordinary space.

[2]) Though we do not wish to push the general investigation any further we still mention the following theorem:

"The net of S_{2^n-1} with the power partition cycle 2^n is built up of two central symmetic constituents only".

Let us take the polytope (221000) of S_5. This polytope can belong to a net the period r of which is either 2 or 3. In the first case we find ${}_2(221000)$ which can be reduced to ${}_2(100000)$ by going two steps backward; in the second case we have ${}_3(221000)$ which passes into ${}_3(211100)$ also by going two steps backward.

Or, let us go back to the constituent

$$(\overset{\rho_r}{\overline{r}}, \overset{\rho_{l-1}}{\overline{r-1}}, \ldots\ldots, \overset{\rho_o}{\overline{r-l}}, \overset{\rho_{r-1}}{\overline{r-l-1}}, \ldots., \overset{\rho_{l+1}}{\overline{1}}, \overset{\rho_l-\rho_r}{\overline{0}})$$

used in art. 29 in order to make the proof as general as possible. This constituent can belong to two nets, one with the period r, an other with the period $r+1$; the two partition cycles of these nets are

$$\left.\begin{array}{l} {}_r(\rho_{l-1}, \rho_{l-2}, \ldots, \rho_o, \rho_{r-1}, \ldots, \rho_{l+1}, \quad \rho_l \quad) \\ {}_{r+1}(\rho_r, \rho_{l-1}, \rho_{l-2}, \ldots, \rho_o, \rho_{r-1}, \ldots, \rho_{l+1}, \rho_l - \rho_r) \end{array}\right\}$$

and may be reduced to

$$\left.\begin{array}{l} {}_r(\rho_{r-1}, \rho_{r-2}, \ldots, \rho_{l+1}, \quad \rho_l \quad, \rho_{l-1} \ldots, \rho_1, \rho_0) \\ {}_{r+1}(\rho_r, \rho_{r-1}, \rho_{r-2}, \ldots, \rho_{l+1}, \rho_l - \rho_r, \rho_{l-1} \ldots, \rho_1, \rho_0) \end{array}\right\}.$$

In the case of an asymmetric constituent it may happen as we have seen that a definitely orientated one occurs in two different nets, if we consider as different two nets as 5_{VI}^a, 5_{VI}^b which are each others reversions. So under this point of view the two positions of ${}_3(221000)$ occur together in three different nets. But the statement of the theorem about *each* of the two positions of an asymmetric constituent holds under any point of view.

A second remark refers to the expansion symbols used in the table. In order to bring the two different orientations of the asymmetric constituents into evidence we have introduced the expansion symbols provided with the negative sign. But the law of succession of the different constituents of each net proceeding in the list from column to column would have been much more evident if we had stuck to expansion symbols without sign. Then the order of succession in the case of net 5_I would have been e_0, e_1, e_2, e_3, e_4 leading to the supposition that in general at each step the index of each e increases by unity, an illusion which is already destroyed by the series $e_o, e_oe_1, e_1e_2, e_2e_3, e_3e_4, e_4$. At any rate this second remark places us before the question by which rule the expansion symbols of the constituents of a net can be deduced from the partition cycle. The answer may be given in the form of:

THEOREM XXI. "The constituents of the net of S_n corresponding to the partition cycle ${}_r(\rho_{r-1}, \rho_{r-2}, \ldots, \rho_0)$ are found by applying to the system of digits consisting of the series

$$-1, \rho_{r-1}-1, \rho_{r-1}+\rho_{r+2}-1, \ldots\ldots, n-\rho_0$$

preceded by the repetition of all its terms after having subtracted $n+1$ from each (of which repetition the negative terms with an absolute value surpassing n may be omitted) a number of n times the process of increasing all the digits by unity and throwing out (as soon as they appear) digits surpassing $n-1$ and (afterwards when the process is finished) all the negative digits. Then the $n+1$ rows obtained represent the indices of the e operations to be applied to $(\overline{00\ldots0}^{\,n+1})$ in order to obtain the expansion symbols of the constituents".

Before proving this general rule we elucidate its meaning by applying it to an example, for which we choose the case $5_{VI}{}^{b}$. Here the series is $-1, 0, 3$ which has to be proceeded by -3. So, if we indicate in heavy type the figures which are to be kept, the operation is

$$\begin{array}{cccc}
-3 & -1 & \mathbf{0} & \mathbf{3} \\
-2 & \mathbf{0} & \mathbf{1} & \mathbf{4} \\
-1 & \mathbf{1} & \mathbf{2} & \\
\mathbf{0} & \mathbf{2} & \mathbf{3} & \\
\mathbf{1} & \mathbf{3} & \mathbf{4} & \\
\mathbf{2} & \mathbf{4} & &
\end{array}$$

giving $e_0\, e_3$, $e_0\, e_1\, e_4$, $e_1\, e_2$, $e_0\, e_2\, e_3$, $e_1\, e_3\, e_4$, $e_2\, e_4$ for the six expansion symbols of $5_{VI}{}^{b}$. As we have $e_1\, e_3\, e_4 = -e_0\, e_1\, e_3$ and $e_2\, e_4 = -e_0\, e_2$ this series is the same as that inscribed in the table.

The proof of this general theorem splits up into three parts. In the first we show that the top row corresponds to the constituent for which ${}_r(\overline{r-1}^{\,\rho_{r-1}}, \overline{r-2}^{\,\rho_{r-2}}, \ldots, \overline{1}^{\,\rho_1}, \overline{0}^{\,\rho_0})$ is the zero symbol. In the second we explain that the addition of a unit to all the digits corresponds to what happens to the digits in the processes of art. 27 but for the transplantation of the digit at the end to the beginning. In the third we will be concerned with the influence of that transplantation.

The first and the second parts are mere consequences of theorem IX. In the case of the zero symbol ${}_r(\overline{r-1}^{\,\rho_{r-1}}, \overline{r-2}^{\,\rho_{r-2}}, \ldots, \overline{1}^{\,\rho_1}, \overline{0}^{\,\rho_0})$ the unit intervals present themselves behind the digits of rank

$$\rho_{r-1}, \rho_{r-1}+\rho_{r-2}, \ldots\ldots, n+1-\rho_0$$

and this proves in connection with theorem IX the first part. Moreover the circular permutation over one digit to the right hap-

pening at each step of the two processes of art. 27 (see the example following theorem XIII) changes the ranks $k+1$ and $k+2$ of two adjacent digits into $k+2$ and $k+3$, i. e. — according to theorem IX — the operation e_{k+1} is still to be applied or has already been performed on the new constituent according to the operation e_k being still to be applied or having already been performed on the original constituent; i. e. if e_k occurs in the e-symbol of the original constituent, e_{k+1} must occur in the e-symbol of the new one, what proves the second part.

In the third part we have to consider all the possible cases of the transplantation of a digit from the end to the beginning; these cases, four in number, are the following:

$(r-1, .., 1, 0)$ becomes $(r-1, r-2, .., 0)$ .. loss of e_{n-1}, gain of e_0,
$(r-1, .., 0, 0)$ „ $(r, r-1, \ldots.., 0)$ .. gain of e_0,
$(r, \ldots\ldots, 1, 0)$ „ $(r-1, r-2, .., 0)$ .. loss of e_{n-1},
$(r, \ldots\ldots, 0, 0)$ „ $(r, r, \ldots\ldots\ldots, 0)$ .. neither loss nor gain.

So we find the two rules:

1°. If e_{n-1} appears in the symbol of the original constituent it falls out in the next one, though an other e_{n-1} may be introduced (if e_{n-2} was contained also in the original symbol).

2°. If the number of the operation factors e_k is $r-1$ the symbol e_0 appears in the next constituent.

But this is also the effect of the operation indicated in the theorem, the first rule being a consequence of the omission of the digits surpassing $n-1$, the second being deducable from the repetition of the series $-1, \rho_{r-1}-1, \rho_{r-1}+\rho_{r-2}-1$, etc. If, in order to add still one word about the second rule, G_{p-1}, G_p, G_{p+1} indicate three constituents, consecutive in the sense of the theorem, and the e-symbol of G_p bears only $r-1$ expansion factors, then the e-symbol of G_{p-1} contains $n-1$ and therefore also $-(n+1)+(n-1)=-2$, before the negative digits have been omitted; this -2 becomes -1 for G_p and 0 for G_{p+1}.

33. The theorem XXI enables us to show how the "principal" net of S_n, i. e. the net with the period $r=1$ always inscribed first, can be transformed successively into all the other ones.

The result for S_3 is given in the following table, in the left half in the symbols to be applied to the different constituents of $N(T, O)$, in the right half by the results of this application.

	Constituents			Vertex	Constituents			Vertex
I	e_0	e_1	e_2	gap	T	O	$-T$	gap
II	—	0	1	e_2	T	tT	$-tT$	$-T$
	1	2	—	e_1	tT	$-tT$	$-T$	T
III	2	—	0	e_1	CO	O	CO	O
IV	1	02	1	$e_0\, e_2$	tT	tO	$-tT$	CO
	2	0	01	$e_1\, e_2$	CO	tT	tO	$-tT$
	12	2	0	$e_0\, e_1$	tO	$-tT$	CO	tT
V	12	02	01	—	tO	tO	tO	—

According to this table the principal net $N(T, O)$ can be transformed into the net 3_{II} either by applying to O and $-T$ the operations e_0 and e_1 or by applying to T and O the operations e_1 and e_2; these two transformations are of the same kind; as they pass into each other by interchanging the two sets of tetrahedra and at the same time the two sets of four non adjacent faces of each octahedron in contact with them. Whilst each of the two nets 3_{III} and 3_V can be deduced in one way only, there are three manners of deduction of net 3_{IV}; of these the first stands by itself and the second and the third pass into each other by the indicated interchange of the two sets of tetrahedra, etc.

The table for S_4 is the following

	Constituents				Vertex	Constituents				Vertex
I	e_0	e_1	e_2	e_3	gap	e_0	e_1	e_2	e_3	gap
II	—	0	1	2	e_3	e_0	$e_0\, e_1$	$e_1\, e_2$	$-e_0\, e_1$	$-e_0$
	1	2	3	—	e_0	$e_0\, e_1$	$e_1\, e_2$	$-e_0\, e_1$	$-e_0$	e_0
III	2	3	—	0	e_1	$e_0\, e_2$	$-e_0\, e_2$	$-e_1$	$e_0\, e_3$	e_1
	3	—	0	1	e_2	$e_0\, e_3$	e_1	$e_0\, e_2$	$-e_0\, e_2$	$-e_1$
IV	1	02	13	2	$e_0\, e_3$	$e_0\, e_1$	$e_0\, e_1\, e_2$	$-e_0 e_1 e_2$	$-e_0\, e_1$	$e_0\, e_3$
	3	0	01	12	$e_2\, e_3$	$e_0\, e_3$	$e_0\, e_1$	$e_0\, e_1\, e_2$	$-e_0 e_1 e_2$	$-e_0\, e_1$
	12	23	3	0	$e_0\, e_1$	$e_0\, e_1\, e_2$	$-e_0 e_1 e_2$	$-e_0\, e_1$	$e_0\, e_3$	$e_0\, e_1$
V	2	03	1	02	$e_1\, e_3$	$e_0\, e_2$	$e_0\, e_1\, e_3$	$e_1\, e_2$	$-e_0 e_1 e_3$	$-e_0\, e_2$
	13	2	03	1	$e_0\, e_2$	$e_0\, e_1\, e_3$	$e_1\, e_2$	$-e_0 e_1 e_3$	$-e_0\, e_2$	$e_0\, e_2$
	23	3	0	01	$e_1\, e_2$	$-e_0 e_1 e_3$	$-e_0\, e_1$	$e_0\, e_2$	$e_0\, e_1\, e_3$	$e_1\, e_2$
VI	12	023	13	02	$e_0\, e_1\, e_3$	$e_0\, e_1\, e_2$	$e_0\, e_1 e_2 e_3$	$-e_0 e_1 e_2$	$-e_0 e_1 e_3$	$e_0 e_1 e_3$
	13	02	013	12	$e_0\, e_2\, e_3$	$e_0\, e_1\, e_3$	$e_0\, e_1\, e_2$	$e_0\, e_1\, e_2\, e_3$	$-e_0 e_1 e_2$	$-e_0 e_1 e_3$
	23	03	01	012	$e_1\, e_2\, e_3$	$-e_0 e_1 e_3$	$e_0\, e_1\, e_3$	$e_0\, e_1\, e_2$	$e_0\, e_1\, e_2\, e_3$	$-e_0 e_1 e_2$
	123	23	03	01	$e_0\, e_1\, e_2$	$e_0 e_1 e_2 e_3$	$-e_0 e_1 e_2$	$-e_0 e_1 e_3$	$e_0\, e_1\, e_3$	$e_0 e_1 e_2$
VII	123	023	013	012	—	$e_0 e_1 e_2 e_3$	$e_0\, e_1\, e_2\, e_3$	$e_0\, e_1\, e_2\, e_3$	$e_0\, e_1\, e_2\, e_3$	—

In the following table for S_5 we give only the indices of the e-symbols which are to be applied to the constituent of the principal net in order to obtain all the other ones.

I	Constituents 0	1	2	3	4	Vertex gap
II	—	0	1	2	3	4
	1	2	3	4	—	0
III	2	3	4	—	0	1
	4	—	0	1	2	3
IV	3	4	—	0	1	2
V	1	02	13	24	3	04
	4	0	01	12	23	34
	12	23	34	4	0	01
VI[a]	2	03	14	2	03	14
	13	24	3	04	1	02
	34	4	0	01	12	23
VI[b]	3	04	1	02	13	24
	14	2	03	14	2	03
	23	34	4	0	01	12
VII	24	3	04	1	02	13

I	Constituents 0	1	2	3	4	Vertex gap
VIII	12	023	134	24	03	014
	14	02	013	124	23	034
	34	04	01	012	123	234
	123	234	34	04	01	012
IX	13	024	13	024	13	024
	24	03	014	12	023	134
	124	23	034	14	02	013
	234	34	04	01	012	123
X	23	34	14	02	013	124
	134	24	03	014	12	023
XI	123	0234	134	024	013	0124
	124	023	0134	124	023	0134
	134	024	013	0124	123	0234
	234	034	014	012	0123	1234
	1234	234	034	014	012	0123
XII	1234	0234	0134	0124	0123	

We only remark here that the number of ways in which the principal net can be transformed into any other one is equal to the number of different cyclical permutations of the partition symbol of the latter, if we make allowance for the fact that two of these ways may be essentially the same as they pass into each other by interchanging the different positions of the constituents without central symmetry, etc.

34. In the outset of this paragraph (art. 22) we have excluded prismatic nets, restricting ourselves to uniform ones; moreover we have disregarded 1°. all cases in which not all the constituents are of simplex extraction (the hybridous nets of art. 24, *b*) and 2°. the nets with two systems of vertices (art. 25, *c*). Now that our general considerations about simplex nets are come to a close we wish to add a few words about these two exceptional groups of nets.

Hybridous nets. In order not to become too circumstantial we only *mention* the decomposing symbols of the three plane hybridous nets indicated in art. 24. They are (in the notation of art. 24):

$$N(p_3; p_4; p_6) \ldots \begin{cases} (a_1(1+\sqrt{3})+\mathbf{1},\ a_2(1+\sqrt{3})+\mathbf{0},\ a_3(1+\sqrt{3})+\mathbf{0}),\ \Sigma a_i=0, \\ (a_1(1+\sqrt{3})+\tfrac{1}{3}\sqrt{3}+\mathbf{0},\ a_2(1+\sqrt{3})+\tfrac{1}{3}\sqrt{3}+\mathbf{1}, \\ \qquad a_3(1+\sqrt{3})+\tfrac{1}{3}\sqrt{3}+\mathbf{1}),\ \Sigma a_i=-1. \end{cases}$$

$$N(p_6; p_4; p_3) \ldots \quad (a_1(1+\tfrac{1}{3}\sqrt{3})+\mathbf{2},\ a_2(1+\tfrac{1}{3}\sqrt{3})+\mathbf{1},$$
$$a_3(1+\tfrac{1}{3}\sqrt{3})+\mathbf{0}),\ \Sigma a_i=0.$$

$$N(p_6; p_4; p_{12}) \ldots \quad (a_1(1+\sqrt{3})+\mathbf{2},\ a_2(1+\sqrt{3})+\mathbf{1},\ a_3(1+\sqrt{3})+\mathbf{0}),\ \Sigma a_i=0,$$

the a_i different from each other with respect to mod. 3.

$$N(p_3; -; p_{12}) \ldots \begin{cases} (a_1(2+\sqrt{3})+\mathbf{1},\ a_2(2+\sqrt{3})+\mathbf{0},\ a_3(2+\sqrt{3})+\mathbf{0}),\ \Sigma a_i=0, \\ ((a_1+\tfrac{1}{3}\sqrt{3})(2+\sqrt{3})+\mathbf{0},\ (a_2+\tfrac{1}{3}\sqrt{3})(2+\sqrt{3})+\mathbf{1}, \\ \qquad (a_3+\tfrac{1}{3}\sqrt{3})(2+\sqrt{3})+\mathbf{1}),\ \Sigma a_i=-2. \end{cases}$$

In space we find two hybridous nets. If $N(A; B)$ represents a net with the polyhedric constituents A, B, the first being of body, the second of vertex import, these two nets and their generation are indicated by the equations

$$e_2 N(T, \underline{O}) = N(T, RCO; C),\ e_1 e_2 N(T, \underline{O}) = N(tT, tCO; tC),$$

the stroke under O referring to this that the expansions are to be applied to O. Here we even abstain from mentioning decomposing symbols.

Which prospect opens hyperspace for the hunting up of hybridous simplex nets? Very probably none at all. For the most powerful instrument in the plane, the operation e_n, is quite ineffective in ordinary space already, whilst the two hybridous nets of that space are due to the special character of the octahedron as simplex polyhedron.

Nets with two kind of vertices. Neither is it probable that hyperspace contains nets with a constituent occurring in such a manner in two different orientations that any vertex of the net only belongs to one polytope of one of the two sets; for in S_2 and in S_3 the only nets admitting this particularity are precisely hybridous nets, the net $N(p_3, p_{12})$ with respect to p_3, the net $e_2 N(T, \underline{O})$ with respect to T and the net $e_1 e_2 N(T, \underline{O})$ with respect to tT.

35. We finish this paragraph by mentioning other generations of the nets n_I and n_{II} of the theorems XVIII and XIX.

"If we start from a simplex $S(n+1)^{(1)}$ of S_n and complete the $n+1$ spaces S_{n-1} bearing the $n-1$-dimensional limits $S(n)^{(1)}$ to $n+1$ systems of equidistant parallel spaces S_{n-1}, the distance between

any two adjacent parallel spaces S_{n-1} being either the height of $S(n+1)^{(1)}$ or twice that height, we get either the net n_I or the net n_{II}."

"If we intersect a net of measure polytopes M_{n+1} of space S_{n+1} by a space S_n normal to a diagonal of a measure polytope and we make that S_n to pass either through a vertex or through the centre of an edge of that polytope we generate either a net n_I or a net n_{II}. In order to obtain nets n_I and n_{II} with length of edge unity we must start in the first case from a net $N(M_{n+1}^{\frac{1}{2}\sqrt{2}})$, in the second case from a net $N(M_{n+1}^{\sqrt{2}})$."

The first generation is easily proved, if we consider the cases of the triangle net $N(p_3)$ and the triangle and hexagon net $N(p_3, p_6)$ of the plane and the cases of the net $N(T, O)$ and the net $N(T, tT)$ of threedimensional space first.

But the second generation, used already in two different papers, [1]) has this great advantage that it furnishes at the same time an easy method of deducing the character of the different constituents. We only trace this method here, as the different constituents have been found otherwise already.

The generation itself shows that all the constituents are sections of the measure polytope M_{n+1} by a space S_n normal to a diagonal. In the first of the two papers quoted just now is demonstrated that "the section of M_{n+1} by a space S_n normal to a diagonal can always be regarded as a part of that space S_n enclosed by two definite, concentric, oppositely orientated, regular simplexes $S(n+1)$ of that space", i. e. that this section is a "regularly truncated regular simplex". Moreover the second of the two papers indicates how to find the amount of these truncations, whilst finally the theorem V, or rather its inversion, teaches how to deduce the zero symbol from the truncation numbers.

F. *Polarity.*

36. If we polarize one of the regular or one of the Archimedian semiregular polyhedra with respect to any concentric sphere, i. e. if we replace that polyhedron characterized by its vertices by the polyhedron included by the polar planes of these vertices with respect to that sphere, we pass from a body with one kind of vertex and edges of the same length to a body with one kind of face and equal dihedral angles. We suppose the simple laws of this "inversion" to be known; so we state only that the lines bearing the edges of the

[1]) *Proceedings* of the Academy of Amsterdam, vol. X, pp. 485 and 688.

new body are the reciprocal polars of the lines bearing the edges of the original one, that the vertices of the new body are the poles of the planes bearing the faces of the original one, etc.

This definition of reciprocal polyhedra can be extended immediately to space S_n, where we have to use a concentric spherical space (with ∞^{n-1} points) as polarisator. If this concentric spherical space is the circumscribed one, the limiting spaces S_{n-1} of the new polytope pass through the corresponding vertices of the original one and are normal in these points to the lines joining these points to the centre. We use this most simple disposition in order to show that the property of having one length of edge is transformed into that of the equality of the dispatial angles. To that end we consider (fig. 11) the plane determined by any edge AB and the centre O of the original polytope and remark that the polar spaces S_{n-1} of A and B project themselves onto that plane in the lines a and b, in A and B normal to OA and OB respectively; so the space of intersection S_{n-2} of these two spaces S_{n-1} projects itself in the point C common to a and b and the angle ACB is the dispatial angle between the two spaces S_{n-1}; but this angle is the supplement of the angle AOB which is constant, $OA = OB$ and AB being constant.

By applying this inversion to any semiregular polytope of simplex extraction the characteristic number symbol of it is inverted too. So the symbol (15, 60, 80, 45, 12) of $ce_1\ S(6)$ — see the table — passes into (12, 45, 80, 60, 15).[1]

If, in inverting a definite polytope of simplex descent in S_n, we assume as polarisator the imaginary spherical space for which the vertices of the simplex from which the polytope was derived admit as polar spaces S_{n-1} the opposite limiting spaces S_{n-1} of that simplex, and $(a_1, a_2 \ldots . a_{n+1})$ is the coordinate symbol of the

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[1]) It is a very good exercise to deduce the limiting bodies of the reciprocal polytopes of S_4 by polarizing the properties of the edges passing through the vertices of the original simplex polytopes. So, if $Le_1\, e_2\, e_3$ stands for "the limiting bodies of the reciprocal polytope of $e_1\, e_2\, e_3\, S$ (5)", if $T\,(1_3,\ 1_{2+1},\ 2_{1+1+1})$ indicates a tetrahedron of which one vertex bears three equal edges, one two equal and one unequal edges, two three different edges, if $P^1{}_{\text{deltoid}}$ means pyramid on a deltoid base, $P^2{}_{2+1}$ double pyramid on an isosceles triangle as base, Rh rhombohedron, the results to be obtained are represented by the equations

$Le_1 = 20\ T\,(1_2,\ 3_{2+1})$,	$Le_1\, e_3 = L\,(-e_2\, e_3) = 60\ P^1{}_{\text{deltoid}}$,
$Le_2 = 30\ P^2{}_{2+1}$,	$Le_1\, e_2\, e_3 = 120\ T\,(2_{2+1},\ 2_{2+1})$,
$Le_3 = 20\ Rh$,	$Lce_1 = 10\ P^2{}_3$,
$Le_1\, e_2 = 60\ T\,(1_3,\ 1_{2+1},\ 2_{1+1+1})$,	$Lce_1\, e_2 = 30\ T\,(4_{2+1})$.

polytope in true value coordinates, this symbol also represents all the limiting spaces S_{n-1} of the new polytope in space coordinates, i. e. that these spaces S_{n-1} are represented by the equations $b_1 x_1 + b_2 n_2 + \ldots + b_{n+1} x_{n+1} = 0$, where $b_0, b_1, \ldots, b_{n+1}$ stands for any permutation of the $n+1$ digits a_i. [1])

Finally it is easy to see in what manner the process of truncation is transformed by inversion. As we have no intention of studying the new system of semiregular polytopes for itself, it may suffice here to remark that truncation at a limit $(l)_p$, which implies the determination of the intersection of a definite space S_{n-1} with the limits $(l)_{p+1}$ passing through that $(l)_p$, is transformed into the assumption of a point in the line joining the centre of a limit $(l)_{n-p-1}$ of the new polytope to the centre O of that polytope, which implies that this point is joined to all the limits $(l)_{n-p-2}$ of that $(l)_{n-p-1}$ by new limits $(l)_{n-p-1}$ replacing the chosen one, etc.

37. We now prove the theorem:

Theorem XXII. "Any polytope $(P)_n$ of simplex descent in S_n has the property that the vertices V_i adjacent to any arbitrary vertex V lie in the same space S_{n-1} normal to the line joining that vertex V to the centre O of the polytope. The system of the spaces S_{n-1} corresponding in this way to the different vertices V of $(P)_n$ include an other polytope $(P)'_n$, the reciprocal polar of $(P)_n$ with respect to a certain spherical space with O as centre".

In order to prove the theorem we consider the polytope with the zero symbol $(a_1, a_2, \ldots a_{n+1})$ and in connection with it the linear expression

$$a_1 x_1 + a_2 x_2 + \ldots + a_{n+1} x_{n+1}.$$

This expression assumes the value

$$a^2_1 + a^2_2 + \ldots + a^2_{n+1}$$

for the pattern vertex V and the same value diminished by unity for each of the points V_i adjacent to V. For we pass from the pattern vertex V to any vertex V_i adjacent to it by making two digits p and $p-1$ interchange places and by this process the sum $p^2 + (p-1)^2$ contained in $\sum\limits_1^{n+1} a^2_i$ is replaced by $p(p-1) + (p-1)p = 2p^2 - 2p$. So the coordinates of the points V_i adjacent to the

[1]) Compare "*Nieuw Archief voor Wiskunde*", vol IX, p. 138—141.

pattern vertex satisfy the equation $\sum_1^{n+1} a_i x_i = \sum_1^{n+1} a^2_i - 1$; as this equation represents a space S_{n-1} normal to the line from the centre of the coordinate simplex to the pattern vertex [1]), the first part of the theorem is proved.

From the regularity of the considered polytope it can be deduced that the distance OP' from the centre O to the space S_{n-1}c ontaining the vertices V_i adjacent to any vertex V does not change with that vertex. So all the vertices of the considered polytope are transformed into the spaces S_{n+1} containing their adjacent vertices by means of an inversion with respect to the spherical space with O as centre and $\sqrt{OP.\,OP'}$ as radius.

38. If we use the symbol $S_0(n+1)^{(1)}$ introduced in art. 21 we have:

Theorem XXIII. "The two polytopes

$$e_a e_b e_c \ldots e_r e_s e_t S_0 (n+1)^{(1)},\quad e_{a'} e_{b'} e_{c'} \ldots e_{r'} e_{s'} e_{t'} S_0 (n+1)^{(1)}$$

are equal and concentric, but of opposite orientation, if and only if we have generally

$$a+t'=b+s'=c+r'= \ldots . =r+c'=s+b'=t+a'=n-1"$$

"For $a=a'$, $b=b'$, $c=c'$, $\ldots$, $r=r'$, $s=s'$, $t=t'$ the polytope in which the two given ones coincide is central symmetric, if and only if we have

$$a+t=b+s=c+r= \ldots =n-1$$

under which conditions there may be an unpaired middle expansion $e_{\frac{n-1}{2}}$ for n odd".

This theorem gives in analytical form the results published in a joint paper of M[rs] Stott and myself [2]), already quoted on page 17, as far as the simplex offspring is concerned; for the supposition that the reciprocal polytopes A and A' mentioned in art. 3 of that paper are $e_0 S_0(n+1)^{(1)}$ and $e_{n-1} S_0(n+1)^{(1)}$, i. e two concentric and equal simplexes $S(n+1)^{(1)}$ of opposite orientation, specializes the general results found there to the simplex theorem stated just now here. To prove the latter analytically we have only to write out the result of the operations $e_a e_b e_c \ldots e_r e_s e_t$ and $e_{a'} e_{b'} e_{c'} \ldots e_{r'} e_{s'} e_{t'}$

[1]) Compare "*Nieuw Archief voor Wiskunde*", vol. IX, p. 140, remark I.

[2]) Reciprocity in connection with semiregular polytopes and nets, "*Proceedings* of the Academy of Amsterdam", September, 1910.

on $S_0(n+1)^{(1)}$ and to investigate under what circumstances the zero symbol of the one is the inversion of that of the other. If each of the two products

$$e_a\, e_b\, e_c \ldots e_r\, e_s\, e_t \quad , \quad e_{a'}\, e_{b'}\, e_{c'} \ldots e_{r'}\, e_{s'}\, e_{t'} \ ,$$

where we have

$$a<b<c<\ldots<r<s<t \quad , \quad a'<b'<c'<\ldots<r'<s'<t'$$

bears k factors, the two results are represented by

$$(\overbrace{k,k,..k}^{a+1},\overbrace{k-1,k-1,..k-1}^{b-a},\overbrace{k-2,k-2,..k-2}^{c-b},\ldots\overbrace{22..2}^{s-r},\overbrace{11..1}^{t-s},\overbrace{00..0}^{n-t})$$

and the same expression in which the $a, b, c, \ldots r, s, t$ are dashed. So the conditions are

$$a+1=n-t',\ b-a=t'-s',\ c-b=s'-r',\ \ldots$$
$$\ldots,\ s-r=c'-b',\ t-s=b'-a,\ n-t=a'+1,$$

giving immediately

$$a+t'=b+s'=c+r'=\ldots \quad =r+c'=s+b'=t+a'=n-1.$$

So the first part is proved and the second is deduced from this by suppression of the dashes. In this second part the unpaired middle expansion $e_{\frac{n-1}{2}}$ occurs, if and only if both n and k are odd.

It is an easy task to return to the e and c symbols referring to the simplex $(1\,\overbrace{00\,..\,0}^{n})$; to that end we have to omit the e_0 symbol and to add c to any expansion form, where e_0 is lacking.

In doing so we arrive for $n=3, 4, 5$ by means of the first part of the theorem to all the cases, as $e_2\, e_3\, S(5) = -\, e_1\, e_3\, S(5)$, of equal and concentric polytopes of opposite orientation mentioned in the table, and by means of the second part to all the cases, as $ce_2\ S(6)$, of central symmetry.

39. In the joint paper of M[rs] Stott and myself quoted in the preceding article, the notion of reciprocal polytopes has been extended to that of reciprocal nets by considering a net of S_n as a polytope with an infinite number of limits $(l)_n$ in S_{n+1}. In this case the centre of the circumscribed spherical space of the polytope lies at infinity in the direction of the normal to the space S_n bearing the space filling, from which it ensues that the poles of the limits $(l)_n$ coincide with the centres of these polytopes. So one obtains a net reciprocal to a given one by considering the centres

of the polytopes of the given net as vertices (see art. 51 of ANDREINI's memoir, quoted in art. 22).

Under what circumstances polarization of a simplex net leads to an other simplex net? The answer to this question is: "this only happens in the plane with the nets $N(p_3)$, $N(p_6)$, $N(p_3, p_6)$ and the for two reasons discarded net $N(p_3, p_{12})$". For, if otherwise the net contains two or more different constituents the reciprocal net will contain two or more differents kinds of vertices, and if the net is formed by one constituent only and this self space filler is partially regular the vertices of the new net will be partially regular. So the only possible case of *two reciprocal simplex nets* is that of the pair $N(p_3)$ and $N(p_6)$ in the plane, the centres of the two sets of triangles of $N(p_3)$ being the vertices of an $N(p_6)$, the centres of the hexagons of $N(p_6)$ being the vertices of an $N(p_3)$.

In the treatise "Sulle reti, ecc." quoted once more above M[r]. ANDREINI has indicated how to draw up a complete list of all the reciprocal nets of threedimensional space; in this research he comes to the remarkable result (art. 59) that the rhombic dodecahedron and some other less regular polyhedra into which this semiregular polyhedron of the second kind can be decomposed form the constituents of the different reciprocal nets. If we restrict ourselves to the cases concerned with nets of simplex extraction this result is that the constituent of the reciprocal net of

$N(T, O)$	is the rhombic dodecahedron RD,
$N(T, tT)$	,, the rhombohedron ($\frac{1}{4}$ RD),
$N(O, CO)$	,, a double pyramid on a square ($\frac{1}{6}$ RD),
$N(tT, tO, CO)$	,, a pyramid on a lozenge ($\frac{1}{12}$ RD),
$N(tO)$	,, a tetrahedron limited by four equal isosceles triangles ($\frac{1}{24}$ RD).

What corresponds to this remarkable result in space S_n? It goes without saying that this question deserves an answer. But that answer can only be fragmentary, unless we surpass the limits between which we wish to confine ourselves in this paper. So all we can do now is to express the hope that we may be able to give a complete answer to that question in a new paper of its own. Only we cannot retain the remark that the constituent of the reciprocal net of the net corresponding to the undivided partition $n+1$ of $n+1$ and that of the reciprocal net of the net corresponding to the partition of $n+1$ consisting of units only are very interesting polytopes, worthy of study for their own sake.

G. *Symmetry, considerations of the theory of groups, regularity.*

40. We begin by determining the spaces S_{n-1} of symmetry which may be indicated by Sy_{n-1} and we consider to that end successively the case of the simplex $S(n+1)$ of S_n and that of any polytope $(P)_s$ deduced from that simplex $S(n+1)$ by the operations of expansion and contraction.

Case of the simplex. The vertices of $S(n+1)$ lying outside a space of symmetry Sy_{n-1} of this $S(n+1)$ occur in couples. Now there must be at least one of these couples, as Sy_{n-1} cannot contain all the vertices of $S(n+1)$, and on the other hand there cannot be more than one of these couples, as $S(n+1)$ does not admit parallel edges. So any space Sy_{n-1} must bisect orthogonally one edge of $S(n+1)$, i. e. the number of spaces Sy_{n-1} is $\frac{1}{2}n(n+1)$.

It is not at all difficult to indicate the equations of the $\frac{1}{2}n(n+1)$ spaces Sy_{n-1} of $(\overbrace{1\,00\,..0}^{n})$. For the space S_{n-1} bisecting normally the edge $A_k A_l$ joining the points A_k and A_l with the coordinates $(x_k = 1,\ x_{not\ k} = 0)$ and $(x_l = 1,\ x_{not\ l} = 0)$ is represented by the equation $x_k = x_l$.

Case of the polytope $(P)_s$ deduced from the simplex. It goes without saying that the $\frac{1}{2}n(n+1)$ spaces of symmetry $x_k = x_l$ of $S(n+1)$ are at the same time spaces Sy_{n-1} for any polytope $(P)_s$ derived from that $S(n+1)$ by the operations e and c, and that any two limits of that $(P)_s$ which are each others mirror images with respect to any of these Sy_{n-1} are of the same import. So the only question is, if the polytope $(P)_s$ can possess a space of symmetry which is no Sy_{n-1} for the $S(n+1)$ from which the $(P)_s$ has been derived. To answer this question we suppose there *is* such a space Sy_{n-1} and we examine the consequences to which this supposition leads. According to this supposition $(P)_s$ is its own mirror image with respect to that definite Sy_{n-1}, which may be represented by the symbol $\overline{Sy}_{n-1}$, whilst the mirror image of the simplex $S(n+1)$ from which $(P)_s$ has been deduced is an other simplex $S'(n+1)$ concentric to $S(n+1)$. But then the figure consisting of $(P)_s$ on one hand and the two simplexes $S(n+1)$, $S'(n+1)$ on the other is symmetric with respect to $\overline{Sy}_{n-1}$; so it must be possible to deduce $(P)_s$ by the same set of expansion operations from the new simplex $S'(n+1)$. From this we can draw two conclusions, one with respect to the two simplexes, an other with respect to $(P)_s$. If we can deduce the same polytope $(P)_s$ from two different simplexes, these simplexes must be concentric and *oppositely orientated*; if we

can do so by means of the *same* set of expansions, $(P)_s$ must be *central symmetric*. So we have to solve first the new question if the simplex $S(n+1)$ admits a space $\overline{S}_{n-1}$ reflecting it into a concentric simplex $S'(n+1)$ oppositely orientated to $S(n+1)$. We once more suppose that there *is* such a space $\overline{S}_{n-1}$ and we examine the consequences of this supposition. Let $A_1 A_2 \ldots A_n$ be the given simplex and $A_1' A_2' \ldots A'_{n+1}$ the concentric simplex of opposite orientation, the common centre O being at the same time centre of the $n+1$ segments $A_i A_i'$, $(i = 1, 2, \ldots, n+1)$ Then the image of A_1 wit respect to $\overline{S}_{n-1}$ must be either A'_1 or one of the other vertices of $S'(n+1)$, say A'_2. We consider these two different cases each for itself.

If A'_1 is the mirror image of A_1, the space $\overline{S}_{n-1}$ is normal to the line $A_1 O$ joining in $S(n+1)$, if produced, the vertex A_1 with the centre M_1 of the opposite limit $S(n)$, which line $A_1 M_1$ may be called a "first transversal" of $S(n+1)$; this $S(n)$ with the vertices $A_2 A_3 \ldots A_{n+1}$ is contained in a space S_{n-1} parallel to $\overline{S}_{n-1}$, whilst the mirror image of it is the limit $S'(n)$ of $S'(n+1)$ opposite to A'_1, with the vertices $A'_2, A'_3, \ldots, A'_{n+1}$. So, as a whole, these $S(n)$ and $S'(n)$ have to be at the same time equipollent and oppositely orientated to each other, equipollent as reflections of figures lying in spaces S_{n-1} and S'_{n-1} parallel to the mirror $\overline{S}_{n-1}$, oppositely orientated as corresponding parts of the oppositely orientated simplexes $S(n+1)$ and $S'(n+1)$. This is impossible for $n > 2$, e. g. two triangles lying in parallel planes cannot be equipollent and oppositely orientated at the same time. Now the case $n = 1$, meaningless in itself, leads to two conciding simplexes, i. e. to a point of symmetry of the simplex of the linear domain, the line segment. So the case $n = 2$ of the triangle $A_1 A_2 A_3$ with the lines through O parallel to the sides is the only remaining one.

If A'_2 (fig. 12) is the mirror image of A_1, we consider the triangle $A_1 A_2 A'_2$ with a right angle in A_1, as the centre O of $A_2 A'_2$ is at equal distance from the three vertices; if M_{12} is the centre of $A_1 A_2$, the line $M_{12} O$, parallel to $A_1 A'_2$, passes, if produced, through the centre of the limit $S(n-1)$ of $S(n+1)$ containing $A_3, A_4, \ldots, A_{n+1}$ as vertices and may therefore be called a "second transversal" of $S(n+1)$. Now the mirror $\overline{S}_{n-1}$ bisects $A_1 A_2$ orthogonally and is therefore normal to the second transversal $M_{12} O$, while the limits $S(n-1)$ with the vertices $A_3 A_4, \ldots, A_{n+1}$ and $S'(n-1)$ with the vertices $A'_3 A'_4 \ldots A'_{n+1}$ lie in parallel spaces S_{n-2} and S'_{n-2}. Here too these limits have to be at the same time equipollent and oppositely orientated to each other, which is impossible for $n-1 > 2$. So we find here

two cases, the case $n=2$ found above and the case $n=3$ of the tetrahedron $A_1 A_2 A_3 A_4$ with the planes through O parallel to a pair of opposite edges. So we have proved the general theorem:

THEOREM XXIV. "The simplex $(1\,\overbrace{00\,..\,0}^{n})$ of S_n and the polytopes deduced from it by expansion and contraction admit $\frac{1}{2}n(n+1)$ spaces Sy_{n-1} of symmetry, the spaces $x_i = x_k$. Moreover in the plane the $e_1(p_3)$ admits the three new axes of symmetry $x_1 = y$ of the hexagon, whilst in space the $ce_1 T = O$, $e_2 T = CO$, $e_1 e_2 T = tO$ admit the new planes of symmetry $x_i + x_j = x_k + x_l$ of the octahedron".

41. We now prove the following theorem [1]:

THEOREM XXV. "The order of the group of anallagmatic displacements of the simplex $S(n+1)$ of S_n and of the polytopes deduced from it by expansion and contraction is $\frac{1}{2}(n+1)!$"

"The order of the extended group of anallagmatic displacements of these polytopes, reflexions with respect to spaces Sy_{n-1} of symmetry included, is $(n+1)!$ In this extended group the first group of order $\frac{1}{2}(n+1)!$ forms a perfect subgroup".

"For $n=2$ and $n=3$ these general results have to be completed in the generally known way".

The simplest proof of this theorem is connected with the remark that reflexion of the polytopes with respect to any space Sy_{n-1} corresponds to the interchanging of any pair of vertices of the simplex. So the order of the group of reflexions (and anallagmatic displacements) is equal to the number of permutations of the $n+1$ vertices of $S(n+1)$, i. e. $(n+1)!$, and the group of the anallagmatic displacements is of an order half as large, i. e. of order $\frac{1}{2}(n+1)!$

For the cases $n=2$ and $n=3$ we refer to F. KLEIN's "Vorlesungen über das Ikosaeder" (Leipsic, Teubner, 1884).

42. The manner in which the polytopes considered here have been derived from the simplex is a guarantee that all the vertices are of the same kind and all the edges have the same length. But this is all that can be asserted; so e. g. the polyhedron tT has *two* kinds of edges, edges common to two hexagons lying in planes including a definite acute angle and edges common to a hexagon and a triangle lying in planes including the obtuse supplementary angle. So in judging of the regularity we have to look at the edges from two different points of view; we must not only take into account the length but also consider angles on or faces through the edges, etc.

[1]) Compare *Report of the British Association*, 1894, p. 563.

In his dissertation — which is about to appear — M^r. E. L. Elte has created an artificial system [1]) according to which it is possible to count the degree of regularity of the partially regular polytopes deduced from the regular polytopes by regular truncation. In this system the regularity of such a polytope is expressed by a fraction, the denominator of which is equal to the number of dimensions, while each group of limiting elements as vertices, edges, faces, etc. may contribute a unit to the numerator. With the exception of the group of vertices [2]) every group of limiting elements has this unit subdivided into two halves, one half for equality of form, the other half for equality of position with respect to the surroundings; moreover only *successive* contributions count, beginning at the vertices. So in the case of tT the contributions of vertices and edges are 1, $\frac{1}{2}$ and the degree of regularity is $\frac{1+\frac{1}{2}}{3} = \frac{1}{2}$ and this is the case with all the Archimedian semiregular polyhedra, except CO and ID, where the dihedral angles on the edges are equal and the degree of regularity is $\frac{1+1}{3} = \frac{2}{3}$.

Of the two halves corresponding to equality of form and to equality of position with respect to the surroundings the first needs no explanation, while the second may seem rather difficult to grasp. But this second half also will become clear, if we indicate it as follows. Equality of vertices means that the figures formed by the systems of edges concurring in the different vertices (vertex polyangles) are congruent, equality of edges means that the edges have the same length (first $\frac{1}{2}$) and that the figures formed by the systems of intersecting lines of the faces passing through the different edges with spaces S_{n-1} normal to the edges (edge polyangles) are congruent (second $\frac{1}{2}$), equality of faces means that the faces are congruent (first $\frac{1}{2}$) and that the figures formed by the systems of intersecting lines of the limiting threedimensional spaces passing through the faces with spaces S_{n-2} normal to the faces (face polyangles) are congruent (second $\frac{1}{2}$), etc.

So we will be able to determine the regularity fraction of a given polytope derived from the simplex in the scale of M^r. Elte, if we have found the *different subgroups* of each of the limiting

[1]) We can only give a glimpse of the system here. For more particulars we must refer to the dissertation written in English.

[2]) If we count from the other side (see the next page) we must say: "with the exception of the group of limits $(l)_{n-1}$", etc.

elements $(l)_1, (l)_2, \ldots, (l)_{n-1}$. So this research is closely related with theorem III of art. 10 which enables us to find the subgroups of the same system of limiting elements l_d characterized by *different symbols*, the more so as we have the theorem:

THEOREM XXVI. "Any two limiting elements of the same group $(l)_d$ belong to the same subgroup or to different subgroups, in the sense of the scale of regularity, according to their zero symbols being equal or different, if we consider two different zero symbols of a central symmetric polytope as being equal when they pass into each other by inversion."

This theorem is nearly self evident. A rigid proof of it can be based on the consideration of the limits $(l)_{n-1}$ passing through the $(l)_d$. So in the case of the form (321100) treated in art. 11 the different unextended edge symbols (32), (21), (10) correspond to subgroups of edges with different positions in relation to the surroundings. For, if we consider the four groups (32110), (321)(100), (32)(1100), (21100) of limiting polytopes it is immediately evident that the second group distinguishes (10) from the others, that the fourth group distinguishes (32) from the others, whilst the third group alone shows already that no two of the three subgroups of edges can be equal.

We remarked above that we count the contributions to the numerator of the regularity fraction beginning at the vertices and taking in only successive contributions. But the case may present itself that a polytope derived from the simplex shows also some regularity at the side of the limiting elements $(l)_{n-1}$ of the highest number of dimensions. We then indicate two fractions of regularity, one for each side, as will be shown in an example in the next article.

The fifth column of Table I contains the regularity fraction of the different forms obtained in the cases $n = 3, 4, 5$, only counted from the vertex side. In the fourth column the subscripts indicate the numbers of the different subgroups of each limiting element $(l)_d$.

43. We elucidate the theory by applying it to several examples:

a). *Example* **(321100)**. Here we find three different groups of edges. So the vertices contribute 1, the edges contribute $\frac{1}{2}$ to the numerator and the fraction is $\dfrac{1+\frac{1}{2}}{5} = \dfrac{3}{10}$.

b). *Example* **(110000)**. This form has only one kind of edge (10) but two subgroups (110) and (100) of triangular faces. So we find $\dfrac{1+1+\frac{1}{2}}{5} = \dfrac{1}{2}$.

c). *Example* **(111000)**, This central symmetric form has one kind of edge (10), one kind of face (110) = — (100), but two subgroups (1110) = — (1000) = T and (1100) = O of limiting bodies and once more one kind of limiting polytopes (11100) = — (11000). So we find $\left(\frac{3}{5}; \frac{1}{5}\right)$.

Remark. The degree of regularity of the polytopes of S_n found here is at least $\frac{1+\frac{1}{2}}{n} = \frac{3}{2n}$ and therefore for $n = 3$ at least $\frac{1}{2}$. So the Archimedian polyhedra of the stereometry are *semi*regular in the right sense of the word, if we take semiregular to mean that the degree of regularity is $\frac{1}{2}$ at least but less than unity.

44. As the scale used for the determination of the regularity is independent from the number of vertices, edges, faces, etc. of the polytope, the same method may be applied to nets of polytopes, by considering a net in S_n as as polytope limited by an infinite number of limits $(l)_n$ in S_{n+1}. This new application depends only on the problem how to determine the different kinds of vertices, edges, faces, etc. of the net.

All the nets considered here have vertices of the same kind and edges of the same length. So for a net in S_n the fraction of regularity is at least $\frac{1+\frac{1}{2}}{n+1}$, i. e. $\frac{3}{2(n+1)}$. So in the most frequent number of cases in which a constituent of the nets admits two or more differently shaped faces we have only the choice between $\frac{2}{n+1}$ and $\frac{3}{2(n+1)}$ of which the first value corresponds to the case of only one kind of edge, the second to that of two or more differents kinds of edges.

In order to make the determination of the fraction of regularity of the nets in S_4 and S_5 as easy as possible we enumerate in Table III the different limits $(l)_4$, $(l)_3$, $(l)_2$ of the nets in S_4 and the different limits $(l)_5$, $(l)_4$, $(l)_3$, $(l)_2$ of the nets in S_5. In the part corresponding to $n = 4$ we find under the seven headings I, II, . ., VII the subdivisions 4, 3, 2 standing for $(l)_4$, $(l)_3$, $(l)_2$, in the part corresponding to $n = 5$ likewise under I, II, . ., XII the subdivisions 5, 4, 3, 2 standing for $(l)_5$, $(l)_4$, $(l)_3$, $(l)_2$. These limits are indicated in abridged notation: under 5 the symbols 1, ce_1, ce_2, etc. denote $S(6)$, $ce_1S(6)$, $ce_2S(6)$, etc.; under 4 the symbols 1, ce_1, etc. signify $S(5)$, $ce_1S(5)$, etc.

The results of Table III are inscribed in Table II in the sixth column

under the headings $(l)_0, (l)_1, \ldots, (l)_5$; so the number 11 on the lines of the nets $5_{VI}{}^a$, $5_{VI}{}^b$ under $(l)_4$ indicates that in these equal nets of space S_5 the constituents admit together eleven *differently shaped* limits $(l)_4$. What is taken from Table III — and what is self evident — is incribed in small type. The other numbers — inscribed in heavy type —, of which only two correspond to faces, have been found separately. We treat here two of these cases in detail.

Case 4_{III}. Here the constituent (11000) has only one kind of edge. Does this *imply* that the *net* has only one kind of edge? The example of the net 3_{IV} where the CO admits also only one kind of edge, whilst Andreini rightly mentions the fact (see his treatise, p. 32 under n°. 21) that of the five edges concurring in a vertex one is common to 2 tT and 2 tO and each of the four others to tT, tO, CO, must prevent us from jumping too rashly to this conclusion. So we investigate this point and examine if, e. g. in the case of the constituent (21100) with two kinds of edges, (21)100 and 21(10)0 these two edges are different with respect to the net or not. So we enumerate first the different limits $(l)_4$ to which the vertex 21100 is common. They are

$$
\begin{array}{rlllllll}
e_2 \ldots & (\mathbf{2}, & \mathbf{1}, & \mathbf{1}, & \mathbf{0}, & \mathbf{0}) & \\
-e_2 \ldots & (\mathbf{2}, & \mathbf{1}, & \mathbf{1}, & -2+\mathbf{2}, & \mathbf{0}) & \left.\right\} 1 \\
-e_2 \ldots & (\mathbf{2}, & \mathbf{1}, & \mathbf{1}, & \mathbf{0}, & -2+\mathbf{2}) & \\
-ce_1 \ldots & (\mathbf{2}, & \mathbf{1}, & \mathbf{1}, & -2+\mathbf{2}, & -2+\mathbf{2}) & 2 \\
e_3 \ldots & (\mathbf{2}, & -2+\mathbf{3}, & \mathbf{1}, & -2+\mathbf{2}, & -2+\mathbf{2}) & \left.\right\} 3 \\
e_3 \ldots & (\mathbf{2}, & \mathbf{1}, & -2+\mathbf{3}, & -2+\mathbf{2}, & -2+\mathbf{2}) & \\
ce_1 \ldots & (\mathbf{2}, & -2+\mathbf{3}, & -2+\mathbf{3}, & -2+\mathbf{2}, & -2+\mathbf{2}) & \\
e_2 \ldots & (\mathbf{2}, & -2+\mathbf{3}, & -2+\mathbf{3}, & -2+\mathbf{2}, & -4+\mathbf{4}) & \left.\right\} 4 \\
e_2 \ldots & (\mathbf{2}, & -2+\mathbf{3}, & -2+\mathbf{3}, & -4+\mathbf{4}, & -2+\mathbf{2}) & \\
-e_2 \ldots & (\mathbf{2}, & -2+\mathbf{3}, & -2+\mathbf{3}, & -4+\mathbf{4}, & -4+\mathbf{4}) &
\end{array}
$$

Starting from (2 11 00) we have indicated in this list of ten polytopes first the two polytopes deduced from (21100) by varying the form of one of the digits 1, 1, 0, 0, then the only polytope obtained by varying two of the digits, etc., see the curved brackets and the numbers 1, 2, 3, 4 at the right. As we can augment all the interchangeable parts by the same integer provided that we diminish all the unmovable parts by the same amount, we find in this manner all the polytopes to which the chosen vertex 21100 is common, though we leave the first digit 2 alone.

If we denote the ten polytopes of the list by $(P)_1, (P)_2, \ldots, (P)_{10}$

we find that the edge (21) 100 is common to $(P)_1, (P)_2, (P)_3, (P)_4, (P)_6$ and the edge 21(10)0 to $(P)_1, (P)_3, (P)_6, (P)_7, (P)_8$. So both edges are common to $3\,e_2$, ce_1, e_3.

But this fact is not yet decisive, as the possibility exists that the grouping of the sets of five polytopes around the edges (21) and (10) is different. In order to decide this point we draw up the following table of threedimensional contact, where 1, 2, . . ., 10 stand for $(P)_1$, $(P)_2, \ldots, (P)_{10}$ and contact by a prism is indicated by a small asterisk.

1	2	3	4	5 & 6	7	8	9	10
2	1	1	2	1*	1	3	2	4
3	4	4	3	2*	5	5*	5*	5*
5*	5*	5*	5	3*	6	6*	6*	6*
6*	6*	6*	6	4	8	7	7	8
7	9	8	10	7	9	10	10	9
				8*				
				9*				
				10*				

This table shows that if we arrange each of the two sets of five polytopes as follows in three groups

$$\begin{array}{l} P_1, P_4 \;-\; P_2, P_3 \;-\; P_6 \\ P_3, P_7 \;-\; P_1, P_8 \;-\; P_6 \end{array}$$

each polytope is in bodily contact with the polytopes of the other groups of its horizontal row, whilst two polytopes in the same column are equal. So there is no difference whatever in the threedimensional contact, i. e. there is only one kind of edge

Case 5_V. Here the point 321000 is common to the 17 polytopes

$$\begin{array}{lllllll}
e_1 e_2 \ldots (\mathbf{3}, & \mathbf{2}, & \mathbf{1}, & \mathbf{0}, & \mathbf{0}, & \mathbf{0}) & 1 \\
ce_1 e_2 e_3 . (\mathbf{3}, & \mathbf{2}, & \mathbf{1}, & -3+\mathbf{3}, & \mathbf{0}, & \mathbf{0}) & 3 \\
-e_1 e_2 \ldots (\mathbf{3}, & \mathbf{2}, & \mathbf{1}, & -3+\mathbf{3}, & -3+\mathbf{3}, & \mathbf{0}) & 3 \\
-e_1 \ldots . (\mathbf{3}, & \mathbf{2}, & \mathbf{1}, & -3+\mathbf{3}, & -3+\mathbf{3}, & -3+\mathbf{3}) & 1 \\
e_4 \ldots . (\mathbf{3}, & \mathbf{2}, & -3+\mathbf{4}, & -3+\mathbf{3}, & -3+\mathbf{3}, & -3+\mathbf{3}) & 1 \\
e_1 \ldots . (\mathbf{3}, & -3+\mathbf{5}, & -3+\mathbf{4}, & -3+\mathbf{3}, & -3+\mathbf{3}, & -3+\mathbf{3}) & 1 \\
e_1 e_2 \ldots (\mathbf{3}, & -3+\mathbf{5}, & -3+\mathbf{4}, & -3+\mathbf{3}, & -3+\mathbf{3}, & -6+\mathbf{6}) & 3 \\
ce_1 e_2 e_3 . (\mathbf{3}, & -3+\mathbf{5}, & -3+\mathbf{4}, & -3+\mathbf{3}, & -6+\mathbf{6}, & -6+\mathbf{6}) & 3 \\
-e_1 e_2 \ldots (\mathbf{3}, & -3+\mathbf{5}, & -3+\mathbf{4}, & -6+\mathbf{6}, & -6+\mathbf{6}, & -6+\mathbf{6}) & 1
\end{array}$$

In this list we have availed ourselves of the occurrence of the three zeros in order to represent the 17 polytopes by nine symbols. So the second line represents three different polytopes which can be obtained by putting seccessively $-3+\mathbf{3}$ under each of the three zeros. For each line the number at the right indicates how many different polytopes through the point 321000 correspond to that line.

This list shows that we may find here two kinds of edges though we have three groups, edges (32) 1000 common to 9 polytopes, edges 3(21)000 common to 16 polytopes, edges 32(10)00 common to 9 polytopes, as in the first and the last group the sets of 9 polytopes are both $4\, e_1\, e_2$, $3\, ce_1\, e_2\, e_3$, e_1, e_4. By investigating the fourdimensional contact between the nine polytopes of each set can be found whether the edges (32) and (10) belong to the same kind or not.

From the numbers of differently shaped limits the fraction of regularity has been deduced; it is given in the last column of Table II.

45. We finish this part of our memoir concerned with the offspring of the simplex by a remark about what may be called the "circumpolytope" of a net. This polytope, which has for vertices the vertices of the net joined by edges to any arbitrarily chosen vertex of the net, is by its form a criterion for the regularity of the net. If the net admits one kind of edge the circumpolytope must admit one kind of vertex, etc. This circumpolytope is in the cases of the threedimensional nets:

3_I . . . a CO,
3_{II} . . . a prismoid limited by two equilateral and six equal isosceles triangles,
3_{III} . . . a prismoid limited by two squares and eight equal isosceles triangles,
3_{IV} . . . a pyramid on a rectangular base,
3_V . . . a tetrahedron limited by four equal isosceles triangles;
of these five polyhedra only the fourth has vertices of two different kinds.

The theories developed here enable us to find the circumpolytopes corresponding to the different nets of simplex extraction in S_4 and S_5. But instead of deducing these polytopes here we conclude by the following general problem, for the proof of which we refer to the dissertation of M^r. ELTE:

THEOREM XXVII. "If the regularity of the circumpolytope of a simplex net of S_n is expressed by the fraction $\frac{p}{n}$, the regularity of the net itself will be represented in the general case by $\frac{p+1}{n+1}$ and in the special case $p=0$ by $\frac{3}{2(n+1)}$".

(To be continued).

Groningen, September 1911.

Errata.

Page 7, last line of middle column under $n = 4$ omit "$= c\,e_2\,e_3\,S(5)$",
" 27, line 18 from top replace "$(m_1, m_2, m_3, \ldots, 0)$", by "$(m_0, m_1, m_2, \ldots, 0)$".

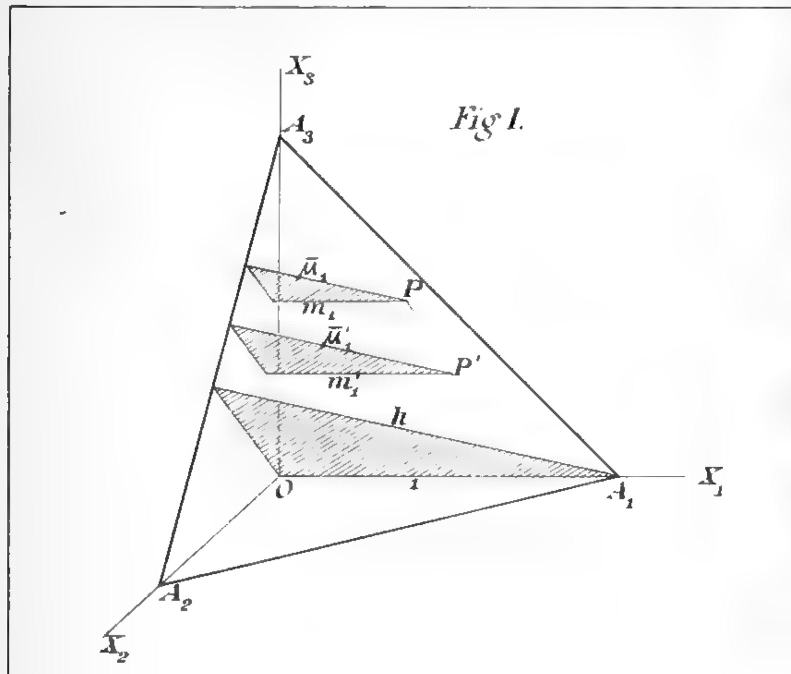

Fig. 1.

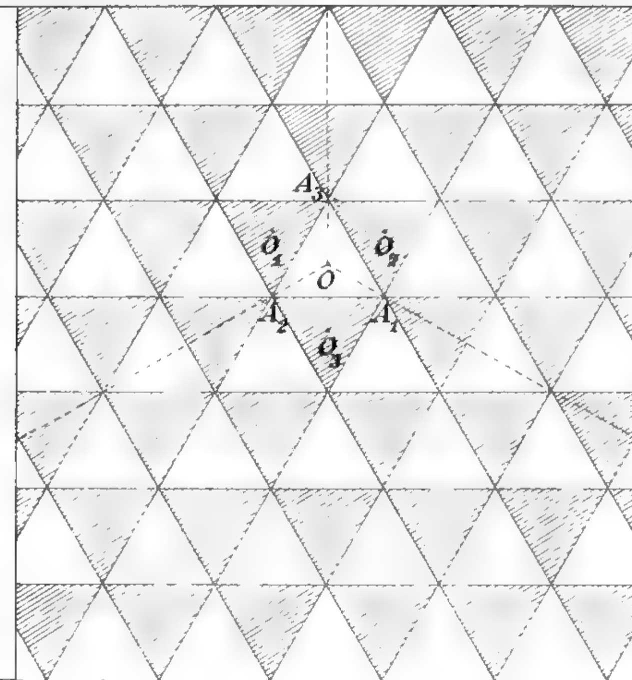

Fig. 5.

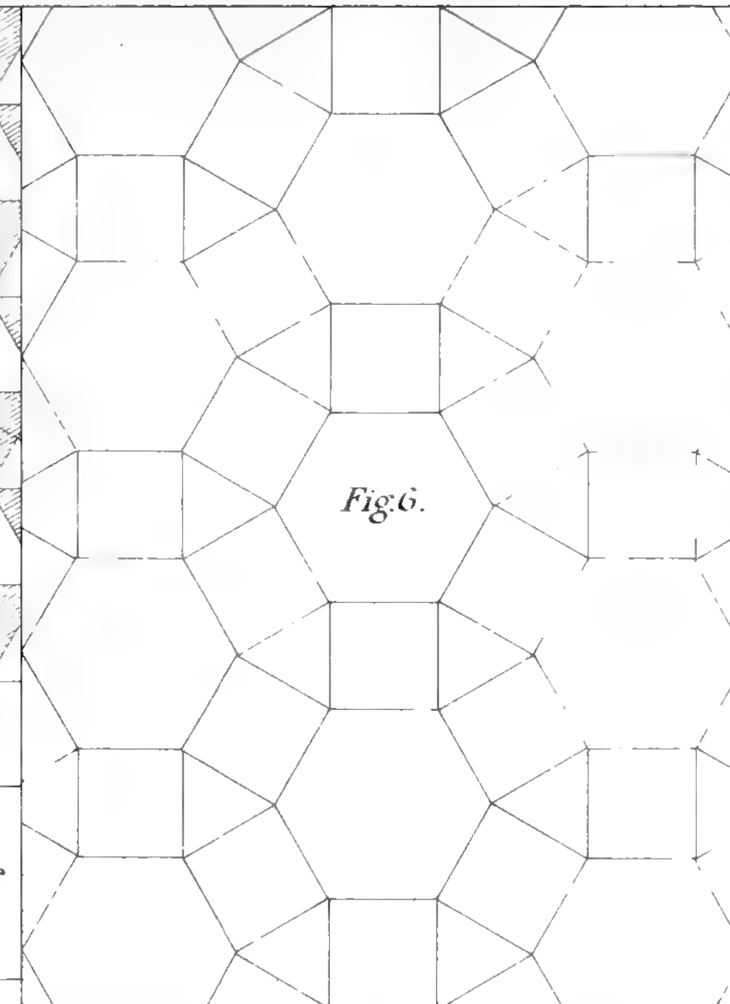
Fig. 6.

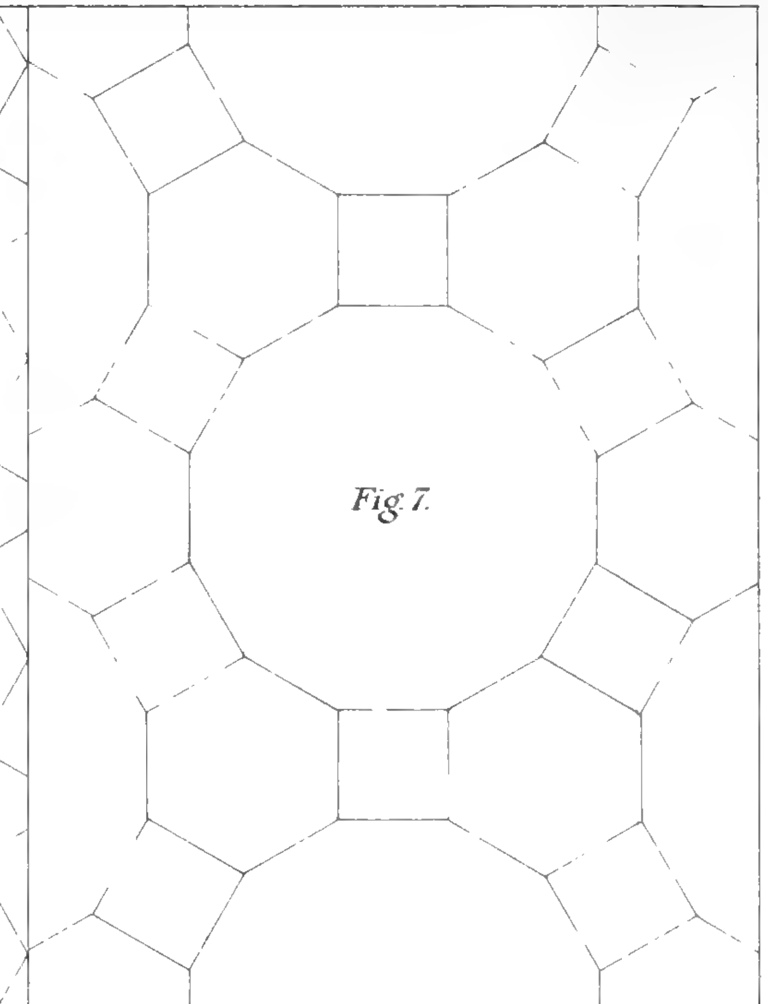
Fig. 7.

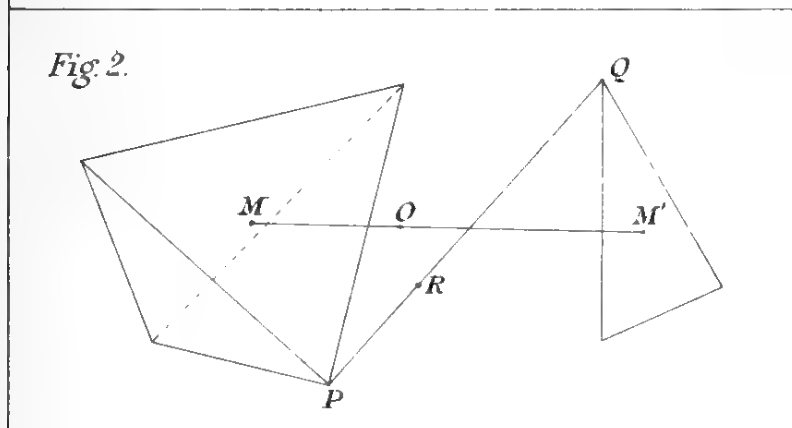

Fig. 2.

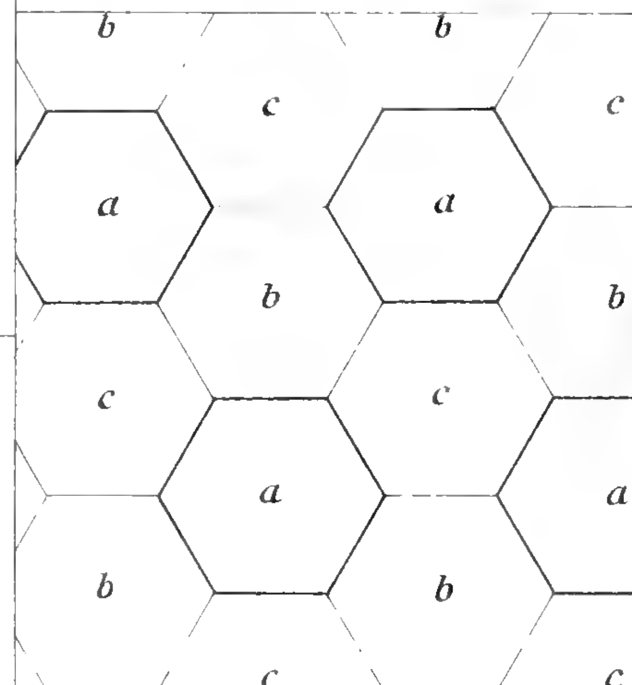

Fig. 10.

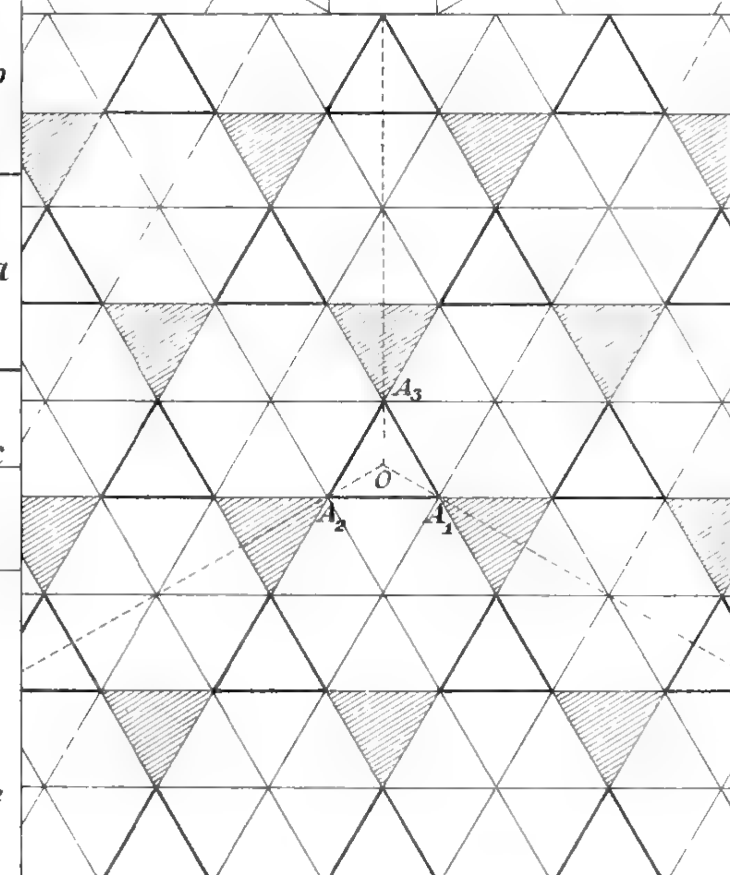

Fig. 9.

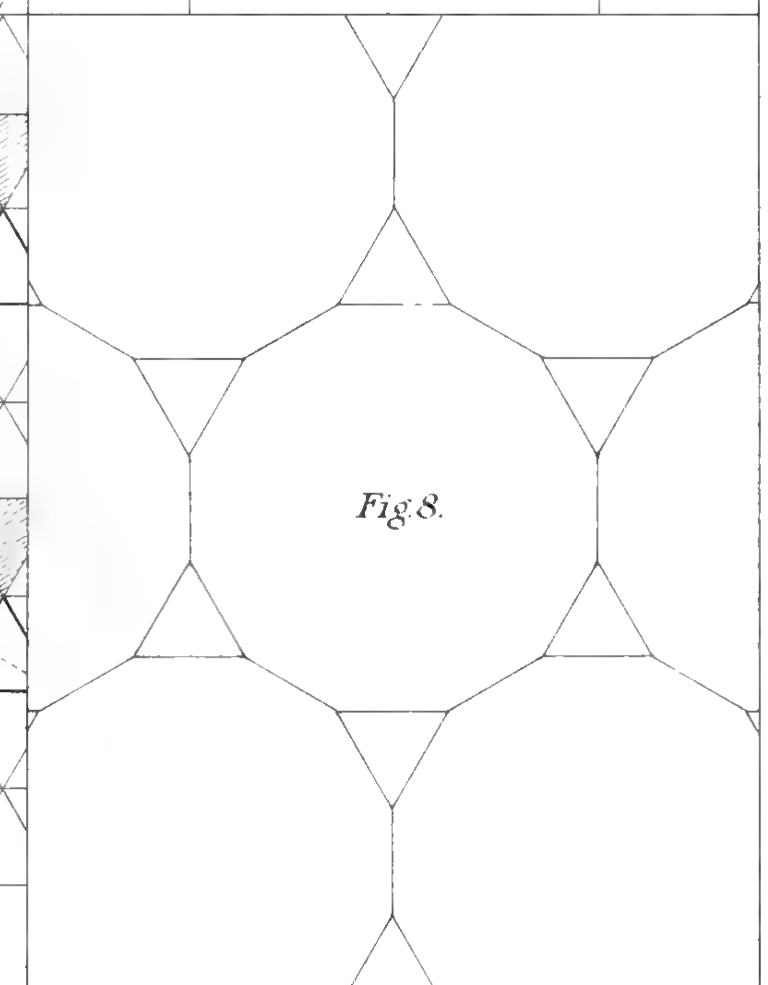
Fig. 8.

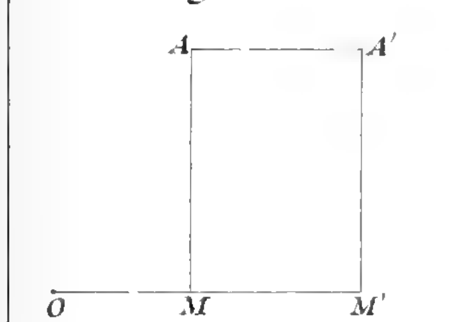

Fig. 3.

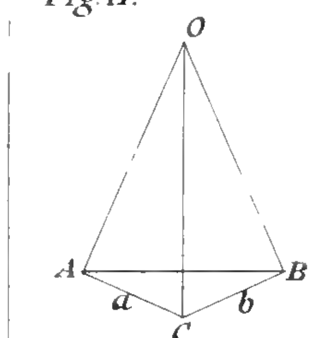

Fig. 11.

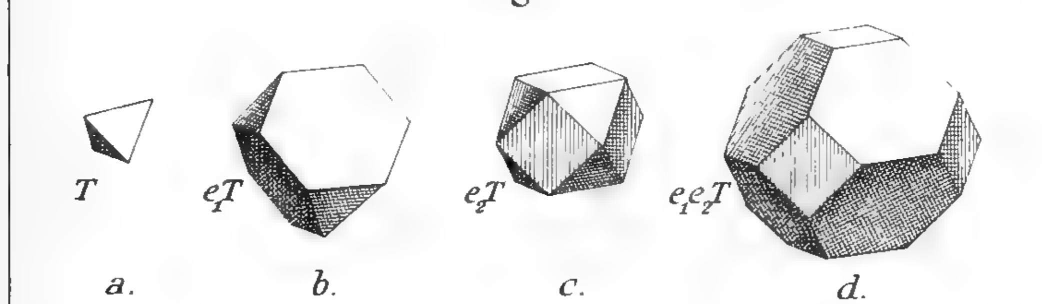

Fig. 4.

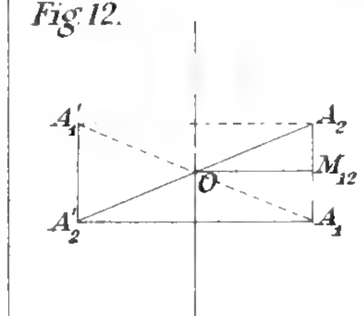

Fig. 12.

J. Bytel lith. P. J. Mulder impr. Leiden

$n = 3$

					4	6	4			
$S(4) = T$	(1000)	0	$(4_1\ \ 6_1\ \ 4_1)$	1	p_3	—	—	1		
$e_1\ S(4) = tT$	(2100)	$\frac{2}{4}$	$(12_1\ \ 18_2\ \ 8_2)$	$\frac{1}{2}$	p_6	—	p_3	3	1	
$e_2\ S(4) = CO$	* (2110)	$\frac{3}{4}$	$(12_1\ \ 24_1\ \ 14_2)$	$\frac{2}{3}$	p_3	p_4	p_3	4	2, 1	c. s.
$e_1 e_2\ S(4) = tO$	* (3210)	$\frac{5}{4}$	$(24_1\ \ 36_2\ \ 14_2)$	$\frac{1}{2}$	p_6	p_4	p_6	6	3, 1	c. s.
$c\ e_1\ S(4) = O$	* (1100)	$\frac{1}{4}$	$(6_1\ \ 12_1\ \ 8_1)$	1	p_3	—	p_3	2	1	c. s.

$n = 4$

					5	10	10	5			
$S(5)$	(10000)	0	$(5_1\ \ 10_1\ \ 10_1\ \ 5_1)$	1	T	—	—	—	1		
$e_1\ S(5)$	(21000)	$\frac{2}{5}$	$(20_1\ \ 40_2\ \ 30_2\ \ 10_2)$	$\frac{3}{8}$	tT	—	—	T	3	1	
$e_2\ S(5)$	(21100)	$\frac{3}{5}$	$(30_1\ \ 90_2\ \ 80_4\ \ 20_3)$	$\frac{3}{8}$	CO	—	P_3	O	4	2, 1	
$e_3\ S(5)$	* (21110)	$\frac{4}{5}$	$(20_1\ \ 60_1\ \ 70_2\ \ 30_2)$	$\frac{1}{2}$	T	P_3	P_3	T	5	3, 2, 1	c. s.
$e_1 e_2\ S(5)$	(32100)	1	$(60_1\ \ 120_3\ \ 80_4\ \ 20_3)$	$\frac{3}{8}$	tO	—	P_3	tT	6	3, 1	
$e_1 e_3\ S(5)$	(32110)	$\frac{6}{5}$	$(60_1\ \ 150_3\ \ 120_5\ \ 30_4)$	$\frac{3}{8}$	tT	P_6	P_3	CO	7	4, 2, 1	$= e_2 e_3\ S(5)$
$e_2 e_3\ S(5)$	(32210)	$\frac{7}{5}$	$(60_1\ \ 150_3\ \ 120_5\ \ 30_4)$	$\frac{3}{8}$	CO	P_3	P_6	tT	8	5, 3, 1	$= e_1 e_3\ S(5)$
$e_1 e_2 e_3\ S(5)$	* (43210)	$\frac{9}{5}$	$(120_1\ \ 240_2\ \ 150_4\ \ 30_2)$	$\frac{3}{8}$	tO	P_6	P_6	tO	10	6, 3, 1	c. s.
$c\ e_1\ S(5)$	(11000)	$\frac{1}{5}$	$(10_1\ \ 30_1\ \ 30_2\ \ 10_2)$	$\frac{5}{8}$	O	—	—	T	2	1	
$c\ e_1 e_2\ S(5)$	* (22100)	$\frac{4}{5}$	$(30_1\ \ 60_1\ \ 40_2\ \ 10_1)$	$\frac{1}{2}$	tT	—	—	tT	5	3, 1	

$n = 5$

					6	15	20	15	6			
$S(6)$	(100000)	0	$(6_1\ \ 15_1\ \ 20_1\ \ 15_1\ \ 6_1)$	1	$S(5)$	—	—	—	—	1		
$e_1\ S(6)$	(210000)	$\frac{2}{6}$	$(30_1\ \ 75_2\ \ 80_2\ \ 45_2\ \ 12_2)$	$\frac{3}{10}$	$e_1\ S(5)$	—	—	—	$S(5)$	3	1	
$e_2\ S(6)$	(211000)	$\frac{3}{6}$	$(60_1\ \ 240_2\ \ 290_4\ \ 135_4\ \ 27_3)$	$\frac{3}{10}$	$e_2\ S(5)$	—	—	P_T	$c\ e_1\ S(5)$	4	2, 1	
$e_3\ S(6)$	(211100)	$\frac{4}{6}$	$(60_1\ \ 270_2\ \ 420_4\ \ 255_5\ \ 47_4)$	$\frac{3}{10}$	$e_3\ S(5)$	—	(3 ; 3)	P_O	$-\ c\ e_1\ S(5)$	5	3, 2, 1	
$e_4\ S(6)$	* (211110)	$\frac{5}{6}$	$(30_1\ \ 120_1\ \ 210_2\ \ 180_2\ \ 62_3)$	$\frac{2}{5}$	$S(5)$	P_T	(3 ; 3)	P_T	$S(5)$	6	4, 3, 2, 1	c. s.
$e_1 e_2\ S(6)$	(321000)	$\frac{5}{6}$	$(120_1\ \ 300_3\ \ 290_4\ \ 135_4\ \ 27_3)$	$\frac{3}{10}$	$e_1 e_2\ S(5)$	—	—	P_T	$e_1\ S(5)$	6	3, 1	
$e_1 e_3\ S(6)$	(321100)	1	$(180_1\ \ 630_3\ \ 720_6\ \ 315_7\ \ 47_4)$	$\frac{3}{10}$	$e_1 e_3\ S(5)$	—	(6 ; 3)	P_O	$e_2\ S(5)$	7	4, 2, 1	
$e_1 e_4\ S(6)$	(321110)	$\frac{7}{6}$	$(120_1\ \ 420_3\ \ 570_5\ \ 330_7\ \ 62_5)$	$\frac{3}{10}$	$e_1\ S(5)$	P_T	(6 ; 3)	P_T	$e_3\ S(5)$	8	5, 3, 2, 1	$= e_3 e_4\ S(6)$
$e_2 e_3\ S(6)$	(322100)	$\frac{7}{6}$	$(180_1\ \ 540_3\ \ 570_6\ \ 255_6\ \ 47_4)$	$\frac{3}{10}$	$e_2 e_3\ S(5)$	—	(3 ; 3)	P_{tT}	$c\ e_1 e_2\ S(5)$	8	5, 3, 1	
$e_2 e_4\ S(6)$	* (322110)	$\frac{8}{6}$	$(180_1\ \ 720_2\ \ 900_4\ \ 420_5\ \ 62_3)$	$\frac{3}{10}$	$e_2\ S(5)$	P_{CO}	(3 ; 3)	P_{CO}	$e_2\ S(5)$	9	6, 4, 2, 1	c. s.
$e_3 e_4\ S(6)$	(322210)	$\frac{9}{6}$	$(120_1\ \ 420_3\ \ 570_5\ \ 330_7\ \ 62_5)$	$\frac{3}{10}$	$e_3\ S(5)$	P_T	(3 ; 6)	P_{tT}	$e_1\ S(5)$	10	7, 5, 3, 1	$= e_1 e_4\ S(6)$
$e_1 e_2 e_3\ S(6)$	(432100)	$\frac{9}{6}$	$(360_1\ \ 900_4\ \ 810_7\ \ 315_7\ \ 47_4)$	$\frac{3}{10}$	$e_1 e_2 e_3\ S(5)$	—	(3 ; 6)	P_{tT}	$e_1 e_2\ S(5)$	10	6, 3, 1	
$e_1 e_2 e_4\ S(6)$	(432110)	$\frac{10}{6}$	$(360_1\ \ 1080_4\ \ 1140_8\ \ 480_9\ \ 62_5)$	$\frac{3}{10}$	$e_1 e_2\ S(5)$	P_{tO}	(6 ; 3)	P_{CO}	$e_1 e_3\ S(5)$	11	7, 4, 2, 1	$= e_2 e_3 e_4\ S(6)$
$e_1 e_3 e_4\ S(6)$	* (432210)	$\frac{11}{6}$	$(360_1\ \ 1080_2\ \ 1080_5\ \ 420_5\ \ 62_3)$	$\frac{3}{10}$	$e_1 e_3\ S(5)$	P_{tT}	(6 ; 6)	P_{tT}	$e_1 e_3\ S(5)$	12	8, 5, 3, 1	c. s.
$e_2 e_3 e_4\ S(6)$	(433210)	2	$(360_1\ \ 1080_4\ \ 1140_8\ \ 480_9\ \ 62_5)$	$\frac{3}{10}$	$e_2 e_3\ S(5)$	P_{CO}	(6 ; 3)	P_{tO}	$e_1 e_2\ S(5)$	13	9, 6, 3, 1	$= e_1 e_2 e_4\ S(6)$
$e_1 e_2 e_3 e_4\ S(6)$	* (543210)	$\frac{14}{6}$	$(720_1\ \ 1800_3\ \ 1560_6\ \ 540_6\ \ 62_3)$	$\frac{3}{10}$	$e_1 e_2 e_3\ S(5)$	P_{tO}	(6 ; 6)	P_{tO}	$e_1 e_2 e_3\ S(5)$	15	10, 6, 3, 1	c. s.
$c\ e_1\ S(6)$	(110000)	$\frac{1}{6}$	$(15_1\ \ 60_1\ \ 80_2\ \ 45_2\ \ 12_2)$	$\frac{1}{2}$	$c\ e_1\ S(5)$	—	—	—	$S(5)$	2	1	
$c\ e_2\ S(6)$	* (111000)	$\frac{2}{6}$	$(20_1\ \ 90_1\ \ 120_1\ \ 60_2\ \ 12_1)$	$\frac{3}{5}$	$-\ c\ e_1\ S(5)$	—	—	—	$c\ e_1\ S(5)$	3	2, 1	c. s.
$c\ e_1 e_2\ S(6)$	(221000)	$\frac{4}{6}$	$(60_1\ \ 150_2\ \ 140_3\ \ 60_3\ \ 12_2)$	$\frac{3}{10}$	$c\ e_1 e_2\ S(5)$	—	—	—	$e_1\ S(5)$	5	3, 1	
$c\ e_1 e_3\ S(6)$	* (221100)	$\frac{5}{6}$	$(90_1\ \ 360_1\ \ 420_3\ \ 180_3\ \ 32_2)$	$\frac{2}{5}$	$e_2\ S(5)$	—	—	—	$e_2\ S(5)$	6	4, 2, 1	c. s.
$c\ e_1 e_2 e_3\ S(6)$	* (332100)	$\frac{8}{6}$	$(180_1\ \ 450_2\ \ 420_3\ \ 180_3\ \ 32_2)$	$\frac{3}{10}$	$e_1 e_2\ S(5)$	—	—	—	$e_1 e_2\ S(5)$	9	6, 3, 1	c. s.

$n = 2$

							$(l)_0$	$(l)_1$	$(l)_2$	$(l)_3$	$(l)_4$	$(l)_5$	
I	1	3	(a_1, a_2, a_3)	$(100) = p_3$	$(110) = -p_3$		1	1	1				1
II	2	1, 2	$(2a_1+1, 2a_2, 2a_3)$	$(100) = p_3$	$(210) = {}^*p_6$	$(110) = -p_3$	1	1	2				$\frac{2}{3}$
III	3	1, 1, 1	$(3a_1+2, 3a_2+1, 3a_3)$	$(210) = {}^*p_6$			1	1	1				1

$n = 3$

								$(l)_0$	$(l)_1$	$(l)_2$	$(l)_3$	$(l)_4$	$(l)_5$	
I	1	4	(a_1, a_2, a_3, a_4)	$(1000) = T$	$(1100) = {}^*O$	$(1110) = -T$		1	1	1	2			$\frac{3}{4}$
II	2	1, 3	$(2a_1+1, 2a_2, 2a_3, 2a_4)$	$(1000) = T$	$(2100) = tT$	$(2210) = -tT$	$(1110) = -T$	1	1	2	2			$\frac{1}{2}$
III	2	2, 2	$(2a_1+1, 2a_2+1, 2a_3, 2a_4)$	$(1100) = {}^*O$	$(2110) = {}^*CO$			1	1	2	2			$\frac{1}{2}$
IV	3	1, 1, 2	$(3a_1+2, 3a_2+1, 3a_3, 3a_4)$	$(2100) = tT$	$(3210) = {}^*tO$	$(2210) = -tT$	$(2110) = {}^*CO$	1	2	3	3			$\frac{3}{8}$
V	4	1, 1, 1, 1	$(4a_1+3, 4a_2+2, 4a_3+1, 4a_4)$	$(3210) = {}^*tO$				1	1	2	1			$\frac{1}{2}$

$n = 4$

									$(l)_0$	$(l)_1$	$(l)_2$	$(l)_3$	$(l)_4$	$(l)_5$	
I	1	5	$(a_1, a_2, a_3, a_4, a_5)$	$(10000) = e_0$	$(11000) = e_1$	$(11100) = -e_1$	$(11110) = -e_0$		1	1	**1**	2	2		$\frac{3}{5}$
II	2	1, 4	$(2a_1+1, 2a_2, 2a_3, 2a_4, 2a_5)$	$(10000) = e_0$	$(21000) = e_0 e_1$	$(22100) = {}^* e_1 e_2$	$(22210) = -e_0 e_1$	$(11110) = -e_0$	1	**1**	2	2	3		$\frac{2}{5}$
III	2	2, 3	$(2a_1+1, 2a_2+1, 2a_3, 2a_4, 2a_5)$	$(11000) = e_1$	$(21100) = e_0 e_2$	$(22110) = -e_0 e_2$	$(11106) = -e_1$	$(21110) = {}^* e_0 e_3$	1	**1**	2	4	3		$\frac{2}{5}$
IV	3	1, 1, 3	$(3a_1+2, 3a_2+1, 3a_3, 3a_4, 3a_5)$	$(21000) = e_0 e_1$	$(32100) = e_0 e_1 e_2$	$(33210) = -e_0 e_1 e_2$	$(22210) = -e_0 e_1$	$(21110) = {}^* e_0 e_3$	1	**2**	3	4	3		$\frac{3}{10}$
V	3	1, 2, 2	$(3a_1+2, 3a_2+1, 3a_3+1, 3a_4, 3a_5)$	$(21100) = e_0 e_2$	$(32110) = e_0 e_1 e_3$	$(22100) = {}^* e_1 e_2$	$(32210) = -e_0 e_1 e_3$	$(22110) = -e_0 e_2$	1	**3**	3	5	3		$\frac{3}{10}$
VI	4	1, 1, 1, 2	$(4a_1+3, 4a_2+2, 4a_3+1, 4a_4, 4a_5)$	$(32100) = e_0 e_1 e_2$	$(43210) = {}^* e_0 e_1 e_2 e_3$	$(33210) = -e_0 e_1 e_2$	$(32210) = -e_0 e_1 e_3$	$(32110) = e_0 e_1 e_3$	1	**2**	3	5	3		$\frac{3}{10}$
VII	5	1, 1, 1, 1, 1	$(5a_1+4, 5a_2+3, 5a_3+2, 5a_4+1, 5a_5)$	$(43210) = {}^* e_0 e_1 e_2 e_3$					1	1	2	2	1		$\frac{2}{5}$

$n = 5$

										$(l)_0$	$(l)_1$	$(l)_2$	$(l)_3$	$(l)_4$	$(l)_5$	
I	1	6	$(a_1, a_2, a_3, a_4, a_5, a_6)$	$(100000) = e_0$	$(110000) = e_1$	$(111000) = {}^* e_2$	$(111100) = -e_1$	$(111110) = -e_0$		1	1	**1**	2	2	3	$\frac{1}{2}$
II	2	1, 5	$(2a_1+1, 2a_2, 2a_3, 2a_4, 2a_5, 2a_6)$	$(100000) = e_0$	$(210000) = e_0 e_1$	$(221000) = e_1 e_2$	$(222100) = -e_1 e_2$	$(222210) = -e_0 e_1$	$(111110) = -e_0$	1	**1**	2	2	3	3	$\frac{1}{3}$
III	2	2, 4	$(2a_1+1, 2a_2+1, 2a_3, 2a_4, 2a_5, 2a_6)$	$(110000) = e_1$	$(211600) = e_0 e_2$	$(221100) = {}^* e_1 e_3$	$(222110) = -e_0 e_2$	$(111100) = -e_1$	$(211110) = {}^* e_0 e_4$	1	**1**	2	4	5	4	$\frac{1}{3}$
IV	2	3, 3	$(2a_1+1, 2a_2+1, 2a_3+1, 2a_4, 2a_5, 2a_6)$	$(111000) = {}^* e_2$	$(211100) = e_0 e_3$	$(221110) = -e_0 e_3$				1	**1**	2	3	4	2	$\frac{1}{3}$
V	3	1, 1, 4	$(3a_1+2, 3a_2+1, 3a_3, 3a_4, 3a_5, 3a_6)$	$(210000) = e_0 e_1$	$(321000) = e_0 e_1 e_2$	$(332100) = {}^* e_1 e_2 e_3$	$(333210) = -e_0 e_1 e_2$	$(222210) = -e_0 e_1$	$(211110) = {}^* e_0 e_4$	1	**2**	3	4	5	4	$\frac{1}{4}$
VI^a	3	1, 2, 3	$(3a_1+2, 3a_2+1, 3a_3+1, 3a_4, 3a_5, 3a_6)$	$(211000) = e_0 e_2$	$(321100) = e_0 e_1 e_3$	$(332110) = -e_0 e_2 e_3$	$(222100) = -e_1 e_2$	$(322210) = -e_0 e_1 e_4$	$(221110) = -e_0 e_3$	1	**3**	3	6	11	6	$\frac{1}{4}$
VI^b	3	1, 3, 2	$(3a_1+2, 3a_2+1, 3a_3+1, 3a_4+1, 3a_5, 3a_6)$	$(211100) = e_0 e_3$	$(321110) = e_0 e_1 e_4$	$(221000) = e_1 e_2$	$(322100) = e_0 e_2 e_3$	$(332210) = -e_0 e_1 e_3$	$(222110) = -e_0 e_2$	1	**3**	3	6	11	6	$\frac{1}{4}$
VII	3	2, 2, 2	$(3a_1+2, 3a_2+2, 3a_3+1, 3a_4+1, 3a_5, 3a_6)$	$(221100) = {}^* e_1 e_3$	$(322110) = {}^* e_0 e_2 e_4$					1	**1**	2	4	3	2	$\frac{1}{3}$
VIII	4	1, 1, 1, 3	$(4a_1+3, 4a_2+2, 4a_3+1, 4a_4, 4a_5, 4a_6)$	$(321000) = e_0 e_1 e_2$	$(432100) = e_0 e_1 e_2 e_3$	$(443210) = -e_0 e_1 e_2 e_3$	$(333210) = -e_0 e_1 e_2$	$(322210) = -e_0 e_1 e_4$	$(321110) = e_0 e_1 e_4$	1	**2**	3	5	7	3	$\frac{1}{4}$
IX	4	1, 1, 2, 2	$(4a_1+3, 4a_2+2, 4a_3+1, 4a_4+1, 4a_5, 4a_6)$	$(321100) = e_0 e_1 e_3$	$(432110) = e_0 e_1 e_2 e_4$	$(332100) = {}^* e_1 e_2 e_3$	$(433210) = -e_0 e_1 e_2 e_4$	$(332210) = -e_0 e_1 e_3$	$(322110) = {}^* e_0 e_2 e_4$	1	**3**	3	7	8	4	$\frac{1}{4}$
X	4	1, 2, 1, 2	$(4a_1+3, 4a_2+2, 4a_3+2, 4a_4+1, 4a_5, 4a_6)$	$(322100) = e_0 e_2 e_3$	$(432210) = {}^* e_0 e_1 e_3 e_4$	$(332110) = -e_0 e_2 e_3$				1	**2**	3	4	5	2	$\frac{1}{4}$
XI	5	1, 1, 1, 1, 2	$(5a_1+4, 5a_2+3, 5a_3+2, 5a_4+1, 5a_5, 5a_6)$	$(432100) = e_0 e_1 e_2 e_3$	$(543210) = {}^* e_0 e_1 e_2 e_3 e_4$	$(443210) = -e_0 e_1 e_2 e_3$	$(433210) = -e_0 e_1 e_2 e_4$	$(432210) = {}^* e_0 e_1 e_3 e_4$	$(332110) = e_0 e_1 e_2 e_4$	1	**3**	3	6	8	4	$\frac{1}{4}$
XII	6	1, 1, 1, 1, 1, 1	$(6a_1+5, 6a_2+4, 6a_3+3, 6a_4+2, 6a_5+1, 6a_6)$	$(543210) = {}^* e_0 e_1 e_2 e_3 e_4$						1	1	2	3	3	1	$\frac{1}{3}$

$n = 4$

I			II			III			IV			V			VI			VII		
4	**3**	**2**	**4**	**3**	**2**	**4**	**3**	**2**	**4**	**3**	**2**	**4**	**3**	**2**	**4**	**3**	**2**	**4**	**3**	**2**
1	T	p_3	1	T	p_3	ce_1	O	p_3	e_1	tT	p_3	e_2	CO	p_3	$e_1 e_2$	tO	p_3	$e_1 e_2 e_3$	tO	p_4
ce_1	O		e_1	tT	p_6	e_2	T	p_4	$e_1 e_2$	T	p_4	$e_1 e_3$	P_3	p_4	$e_1 e_2 e_3$	P_3	p_4		P_6	p_6
			$ce_1 e_2$			e_3	CO		e_3	tO	p_6	$ce_1 e_2$	O	p_6	$e_1 e_3$	tT	p_6			
							P_3			P_3			tT			P_6				
													P_6			CO				

$n = 5$

I				II				III				IV				V				VI			
5	**4**	**3**	**2**	**5**	**4**	**3**	**2**	**5**	**4**	**3**	**2**	**5**	**4**	**3**	**2**	**5**	**4**	**3**	**2**	**5**	**4**	**3**	**2**
1	1	T	p_3	1	1	T	p_3	ce_1	ce_1	O	p_3	ce_2	ce_1	O	p_3	e_1	e_1	tT	p_3	e_2	e_2	CO	p_3
ce_1	ce_1	O		e_1	e_1	tT	p_6	e_2	1	T	p_4	e_3	e_3	T	p_4	$e_1 e_2$	1	T	p_4	$e_1 e_3$	P_T	P_3	p_4
ce_2				$ce_1 e_2$	$ce_1 e_2$			$ce_1 e_3$	e_2	CO			(3 ; 3)	P_3		$ce_1 e_2 e_3$	$e_1 e_2$	tO	p_6	$e_2 e_3$	ce_1	O	p_6
								e_4	P_T	P_3			P_O			e_4	P_T	P_3		$ce_1 e_2$	$e_1 e_3$	T	
									(3 ; 3)								(3 ; 3)			$e_1 e_4$	(6 ; 3)	tT	
																				e_3	P_O	P_6	
																					(3 ; 3)		
																					P_{tT}		
																					$ce_1 e_2$		
																					e_1		
																					e_3		

VII				VIII				IX				X				XI				XII			
5	**4**	**3**	**2**	**5**	**4**	**3**	**2**	**5**	**4**	**3**	**2**	**5**	**4**	**3**	**2**	**5**	**4**	**3**	**2**	**5**	**4**	**3**	**2**
$ce_1 e_3$	e_2	CO	p_3	$e_1 e_2$	$e_1 e_2$	tO	p_3	$e_1 e_3$	$e_1 e_3$	tT	p_3	$e_2 e_3$	$e_2 e_3$	CO	p_3	$e_1 e_2 e_3$	$e_1 e_2 e_3$	tO	p_3	$e_1 e_2 e_3 e_4$	$e_1 e_2 e_3$	tO	p_4
$e_2 e_4$	P_{CO}	P_3	p_4	$e_1 e_2 e_3$	P_T	P_3	p_4	$e_1 e_2 e_4$	(6 ; 3)	P_6	p_4	$e_1 e_3 e_4$	(3 ; 3)	P_3	p_4	$e_1 e_2 e_3 e_4$	(3 ; 6)	P_6	p_4		P_{tO}	P_6	p_6
	(3 ; 3)	O		$e_1 e_4$	e_1	tT	p_6	$ce_1 e_2 e_3$	P_O	P_3	p_6		P_{tT}	P_6	p_6	$e_1 e_2 e_4$	P_{tT}	P_3	p_6		(6 ; 6)	P_4	
		P_4			$e_1 e_2 e_3$	T		$e_2 e_4$	e_2	CO			$ce_1 e_2$	tT		$e_1 e_3 e_4$	$e_1 e_2$	tT					
					(3 ; 6)	P_6			$e_1 e_2$	O			(6 ; 6)				P_{tO}	P_4					
					P_{tT}				P_{tO}	tO							(6 ; 6)	CO					
					e_3				P_{CO}	P_4							P_{CO}						
									(3 ; 3)								$e_1 e_3$						

THEORETISCHE EN EXPERIMENTEELE ONDERZOEKINGEN OVER PARTIËELE RACEMIE

DOOR

H. DUTILH †
Chem. Docts.

Verhandelingen der Koninklijke Akademie van Wetenschappen te Amsterdam.
(EERSTE SECTIE).
DEEL XI. N°. 4.

AMSTERDAM,
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Theoretische en Experimenteele Onderzoekingen over Partiëele Racemie

DOOR

H. DUTILH †
Chem. Docts.

VOORWOORD.

Den 17^{en} Januari 1910 overleed te Utrecht mijn vriend Herman Dutilh. Kort te voren had Prof. van Romburgh in de Koninklijke Akademie van Wetenschappen te Amsterdam een mededeeling aangeboden, waarin Dutilh voorloopig wees op enkele resultaten van een onderzoek, waarmede hij sinds enkele jaren bezig was en waarop hij weldra tot doctor in de scheikunde hoopte te promoveeren als einde van een studietijd, die met „cum laude" afgelegd candidaats- en doctoraalexamen, zijn leermeesters en zijn studiemakkers zooveel van hem deed verwachten. Het onderzoek was zoo goed als afgesloten en de hoofdstukken II en III, ongeveer in den vorm, waarin ze hier volgen, geschreven. Daar Dutilh en ik sinds jaren gewoon waren dagelijks onze werkzaamheden samen te bespreken, kende ik, althans in hoofdtrekken, zijn opvattingen en toen na zijn dood allen, die hem kenden, gevoelden, dat de vrucht van zooveel arbeid niet voor de wetenschap verloren mocht gaan, verzocht Prof. van Romburgh mij te trachten uit Dutilh's journaal, zijn aanteekeningen en mijn herinnering het ongeschreven gedeelte aan te vullen. Ik heb dat gaarne gedaan. Maar nu het voltooid is gevoel ik sterk, en acht mij verplicht dat hier voorop te stellen, hoe enorm veel beter deze verhandeling geweest zou zijn, indien hij zelf haar geschreven had. Moge, wie het hier volgende leest, en zich een denkbeeld over Dutilh wil vormen, daarmede rekening houden.

De bedoeling van den gestorvene was een hoofdstuk vooraf te doen gaan over de splitsing van racematen in het algemeen en daarvan een aan eigen, oorspronkelijk inzicht getoetst overzicht te geven. Helaas was daarvan nog niets uitgewerkt op papier gebracht en kende ik zijn inzichten niet voldoende om ook maar eenigszins zijn bedoeling met dat hoofdstuk naar waarde weer te geven. Ik heb er dan ook geheel van afgezien.

Wat de positieve conclusies betreft ben ik vrij zeker DUTILH's inzichten aangaande de strychnine-tartraten weergegeven te hebben. Ten opzichte der zure brucine-tartraten hebben wij wel vaak daarover gesproken, zooals ik het hier heb neergeschreven, maar ik durf niet absoluut zeker zeggen of hij, indien hij de zaken zelf neergeschreven had, er zich evenzoo over uitgesproken zou hebben.

Moge ten slotte het geheel zoo zijn geworden, dat het de nagedachtenis van dezen betreurden vriend ter eere zij.

H. R. KRUYT.

September 1910.

HOOFDSTUK I.

INLEIDING.

Alvorens de theoretische kwesties te bespreken, die het hier volgende onderzoek zullen beheerschen, komt het ons gewenscht voor een en ander vooraf te doen gaan aangaande de soort stoffen, wier eigenschappen hier bestudeerd zijn en de bedoelingen, die wij met dit onderzoek hadden.

De partiëele racematen zijn verbindingen waaraan weinig aandacht zou zijn besteed, indien ze niet op merkwaardige wijze een hinderpaal waren geweest bij de bereiding van andere, zeer belangrijke verbindingen. Hun optreden is n.l. beletsel tot de uitvoerbaarheid van een der methoden, door Pasteur ontdekt, om racematen, verbindingen resp. mengsels van optische antipoden, in hun componenten te splitsen. Twee verbindingen, wier moleculairconfiguraties slechts voor een deel spiegelbeelden, voor een ander deel congruente vormen zijn (die nochtans niet met hun spiegelbeeld tot bedekking zijn te brengen), zoodat dus de geheele moleculairconfiguraties noch spiegelbeelden noch congruente vormen zijn, — vertoonen verschil in al hunne physische eigenschappen. Van dat feit maakte Pasteur gebruik bij een zijner splitsingsmethodes. Men kwam echter tot de ervaring, dat deze methode faalde, zoodra, behalve die stoffen met half symetrische constitutie, ook nog hun verbinding optrad.

Heeft men zoo b.v. een racemaat $A_l\,A_d$ (een racemisch zuur b.v.) en laat men daarop inwerken een optisch actieve stof B_l, die in staat is met A een verbinding aan te gaan (een optisch actieve base dus b.v.), dan kunnen zich $A_l\,B_l$ en $A_d\,B_l$ vormen, welke stoffen noch identieke, noch spiegelbeeld-isomere moleculairconfiguraties hebben. Deze lichamen hebben verschillende oplosbaarheid, zoodat bij partiëele stolling dus in het algemeen de minst oplosbare zal uitkristalliseeren; door splitsing der AB verbinding zal dan een optisch actieve A verkregen zijn.

Deze methode faalt natuurlijk in het algemeen als behalve de verbindingen AB ook nog een verbinding van het type $A_l\,A_d\,2\,B_l$

uit de oplossing uit kan kristalliseeren. Aangezien deze verbinding opgebouwd te denken is uit een racemisch gedeelte $A_l\ A_d$ en een optisch-actief gedeelte $(2)B$ heeft men zulk een verbinding een *partieel racemaat* genoemd, een naam over welks juistheid overigens te twisten valt.

Met het oog op de belangrijkheid, zoowel van louter chemisch als van biologisch standpunt, die het vraagstuk der racemaatsplitsing ongetwijfeld toegekend moet worden, als ook om het interessante van het vraagstuk uit physisch chemisch oogpunt, komt het er op aan de stabiliteitscondities dezer partiëele racematen te kennen. In de onderzochte voorbeelden bleek er nl., evenals bij de dubbelzouten en de racematen vaak voorkomt, een temperatuur te bestaan, waarboven, resp. waarbeneden, het partiëele racemaat bij zich instellend evenwicht in zijn componenten uiteen valt; deze temperatuur geeft dan de grens aan, waar de methode van Pasteur toe te passen is. De eerste onderzoekers der partiëele racemie (wij noemen Ladenburg) hebben dan ook dadelijk het gewicht der kennis van deze overgangspunten ingezien en er experimenteel naar gezocht. Hun resultaten worden verder in deze verhandeling uitvoerig besproken.

Op deze onderzoekingen is nu spoedig van theoretische zijde een kritiek van principieelen aard gekomen, nl. van H. W. Bakhuis Roozeboom, wiens verhandelingen evenzeer hieronder een nadere bespreking zal geworden. De uitkomsten van Ladenburg's experimenten verschilden van de door zijn theorie geeischte waarden niet meer dan hij proeffouten zijner methode acht, terwijl van geen andere zijde experimenteel materiaal aangebracht werd. Wij hebben ons daarom ten doel gesteld Ladenburg's experimenten in eenige gevallen te controleeren en zoomogelijk de verhoudingen in een of meer systemen met zoo groot mogelijke zekerheid bloot te leggen.

Allereerst komt het er echter op aan scherp in te zien, wat hier eigenlijk aan de hand is, wat de physisch chemische functies van zulk een overgangspunt zijn, hoe en aan welke functies de kennis omtrent de ligging van zulk een overgangspunt nagegaan kan worden; in het bizonder zullen deze conclusies dan op systemen met racemaat resp. partiëel-racemaatvorming moeten toegepast worden (Hoofdstuk II). Dan rijst de vraag naar een scherpe begripsstelling voor de *partiëele racemie*, waarbij de historie van het ontstaan dezer bestreden term niet kan gemist worden (Hoofdstuk III). Daarna zullen wij de uitkomsten onzer onderzoekingen mededeelen (Hoofdstuk IV en V) en ten slotte de resultaten overzien.

HOOFDSTUK II.

OVER OVERGANGSPUNTEN.

Het is een overbekend feit, dat latere onderzoekers na Pasteur meermalen te vergeefs getracht hebben, om druivenzuur via het natrium-ammoniumzout langs den weg der spontane kristallisatie in *d*- en *l*-wijnsteenzuur te scheiden. Zoo verkreeg Staedel [1]) in 1878 aanvankelijk steeds goed ontwikkelde kristallen uit het monokliene stelsel, waaraan geen hemiëdrische vlakken konden worden waargenomen en wier oplossing optisch inactief was. Eerst uit de moederloog scheidden zich de rhombische kristallen der beide wijnsteenzure zouten van de formule $C_4\,H_4\,O_6\,Na\,NH_4.\ 4\,aq$ af. Staedel kon dit verschijnsel niet verklaren. Reeds tevoren waren dergelijke waarnemingen gedaan. Zoo gaf Mitscherlich [2]) in 1842 op, dat hij door vermengen der oplossing van neutraal druivenzuur-natrium met eene oplossing van iets meer dan één molecuul neutraal kaliumracemaat een in het trikliene stelsel kristalliseerend kalium-natrium racemaat heeft verkregen, dat verschilde van het Seignettezout. Fresenius [3]) vond verder, dat ook zonder overmaat van kaliumracemaat dit trikliene zout met vier moleculen kristalwater, beneden 8° echter met 3 moleculen water wordt verkregen, en Delffs [4]) beschrijft zelfs een dergelijk zout met $4\frac{1}{2}\,aq$, terwijl daartegenover staat, dat volgens Pasteur, wat door Rammelsberg bevestigd is geworden, in dergelijke omstandigheden eene splitsing van het druivenzuur onder afscheiding van de dubbelzouten van *d*-wijnsteenzuur, naast die van *l*-wijnsteenzuur optreedt.

Scacchi [5]) heeft het vraagstuk opgelost door waar te nemen, dat, wanneer de kristallisatie bij verhoogde temperatuur plaats heeft,

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1) Ber. d. D. chem. Ges. **11**, 1752 (1878).
2) Pogg. Ann. **57**, 484 (1842).
3) Lieb. Ann. **53**, 230 (1845).
4) Pogg. Ann. **81**, 304 (1850).
5) Rend. dell' Acad. di Napoli **1865**, 250.

zich het dubbelracemaat $(C_4\,H_4\,O_6\,Na\,NH_4)_2 .\ 2\,H_2\,O$ afzet, terwijl bij gewone temperatuur echter de beide tartraten $C_4\,H_4\,O_6\,Na\,NH_4 .\ 4\,H_2\,O$ uit de oplossing kristalliseeren.

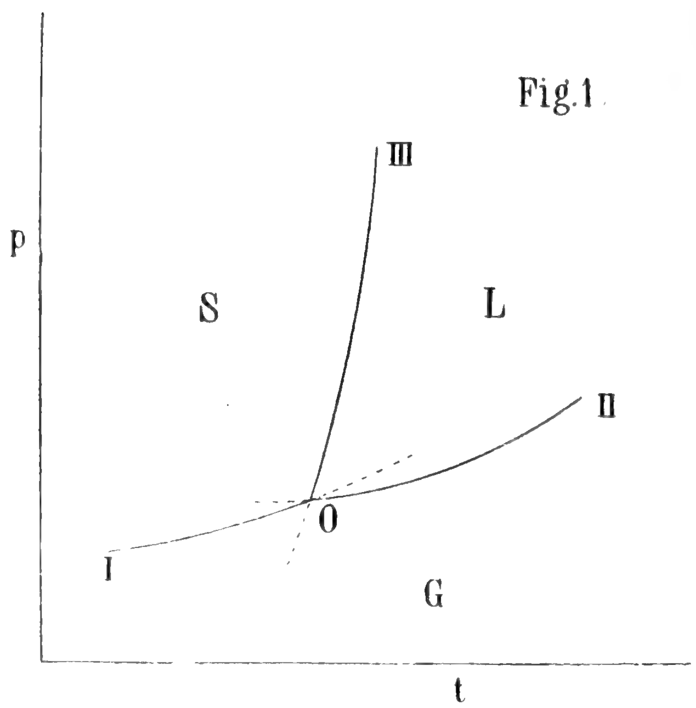
In eene serie van uitgebreide onderzoekingen gelukte het later aan WYROUBOFF [1]) te constateeren, dat de beslissende temperatuur in dit geval 28° bedraagt, zoodat, wanneer oververzadigingsverschijnselen vermeden worden, boven 28° het racemaat, beneden 28° de beide tartraten zich uit de oplossing afzetten.

Die temperatuur van 28° draagt in dit systeem den naam van overgangstemperatuur. Het zal wenschelijk zijn ter behandeling van het verschijnsel der partiëele racemie in dit hoofdstuk en de volgende, de begrippen overgangspunt en overgangstemperatuur van phasentheoretisch standpunt nader te bespreken.

Overgangspunten zijn door BAKHUIS ROOZEBOOM [2]) gedefinieerd als punten, waar in stelsels van n componenten $(n + 2)$ phasen kunnen coexisteeren. Zij zijn gekenmerkt door de samenkomst van $(n + 2)$ curven, die elk op zichzelf het monovariante evenwicht van $(n + 1)$ phasen onderling aangeven.

Het eenvoudigste geval van een overgangspunt doet zich voor in één-component-stelsels als het bekende tripelpunt, waar, bij een bepaalde temperatuur en een bepaalden druk, vast, vloeistof, en gas naast elkaar in nonvariant evenwicht kunnen bestaan.

Fig. 1



Overal, waar in systemen van één component drie phasen kunnen coexisteeren, treft men zulke overgangspunten aan. Er kunnen zich in dergelijke systemen dus nog andere overgangspunten voordoen, n.l. het evenwichtspunt voor $S_1\,S_2\,L$; $S_1\,S_2\,G$; $S_1\,S_2\,S_3$ enz. [3]) [4]).

Dat inderdaad dergelijke punten overgangspunten zijn, dat er dus

[1]) Bull. Soc. Chim. **41**, 210 (1884).

[2]) Zeitschr. f. physik. Chem. **2**, 474, (1888).

[3]) Met S worden vaste phasen aangeduid; L en G geven de vloeistof- resp. gasphase aan.

[4]) Verg. BAKHUIS ROOZEBOOM, Die heterogenen Gleichgew. enz. Ien Heft, Braunschweig (1901).

een overgang (misschien beter omzetting) plaatsgrijpt bij warmtetoevoer of onttrekking van warmte, blijkt wel hieruit, dat men in het geval van het gewone tripelpunt fig. 1, bij constant volumen uitgaande van een punt der lijn I waar de vaste phase naast de gasphase kan bestaan, bij toevoer van warmte, deze lijn in de richting naar O zal volgen; in O gekomen treedt nu naast de phase $S + G$, de vloeistofphase L op. Gaat de smelting met uitzetting gepaard, zooals in fig. 1 wordt verondersteld, en zal de druk constant blijven, dan moet een deel der gasphase verdwijnen, derhalve $S + G \rightarrow L$. Omgekeerd, door onttrekking van warmte in O grijpt plaats de reactic $L \rightarrow S + G$. Iets dergelijks doet zich voor in de andere tripelpunten, die in stelsels van één component kunnen optreden; hunne beteekenis is aldus samen te vatten:

Tripelpunten, d. z. overgangspunten in stelsels van één component, zijn de combinaties van een bepaalden druk met een bepaalde temperatuur, waarbij drie phasen van één component kunnen coexisteeren. In dergelijke punten komen drie curven van monovariant evenwicht samen, overeenkomende met de drie systemen van twee phasen, die uit de drie phasen kunnen worden gevormd.

Bij toe- of afvoer van warmte heeft in die punten door het verdwijnen van één der phasen eene omzetting plaats, waaraan alle phasen deelnemen. In de eene richting verdwijnt steeds ééne bepaalde phase, voor welke het punt dan ook in waarheid het overgangspunt is; in tegengestelde richting daarentegen verdwijnt één der beide andere phasen, afhankelijk van beider hoeveelheid en van het volumen van het systeem.

Het tripelpunt is eene overgangstemperatuur in die richting, waarin slechts ééne curve is gelegen, en voor die phase, wier bestaansgebied tusschen beide andere curven zich uitstrekt.

Meer samengesteld worden de verschijnselen daar, waar in binaire stelsels het quadrupelpunt als overgangspunt optreedt. Het quadrupelpunt ontstaat door de samenkomst van twee evenwichtscurven voor drie phasen van twee stoffen. In dat punt zijn dus 4 phasen met elkaar in evenwicht, zoodat dit punt tevens het snijpunt moet zijn voor nog 2 andere curven van monovariant evenwicht tusschen drie phasen onderling.

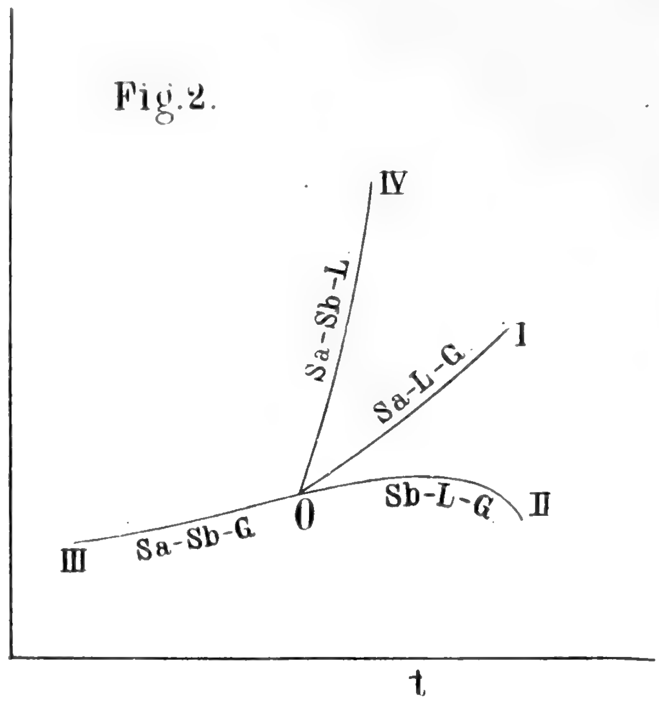
Beginnen wij ook hier weer met het eenvoudigste geval, dan wordt onze aandacht gevraagd voor de coëxistentie van de phasen S_A S_B L G, wier bestaan naast elkaar alleen mogelijk is in het z. g. eutecticum onder dampdruk. Daar men in binaire stelsels, in tegenstelling met unaire, drie vrijheidsgraden heeft n. l. de temperatuur t, den druk p, en de concentratie x, kunnen hier het over-

gangspunt en de curven, die er in samenkomen slechts als eene projectie, b. v. op een $p\,t$-vlak gegeven worden, zoodat in die projectie de curven van monovariant evenwicht (drie phasenlijnen) geen inzicht kunnen geven omtrent de verandering van x met p en t.

Eene dergelijke projectie geeft fig. 2.

Gaan wij uit van een punt der driephasenlijn III, waarop de beide vaste componenten met hun gemeenschappelijken damp in evenwicht zijn, dan zal deze lijn bij warmtetoevoer in de richting naar O worden doorloopen. In O treedt de phase L op, doordat A en B smelten tot L. Stel, deze smelting gaat met uitzetting gepaard; de transformatie in O is dan (v-constant) $S_A + S_B + G \leftrightarrows L$.

Fig. 2.



O is minimum temperatuur voor de phase L. Bij warmteonttrekking gaat men omgekeerd van elk der curven IV, I en II op III over; daarbij verdwijnt L en ontstaat één der phasen S_A, S_B of G.

In tegengestelde richting zal, afhankelijk van het volumen, dat het systeem inneemt, S_A, S_B of G verdwijnen, waardoor men van III op I, II of IV overgaat.

Het punt O is dus weer overgangspunt voor de phase L bij warmteonttrekking onder constanten druk. In fig. 2 is echter niet de eenige mogelijke situatie van het overgangspunt voor S_A, S_B, L en G geteekend. Ik volsta met op te merken, dat een curve als IV ook terugloopend kan zijn; in dat geval heeft de omzetting $S_A + S_B \rightarrow L$ onder contractie plaats en daardoor wordt de transformatie in O: $S_A + S_B \leftrightarrows L + G$. Als dit zoo is, dan kan men uit O naar hoogere en naar lagere temperatuur telkens op 2 curven overgaan, al naarmate de eene of de andere phase verdwijnt. O is dan niet een overgangstemperatuur voor één enkele phase, maar wel in elke richting voor een systeem van 2 phasen. Andere quadrupelpunten kunnen in binaire stelsels ontstaan door de coexistentie van b. v. $S_A\, S_B\, L_1\, L_2$; $S_A\, G\, L_1\, L_2$; $S_A\, S'_A\, S_B\, G$ enz.

Vatten wij deze beschouwing over quadrupelpunten samen, dan resulteert: Een quadrupelpunt geeft in binaire stelsels de eenige waarden van p, t en x aan, waarbij 4 phasen uit twee componenten

in evenwicht kunnen zijn; in dat punt komen 4 curven samen overeenkomstig de 4 systemen van mono-variant evenwicht, die mogelijk zijn. Warmteonttrekking of warmtetoevoer heeft in een quadrupelpunt eene omzetting tengevolge, waaraan alle aanwezige phasen deelnemen. Is één der phasen verbruikt, dan gaat men uit het overgangspunt op één der curven over. Nu kan de vergelijking, die de omzetting in het overgangspunt uitdrukt aan beide zijden twee phasen, of één en drie phasen bevatten. In het eerste geval kan in beide richtingen één van twee phasen verdwijnen in het laatste geval verdwijnt in de eene richting één bepaalde phase, in de andere één der drie overige; welke, hangt van het volumen van het systeem af.

In dit laatste geval is het quadrupelpunt weer een waar overgangspunt voor die ééne phase.

Ingewikkelder nog zijn de verschijnselen in ternaire systemen. Hier geeft het overgangspunt de waarden van druk, temperatuur en concentraties aan, waarbij 5 phasen naast elkaar in evenwicht kunnen bestaan. Daar ik nu bij het hier volgend onderzoek uitsluitend oplosbaarheidsverschijnselen heb bestudeerd van partieelracemische verbindingen, welke eene volkomen analogie moeten vertoonen met de dubbelzouten uit de anorganische en de organische chemie, zooals b.v. astrakaniet en calcium-koperacetaat, zal ik hier alleen het quintupelpunt beschouwen, waarbij evenwicht heerscht tusschen de beide vaste enkelzouten, de oplossing en de gasphase.

Konden wij in binaire stelsels het quadrupelpunt en de curven, die in dat punt elkaar ontmoeten geven als eene $p\ t$ projectie der 4 curven voor mono-variant evenwicht met hun snijpunt, zooals die op het $p\ t\ x$ oppervlak in binaire stelsels zijn gelegen, thans krijgen wij één variabele meer.

Onafhankelijk veranderlijk zijn nu niet alleen p, t en x, maar p, t, x en y, waarin met x en y bedoeld zijn de concentraties aan elk der beide enkelzouten. Hunne evenwichtsvoorwaarden zouden op dezelfde wijze voortgaande dus in de ruimte slechts met behulp van 4 onderling loodrechte assen voor te stellen zijn, waartoe men de 4 dimensionale meetkunde zou moeten te hulp roepen.

Echter zijn de 5 curven van mono-variant evenwicht, die thans in het overgangspunt samen komen, ook in eene $p\ t$ figuur te demonstreeren, waarbij men dan te bedenken heeft, dat langs die curven twee concentraties voortdurend veranderen. Al naar gelang nu een dubbelzout naast oplossing zijn bestaansgebied naar hoogere temperaturen of naar lagere uitstrekt, wordt de voorstelling van het quintupelpunt door fig. 3 of door fig. 4 weergegeven.

Duiden wij de enkelzouten met A en B, het dubbelzout met D, de oplossing met L en de gasphase met G aan. Het is nu mogelijk, dat aan weerszijden van het overgangspunt twee en drie, of één en vier curven zijn gelegen. Na hetgeen voorafgegaan is bij de binaire stelsels zal het duidelijk zijn, dat alleen in het laatste geval het quintupelpunt een overgangspunt voor ééne enkele phase n.l. voor D kan zijn.

Fig. 3 heeft betrekking op astrakaniet, β-pipecolinebitartraat, hydrochinaldinebitartraat en dergelijke dubbelverbindingen, die eerst bij temperaturen, hooger dan het overgangspunt gelegen, naast één hunner bestanddeelen, oplossing en gas stabiel zijn.

Wij zullen nu in de eerste plaats de curven om het punt O beschouwen. De lijn I geeft het evenwicht aan tusschen de beide

Fig. 3.

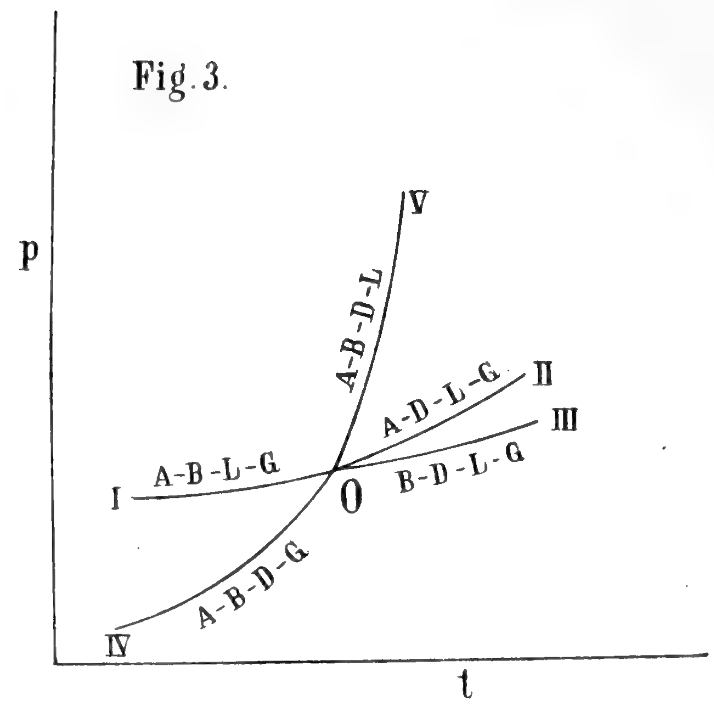
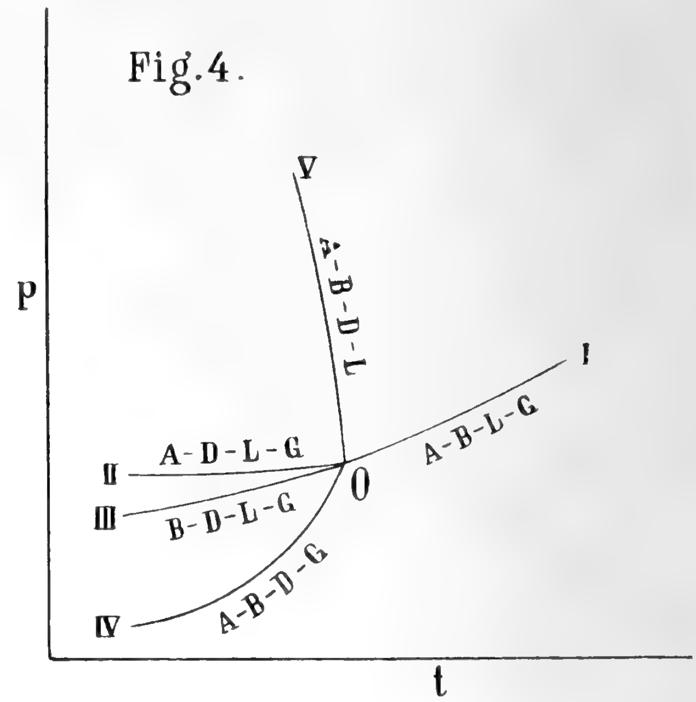


Fig. 4.



enkelzouten, de oplossing en de damp. Warmtetoevoer bij constanten druk voert ons van de lijn I in het veld tusschen I en IV, waarin de aan beide curven gemeenzame phasen i. c. A, B en G zijn gelegen. Derhalve $L \rightleftarrows A + B + G$. In het veld links van curve I bestaan naast elkaar óf $A + B + L$, óf $A + G + L$, óf $B + G + L$, in wier gebieden men komt al naarmate bij de warmteonttrekking onder constanten druk de omzetting $A + B + G \rightarrow L$ het verdwijnen van G, B of A ten gevolge heeft. Dit wordt beheerscht zoowel door het totaal volumen van het systeem, als door de onderlinge verhouding in de quantiteiten der componenten.

De lijn IV is de aaneenschakeling van evenwichten in het phasencomplex bestaande uit beide enkelzouten, dubbelzout en gas. Drukverlaging bij constante temperatuur of temperatuursverhooging bij

constanten druk voert ons in het veld van $B\ D\ G$ en $A\ D\ G$, derhalve wordt de omzetting aangegeven door $A + B \rightleftarrows D + G$.

De lijn V geeft het evenwicht aan tusschen de beide enkelzouten, dubbelzout en oplossing. Hierbij is de omzetting door temperatuursverandering $A + B \rightleftarrows D + L$.

De lijn II is de evenwichtscurve voor één der enkelzouten, dubbelzout, oplossing en damp. Hier is de omzetting $L + A \rightarrow D + G$, wat weer af te leiden is uit de phasen, die in de velden links en rechts van deze curve kunnen bestaan.

Evenzoo is curve III de lijn, die de evenwichten aangeeft tusschen het andere der beide enkelzouten, dubbelzout, oplossing en gas. De omzetting wordt hierop voorgesteld door $L \rightarrow B + D + G$.

Wat is nu de transformatie in O? In het quintupelpunt zijn naast elkaar in evenwicht A, B, D, L en G; bij warmtetoevoer (v-constant) zal men overgaan op één der curven II, III of V en dus zal B, A of G verdwijnen.

De omzetting zal dus zijn: $A + B + G \rightleftarrows D + L$ en zij zal ten einde zijn gekomen, als A, B of G is opgeteerd en men van uit O op één der curven, aan de rechter zijde daarvan, zal kunnen overgaan.

Omgekeerd zal bij afkoeling D of L verdwijnen en daardoor gaat men òf op de curven I òf IV over. Hieruit volgt, dat O niet een overgangstemperatuur voor één enkele phase is; wel in de ééne richting voor een systeem van twee phasen en in de andere richting voor een systeem van drie phasen.

Beschouwt men de systemen, die alleen tusschen I en IV en tusschen V en III kunnen bestaan, dan vindt men $A\ B\ G$ resp. $A\ D\ L$, $B\ D\ L$ en $D\ L\ G$.

Voor $A\ B\ G$ is O een maximumtemperatuur, voor $A + L$ daarentegen een minimumtemperatuur.

Voor het dubbelzout D is echter het punt O geen minimumtemperatuur, immers D kan naast zijne samenstellende bestanddeelen A en B met de gasphase G (curve IV) ook bij lagere temperaturen bestaan.

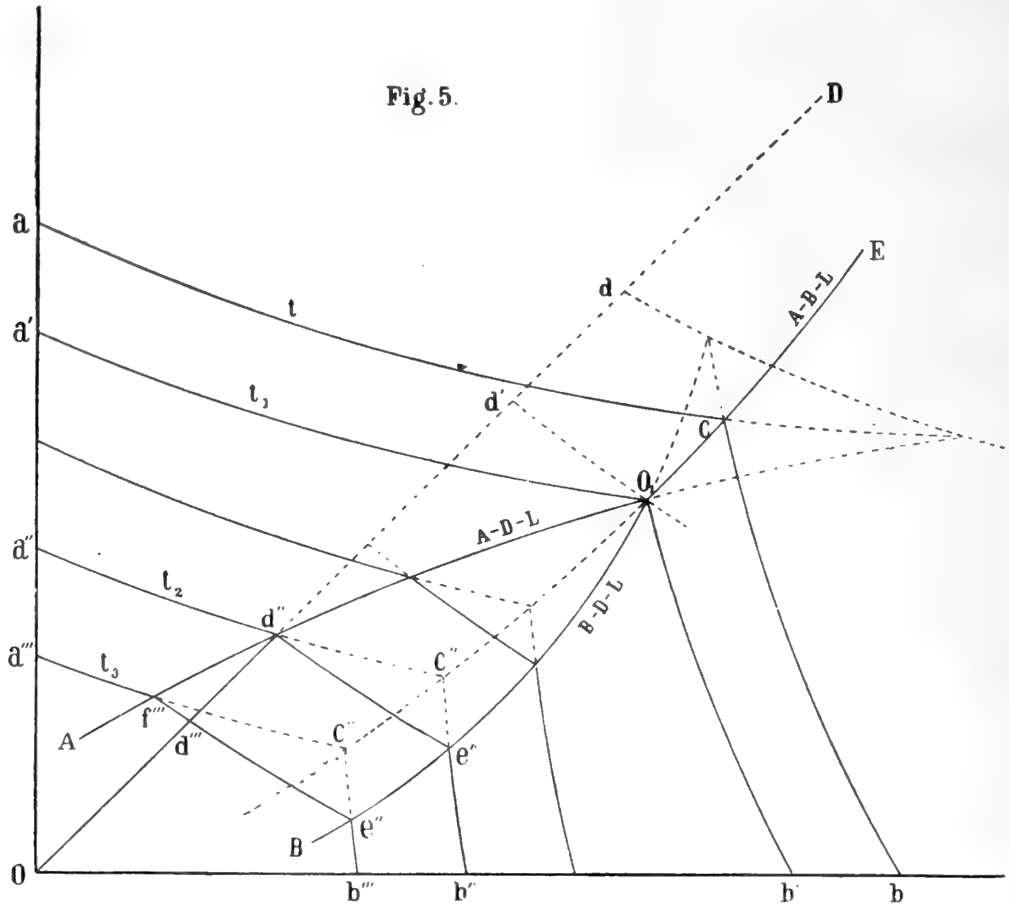
Geheel anders is het verloop der curven van mono-variant evenwicht in fig. 4, die het gedrag van caliumkoperacetaat, neutrale druivenzure strychnine, zure druivenzure brucine enz. weergeeft. Thans heeft men in het temperatuursgebied, beneden O gelegen, 2 curven n.l. voor de evenwichten $A\ D\ L\ G$ en $B\ D\ L\ G$, terwijl zich van O uit naar hoogere temperaturen de curve voor de phasen $A\ B\ L\ G$ uitstrekt.

De curve IV is geheel in overeenstemming en vergelijkbaar met

de gelijknamige curve in fig. 3. Curve V geeft de evenwichten der drie zouten naast hunne smelt bij verschillende temperaturen en drukken. Zij is in het geval van het caliumkoperacetaat experimenteel bepaald en terugloopend gevonden, doordat de omzetting $D \rightarrow A + B + L$, welke bij constanten druk door temperatuursverhooging plaats grijpt, eene contractie tengevolge heeft. Bij de bovengenoemde druivenzure zouten is zij niet langs proefondervindelijken weg vastgesteld, echter is daar haar verloop zeer waarschijnlijk eveneens in overeenstemming met fig. 4.

In deze figuur loopen dus van O uit 4 curven naar lagere temperatuur en éene naar hoogere. Daardoor wordt in O de trans-

Fig. 5.



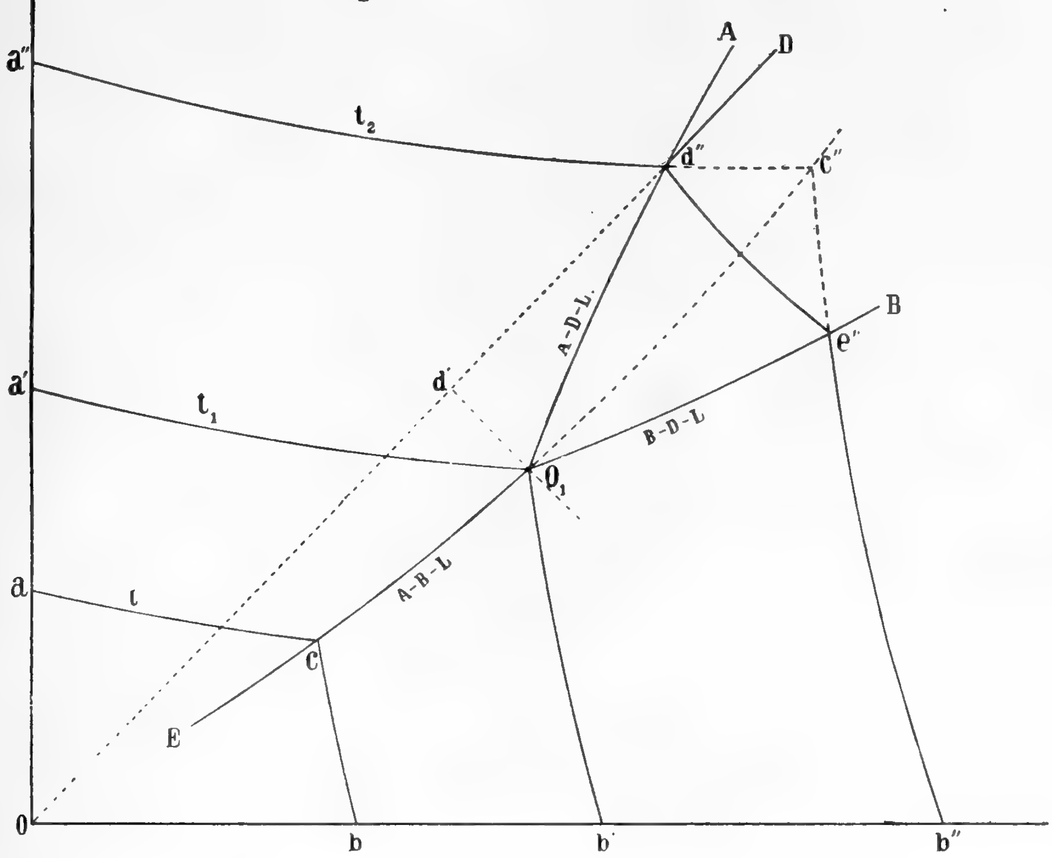
formatie uitgedrukt door het symbool $D \rightarrow A + B + L + G$, wat tot het onmiddellijke gevolg leidt, dat het punt O in waarheid eene overgangstemperatuur voor het dubbelzout is en wel de maximumtemperatuur, waarbij dit zout kan bestaan.

Uit bovenstaande volgt: het quintupelpunt, zooals dat zich bij waterhoudende dubbelzouten in het invariante evenwicht $A\ B\ D\ L\ G$ voordoet, is slechts een overgangspunt voor het dubbelzout, wanneer dit hooger gehydrateerd is dan de beide componenten samen, en zijne omzetting in de beide componenten en oplossing onder contractie plaats heeft. In alle overige gevallen kan het dubbelzout bij temperaturen zoowel hooger als lager dan die van het quintupelpunt bestaan.

Gaan wij nu in plaats van p en t als assen te gebruiken, over op een diagram, waarin x en y, (de concentraties van elk der beide enkelzouten in eene constante hoeveelheid van de 3 components samen) als coördinaten zijn gekozen, dan krijgen we in overeenstemming met de figuren 3 en 4 de figuren 5 en 6 [1]).

Deze figuren gelden voor constanten druk en zijn eene aaneenschakeling van doorsneden, bij een bepaalde temperatuur, van het $p\ t\ x\ y$ oppervlak, dat de evenwichtscondities in ternaire stelsels aangeeft. Legt men de temperatuur vast, dan zal men voor die constante p en t op genoemd oppervlak een lijn krijgen als voor-

Fig. 6.



gesteld is door $a\ c\ b$ in figuur 5. Het punt a geeft de oplosbaarheid

---

[1]) Er zij hier de volgende opmerking ingelascht. Sinds DUTILH's dood is het derde deel der Heterogene Gleichgewichte van de hand van Prof. SCHREINEMAKERS verschenen. Daarin is in volkomen algemeenheid nagegaan welke gevallen mogelijk zijn en daarbij zijn meer mogelijkheden voor den dag gekomen, dan hier behandeld zijn. De twee typen der fig. 3 en 4 zijn dan ook niet de eenig denkbare en aan de fig. 5 en 6 moet nog een derde toegevoegd, die men bij SCHREINEMAKERS l.c. pag. 164 als Fig. 76 vindt aangegeven. Of de behandeling van dat type in dit verband van belang zou zijn, wil ik hier in het midden laten. Maar in elk geval scheen mij een uitbreiding van dit hoofdstuk (mede in overleg met de commissie uit de Akademie) niet gewenscht en bepaal ik mij hier dus tot de verwijzing naar genoemd werk. Kort voor DUTILH's dood heeft hij met Prof. SCHREINEMAKERS overleg gepleegd over deze zaken en waarschijnlijk heeft slechts zijn onverwachte dood hem belet dit hoofdstuk in overeenstemming met die besprekingen te wijzigen. KRUYT.

in eene constante hoeveelheid oplossing van den component A aan bij de temperatuur t en 1 atmosfeerdruk. Op dezelfde wijze wijst het punt b de oplosbaarheid van de component B onder dezelfde omstandigheden aan. De lijn $a\,c$ stelt nu de verandering voor van de oplosbaarheid van A bij toevoeging van B aan de oplossing. Ditzelfde geldt voor de lijn $b\,c$, wat betreft de oplosbaarheid van B in oplossingen, die reeds A bevatten. Eindelijk geeft punt c het gehalte aan A en B aan van de met beide zouten verzadigde oplossing.

Kan echter bij eene temperatuur t_3 (zie fig. 5) het dubbelzout onontleed naast zijne verzadigde oplossing bestaan, dan zal de oplosbaarheidsisotherm voor die temperatuur uit drie takken gevormd worden. Op de lijn $a'''\,f'''$ is de component A „Bodenkörper", op den tak $f'''\,e'''$ de dubbelverbinding, op de lijn $e'''\,b'''$ de component B. d''' geeft de oplosbaarheid der dubbelverbinding aan, als we aannemen, dat D eene aequimoleculaire verbinding van A en B is. Bij temperaturen hooger dan t_3 blijft voorloopig de isotherm uit drie takken bestaan. Hoe hooger echter de temperatuur wordt, des te kleiner wordt de oplosbaarheidscurve van de dubbelverbinding en des te meer breiden zich die der enkelzouten uit, totdat bij eene temperatuur t_1 de oplosbaarheidscurve der dubbelverbinding tot een enkel punt O_1 is ineengekrompen. In dit punt O_1 en bij die temperatuur t_1 kan dus blijkbaar eene oplossing tegelijk verzadigd zijn aan A, B en D. Wij hebben hier dan vier phasen van drie componenten, de druk is vastgelegd en dus is het evenwicht nonvariant. — Dit punt O_1 is strikt genomen niet hetzelfde als het punt O in fig. 4; het komt overeen met het snijpunt van curve V in die figuur met de lijn voor $p = 1$ atm., welk snijpunt in temperatuur zeer dicht bij O gelegen zal zijn. [1])

De oplossing, wier gehalte aan A en B door de coördinaten van het punt O_1 in fig. 5 wordt bepaald, kan derhalve verkregen worden, door bij de temperatuur t_1 hetzij $A + D$, hetzij $B + D$, hetzij $A + B$ tezamen tot verzadigde vloeistof op te lossen. — Merkwaardig is nog de isotherm der temperatuur t_2. Hier snijden de lijnen $a''\,d''$ en $d''\,e''$ elkaar juist op de lijn, die den assenhoek midden doordeelt, zoodat de oplossing, die door d'' wordt aangegeven, verkregen kan worden, door bij t_2 het oplosmiddel aan D of aan $D + A$ te verzadigen. Tot deze temperatuur toe kan het dubbelzout verzadigde oplossingen geven, waarin de verhouding der

[1]) Men zou misschien, ter vermijding van misverstand een punt als O_1 in fig. 5 en fig. 6 het overgangspunt bij 1 atmosfeer druk kunnen noemen.

gehalten aan A en B met de samenstelling van D overeenkomt (zooals d'' bij $t^0{}_3$, enz.). Boven die temperatuur is D niet meer naast zijne oplossing stabiel, maar lost op onder gelijktijdige afzetting van de minst oplosbare zijner componenten. [1]) — Verzadigt men derhalve bij temperaturen boven t_2 het oplosmiddel met het dubbelzout, dan krijgt men niet eene oplossing, wier innerlijke samenstelling met een punt der lijn OD overeenkomt, maar eene oplossing, waarvan het gehalte aan A en aan B wordt weergegeven door een punt der lijn $d'' O_1$. Alle punten der lijn $O_1 A$ geven dus bij verschillende temperaturen de aan $A + D$ verzadigde oplossingen aan. Ditzelfde geldt voor de lijn $O_1 B$ met betrekking tot de aan $B + D$ verzadigde oplossingen. Eindelijk stellen de punten der lijn $O_1 E$ de bij verschillende temperaturen aan $A + B$ verzadigde oplossingen voor.

Vertragingsverschijnselen uitgesloten, zijn dus in de figuren 5 en 6, alleen de getrokken lijnen realiseerbaar. Heeft echter de vorming van het dubbelzout uit zijne beide componenten, resp. zijne ontleding, onder afzetting van de minst oplosbare zijner componenten naast zijne waterige oplossing bij temperaturen, waar die oplossing niet meer stabiel is, uiterst langzaam plaats, dan zal het duidelijk zijn, dat dus ook de verlengingen van de lijn $E O_1$, m. a. w. punten als c'' enz. en de voortzetting van $O d''$ d. z. de punten d', d enz. te verwezenlijken zijn.

Dit doet zich bij partiëele racematen als b.v. druivenzure strychnine inderdaad somtijds voor. In het temperatuurgebied, waar het partiëele racemaat „Bodenkörper" kan zijn van oplossingen met wisselende samenstellingen [2]), kunnen nochtans door vertraging in de vorming der dubbelverbinding uit hare componenten de oplosbaarheidscurven $a''' f'''$, $a'' d''$ en $b''' e'''$, $b'' e''$ resp. tot hunne snijpunten c''' en c'' worden gerealiseerd en evenzoo kunnen punten der lijnen $d'' D$ (fig. 5) en $O d'$ (fig. 6) worden bepaald, hoewel deze eigenlijk moesten gelegen zijn op de lijnen $d'' O_1$.

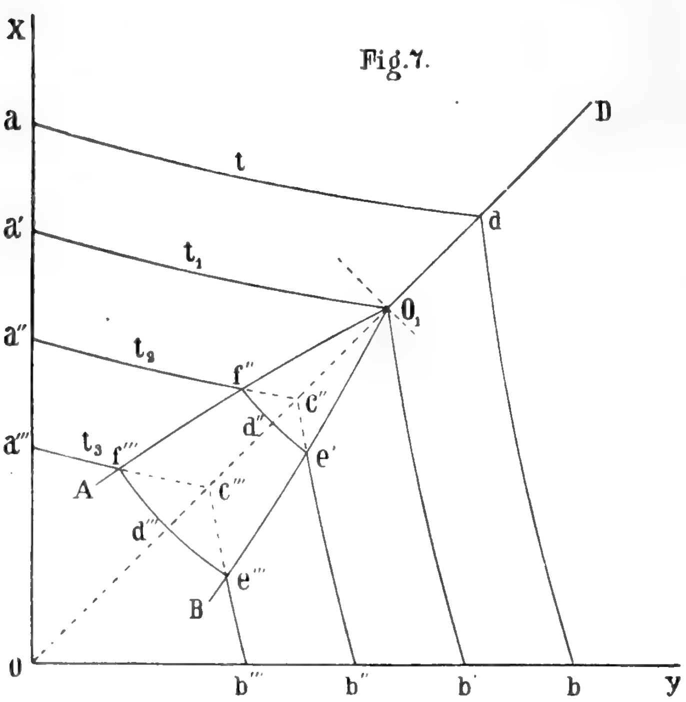
Deze figuren ondergaan geen groote verandering, indien men in plaats van op eene constante hoeveelheid oplossing, x en y betrekt op eene constante hoeveelheid oplosmiddel, als men tenminste, zooals bij de strychnine- en brucinezouten der beide wijnsteenzuren en hunne dubbelverbindingen (i. c. de partiëele racematen), te doen heeft met verbindingen, die eene kleine oplosbaarheid in water bezitten en men hun gedrag in een dergelijk oplosmiddel bestudeert.

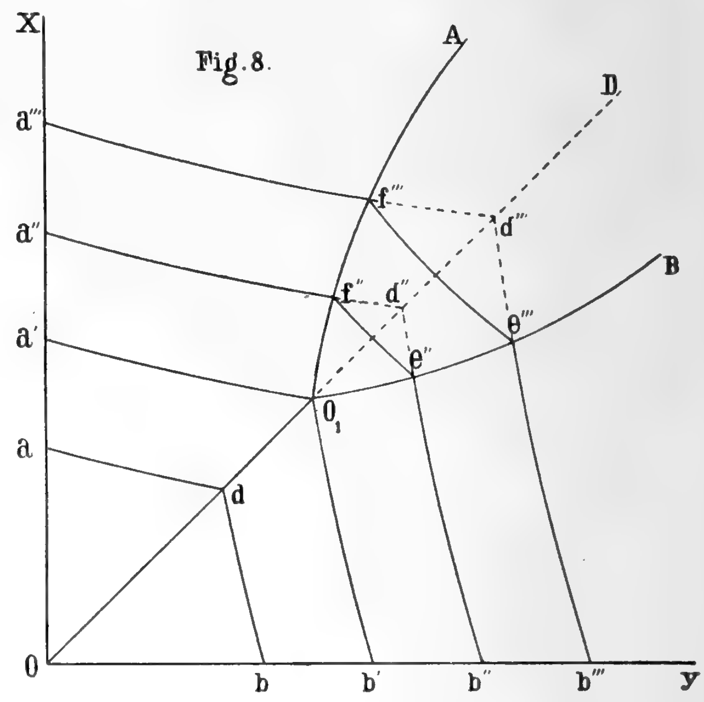
[1]) Hierbij is natuurlijk ondersteld, dat het snijpunt der oplosbaarheidstakken ligt aan de zijde van de concentratieas der meest oplosbare componente, zooals in normale gevallen ook steeds het geval is.

[2]) Men vergelijke de isothermen voor de temperaturen t_2 en t_3 in fig. 5.

Ten slotte zij nog in verband met fig. 5 en 6 opgemerkt, dat het temperatuursgebied gelegen tusschen het overgangspunt en de hoogste resp. laagste temperatuur, waarbij het dubbelzout naast zijne verzadigde oplossing kan bestaan, (welk gebied in fig. 5 resp. fig. 6 is gelegen tusschen de temperaturen t_1 en t_2), den naam heeft verkregen van overgangstraject. [1])

Partiëel-racemische verbindingen als druivenzure-strychnine, die zich naast hunne verzadigde waterige oplossing van zekere temperatuur af kunnen splitsen in *d*-wijnsteenzure-strychnine + *l*-wijnsteenzure-strychnine en wier componenten in alle physische eigenschappen verschillen, zijn volkomen vergelijkbaar in de stabiliteitsverschijnselen naast hunne verzadigde oplossingen in water, met anorganische





dubbelzouten als het astrakaniet, $Na_2 Mg (SO_4)_2 . 4 H_2 O$, dat uiteen kan vallen in $Na_2 SO_4 . 10 H_2 O$ en $Mg SO_4 \; 7 H_2 O$ onder opname van water [2]). Eenvoudiger worden de oplosbaarheidscurven, wanneer men te doen heeft met eigenlijke racematen als natrium ammoniumracemaat, dat zich beneden 27° splitst in *d*-natrium ammonium tartraat + *l*-natrium ammonium tartraat. Hier zijn de componenten, behalve in kristalvorm en in hun gedrag ten opzichte van het gepolariseerde licht, in alle physische eigenschappen en dus ook in hunne oplosbaarheid in hetzelfde medium aan elkaar gelijk. Dit heeft tot gevolg, dat wanneer men hunne oplosbaarheidscurven in een x-y diagram teekent, deze geheel symmetrisch ten opzichte van de lijn, die de samenstelling van het racemaat aangeeft, zullen zijn. Dit is in de figuren 7 en 8 weergegeven.

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[1]) „Umwandlungsintervall". Verg. Meyerhoffer. Zeitschr. f. phys. Chemie **5**, 97 (1891).
[2]) Vgl. Bakhuis Roozeboom, Zeitschr. f. phys. Chem. **28**, 494 (1899).

Deze figuren staan in dezelfde betrekking tot elkaar als fig. 5 en fig. 6. In figuur 7 is het geval voorgesteld, dat het racemaat [1]) bij hoogere temperaturen zich naast zijne verzadigde oplossing in zijne beide componenten *splitst*, terwijl in fig. 8 is ondersteld, dat het racemaat juist eerst bij hoogere temperaturen *in evenwicht kan zijn* met zuivere racemaat-oplossingen. In beide figuren snijden de oplosbaarheidscurven van de d en de l-verbinding in het gebied, waar het conglomeraat (d. i. $D + L$) bestendig is, elkaar steeds op de lijn OD, die weer aequimoleculaire hoeveelheden van D en L in de oplossing aangeeft. Door toevoeging van D aan de verzadigde oplossing van L wordt de oplosbaarheid van dit lichaam op gelijke wijze veranderd als bij toevoeging van L aan de verzadigde D-oplossing de oplosbaarheid van dit zout gewijzigd wordt. Dit heeft tot gevolg, dat de geheele figuur verkregen kan worden door spiegeling van de helft xOD op de lijn OD. Alle $(D + L)$ oplossingen liggen dus op de lijn OD.

Bij de temperatuur t, wordt in beide figuren het racemaat naast zijne verzadigde oplossing bestendig. De overigens nog geheel metastabiele oplosbaarheidscurve van het racemaat gaat hier dus door O, en daarom stelt O weer de samenstelling eener oplossing voor, die verkregen kan worden door bij t°, het oplosmiddel hetzij met $D + L$, met $D + R$, of met $L + R$ te verzadigen.

Bij de temperaturen t_2, t_3 enz. kan het racemaat met oplossingen van verschillende gehalte aan A en B in evenwicht zijn en deze samenstellingen zijn begrensd bij elke temperatuur door twee punten, resp. der lijnen $O_1 A$ en $O_1 B$. De oplosbaarheidscurve van het racemaat is ook weer geheel symmetrisch ten opzichte van de lijn OD, daar de invloed van tevoren opgelost D of L op de oplosbaarheid van R dezelfde zal zijn. Het essentieele verschil in oplosbaarheidsverschijnselen van eigenlijke racematen met partieel-racemische verbindingen en alle andere dubbelzouten is dus hierin gelegen, dat de oplosbaarheidscurven der racematen volkomen symmetrisch zullen zijn ten opzichte van de lijn, die met hunne samenstelling overeenkomt (i. c. de lijn OD), terwijl de oplosbaarheidscurven der laatsten asymmetrisch moeten verloopen. Daardoor kon bij racematen een ontledingstraject niet bestaan; daarentegen moet dit bij partiëele racematen en alle andere dubbelverbindingen steeds voorkomen. Hoewel door LADENBURG en zijne leerlingen tal van oplosbaarheidsbepalingen bij partieel-racemische verbindingen zijn uitgevoerd, met

[1]) Wij zullen in het vervolg de beide componenten kortweg door D en L en de verbinding van beide door R aangeven.

de bedoeling om daardoor het overgangspunt vast te stellen, meen ik, dat zij dit laatste niet voldoende in het oog hebben gehouden. Zij hebben steeds het overgangspunt van het partiëele racemaat allereerst geconstateerd als de temperatuur, bij welke de oplosbaarheid van de beide componenten samen gelijk werd aan die van het racemaat alleen. Dit is juist, wanneer men met de eigenlijke racematen te doen heeft, (wat in het bovenstaande is aangetoond). Betreft het onderzoek echter partieel-racemische dubbelverbindingen, dan moet men figuur 5 of figuur 6 beschouwen, en dan blijkt daaruit, dat zij dus punten als d''', d'', d', d met de punten als c''', c'', O_1, c hebben vergeleken. Weliswaar hebben zij, wanneer zij meenden het overgangspunt te hebben gevonden, de oplosbaarheid van $R + D$ en van $R + L$ bij die temperatuur eveneens bepaald, en deze dan gelijk aan die van R en van $D + L$ bevonden, zoodat dus $(R + D)$, $(R + L)$ en $(D + L)$ in oplosbaarheid aan elkaar gelijk waren. Dit is de voorwaarde voor het overgangspunt; maar een louter toeval blijft de gelijkheid der oplosbaarheid van R aan die der splitsingsproducten bij die temperatuur en principieel onjuist is het, het overgangspunt te gaan zoeken, door de snijding te bepalen van de lijnen, die de oplosbaarheid van R en van $D + L$, in hare afhankelijkheid van de temperatuur, aangeven.

Nog een andere voorstellingswijze van de verhouding van dubbelverbindingen t.o.v. hunne splitsingsproducten in een oplosmiddel is mogelijk. [1]) Men kan nl. als abscissen de temperatuur uitzettende, als ordinaten de totaal-oplosbaarheid der monovariante systemen uitzetten, d. i. de hoeveelheid opgeloste stof der verschillende systemen, waarin twee vaste phasen optreden.

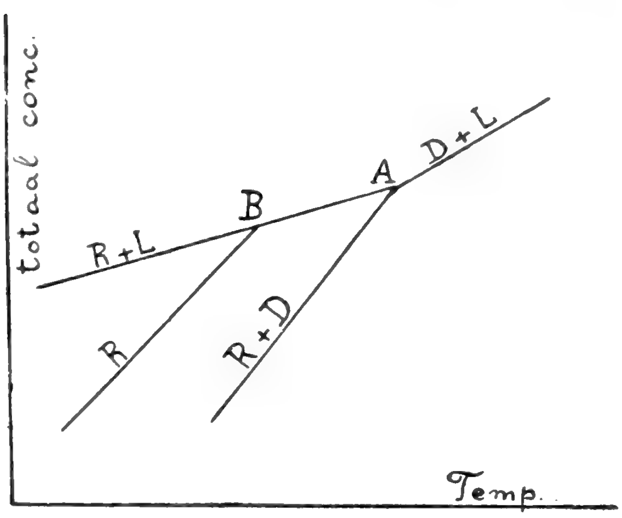

Fig. 9.

Geeft men dan ook nog de oplosbaarheidslijn voor de in ons geval (partieel racemische) dubbelverbinding, dan zullen ook in deze voorstelling twee merkwaardige punten optreden, (zie fig. 9) nl. A, ook hier de snijding van $R + L$, $R + D$ en $D + L$ curve en bovenste grens van het overgangstraject, en B, snijpunt van R en $R + L$ tak en onderste grens van dat traject. Formeel is deze voorstellings-

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[1]) Zie b.v. MEYERHOFFER, Gleichgewichte der Stereomeren, Leipzig—Berlin 1906, S. 48.

wijze juist en kunnen inderdaad de punten A en B beschouwd worden als de snijpunten dezer zoo gemakkelijk te bepalen lijnen. Maar toch zal hier, bij vergelijking met fig. 5 één groot bezwaar aan den dag treden. Trekt men in fig. 5 een lijn die loodrecht staat op de lijn OD dan hebben alle oplossingen, die door de punten van die lijn worden voorgesteld, dezelfde totaal-concentratie, zooals gemakkelijk in te zien is.

Wanneer nu eens de punten zooals f''' en e''', punten dus der $R+L$ resp. $R+D$ curve, eener zelfde isotherm, op zulk een loodlijn op OD lagen, dan zou bij de betreffende temperatuur in fig. 9 de $R+L$ en $R+D$ tak samenvallen. Men kan dus bij deze methode van MEYERHOFFER snijpunten hebben zonder dat die een identiek worden van phasen beduiden, zooals in de punten A en B *wel* het geval is. En aan dit bizonder voorbeeld zal men het algemeene bezwaar begrijpen, dat deze voorstellingswijze haar waarde kan doen verliezen: hoe meer de richting der racemaattakken in de isotherm van fig. 5 (eigenlijk van verbindingslijnen zooals $e''' f'''$) den 45°-stand nadert, zooveel kleiner zal het verschil der bedoelde oplosbaarheden zijn en dus zooveel dichter zullen de curven $R+D$ en $R+L$ in fig. 9 bij elkander komen te liggen, en derhalve *zooveel moeilijker* zal het snijpunt A te bepalen zijn. Eenzelfde bezwaar kan m. m. voor B optreden, vaak voor beiden gelijktijdig.

LADENBURG en DOCTOR hebben de samenstellingen hunner aan $(D+L)$ verzadigde oplossingen bepaald door polarimetrische analyse. Zij bepaalden eerst het specifieke draaiingsvermogen van elk hunner stoffen, dus van D, L en R in hunne afhankelijkheid van de concentratie. Daarna werden de afdampingsresidus der verzadigde $(D+L)$ oplossingen bij verschillende temperaturen op hun draaiingsvermogen onderzocht in oplossingen, waarvan de concentratie nauwkeurig bekend was. De aldus gemeten draaiingshoeken α, werden nu aldus gesplitst $\alpha = \alpha_d + \alpha_l$, waarin α_d en α_l, de draaiingshoeken voorstellen, die D en L elk op zich zelf in de oplossing van het zoutmengsel zouden veroorzaken.

De concentratie aan D werd nu p_d gesteld, die aan L p_l en die aan $(D+L)$ p; dan is $p = p_d + p_l$. Voorts zij l de lengte der polarimeterbuis en s het spec. gew. der oplossing.

Derhalve [1]
$$\alpha = \frac{[\alpha_d] p_d \, l.s}{100} + \frac{[\alpha_l] p_l \, l.s}{100};$$

[1]) Voor $[\alpha]$ moet men feitelijk $[\alpha]_D$ lezen (natriumlicht). Ter voorkoming van verwarring is deze D in de formules weggelaten.

$$\frac{100\,\alpha}{l\,.\,s} = [\alpha_d]\,p_d + [\alpha_l]\,(p - p_d)\,. \quad . \quad . \quad . \qquad \text{I}^{1)}$$

$[\alpha_d]$ in zijne verandering met de concentratie wordt voorgesteld door

$$[\alpha_d] = -A + B\,p_d$$

evenzoo geldt voor $[\alpha_l]$

$$[\alpha_l] = -A_1 + B_1\,p_l = -A_1 + B_1\,(p - p_d)$$

In deze beide vergelijkingen zijn de A's en de B's bepaald door een uitgebreide serie bepalingen.

Substitueert men nu in vergelijking I bovenstaande waarden voor $[\alpha_d]$ en $[\alpha_l]$ dan krijgt men:

$$\frac{100\,\alpha}{l\,.\,s} = -A\,p_d + B\,p^2{}_d - A_1\,(p - p_d) + B_1\,(p - p_d)^2.$$

Men komt dus tot eene vierkantsvergelijking in p_d, waaruit deze en derhalve ook p_l is op te lossen.

Deze rekenwijze is evenwel niet als de juiste te aanvaarden, want oplossingen van $(D + L)$ zijn homogene phasen, waarin krachtens de wet van de massawerking een evenwicht moet worden aangenomen tusschen D, L en R, welk evenwicht bij eene bepaalde temperatuur door de concentraties is bepaald. Wanneer men dus den afgelezen draaiingshoek eenvoudig beschouwt als samengesteld uit de draaiing der D-moleculen met die der L-moleculen in het opgeloste mengsel, dan begaat men een fout, door niet de draaiing, die de R-moleculen veroorzaken, tevens in rekening te brengen.

Is derhalve de totaal-concentratie p, de partiaal-concentratie voor de D-moleculen p_d en die voor de L-moleculen p_l, dan komt aan de R-moleculen eene partiaal-concentratie $(p - p_d - p_l)$ toe en dan worden de bovenstaande vergelijkingen als volgt gewijzigd:

$$\alpha = \alpha_d + \alpha_l + \alpha_r$$

$$\alpha = \frac{[\alpha_d]\,p_d\,l\,.\,s}{100} + \frac{[\alpha_l]\,p_l\,l\,.\,s}{100} + \frac{[\alpha_r]\,(p - p_d - p_l)\,l\,.\,s}{100}$$

$$\frac{100\,\alpha}{l\,.\,s} = [\alpha_d]\,p_d + [\alpha_l]\,p_l + [\alpha_r]\,(p - p_d - p_l)$$

[1]) Verg. G. Doctor, Inaug. Diss. Breslau, p. 62 [1899.]

Voegt men hier weer in $[\alpha_d]$, $[\alpha_l]$ en $[\alpha r]$ in hunne afhankelijkheid van de concentratie der oplossing dan krijgt men opnieuw eene vierkantsvergelijking, ditmaal echter met *twee* onbekenden nl. p_d en p_l, waarvan één enkele polarimeter-bepaling niet de oplossing kan geven. Bij de berekening der polarimetrische analyse-methode van LADENBURG en DOCTOR en de verbeterde rekenwijze, die ik daarop liet volgen, is als vereenvoudiging bovendien nog ondersteld, dat in oplossingen, waarin D-, L- en R-moleculen naast elkaar voorkomen, deze op de grootte van elkaars optisch draaiingsvermogen geen invloed zullen uitoefenen en dat op zulke oplossingen nog de tevoren experimenteel bepaalde formules voor het specifieke draaiingsvermogen van elk der zouten op zich zelf in hunne afhankelijkheid van de concentratie der oplossing mogen worden toegepast. Of dit juist is, moet nog bovendien worden nagegaan. Het zal echter duidelijk zijn, dat de weg, dien LADENBURG en DOCTOR gevolgd hebben, om de innerlijke samenstellingen hunner verzadigde $(D+L)$-oplossingen te leeren kennen, naar alle waarschijnlijkheid niet de juiste is geweest. Het was dus, vooral waar hunne uitkomsten in sommige gevallen verrassend en a priori onwaarschijnlijk waren te noemen, wenschelijk eene andere methode te bedenken, wier resultaten met die van LADENBURG en DOCTOR konden worden vergeleken, en die minder aan bedenkingen onderhevig is. Welke die methode geweest is, zal in het experimenteele gedeelte van dit proefschrift nader worden uiteengezet.

In dit hoofdstuk werd de overgangstemperatuur van dubbelverbindingen in ternaire stelsels eenigszins uitvoerig besproken, omdat slechts met de kennis daarvan het gedrag van partieel-racemische lichamen behoorlijk kan worden bestudeerd. Ik kan er thans toe overgaan, in het volgende hoofdstuk een litteratuur overzicht te geven van de partiëele racemie, waarbij naar volledigheid gestreefd is, en daarin tegelijk wijzen op de theoretische fouten, in de conclusies, uit vorige onderzoekingen getrokken.

HOOFDSTUK III.

DE PARTIËELE RACEMIE.

Historisch Overzicht.

Hoewel de splitsing van racemische zuren met behulp van actieve basen, en die van racemische basen door actieve zuren in tal van gevallen gelukt is, zijn er toch een zeker aantal gevallen bekend, waarbij pogingen, om langs dezen weg eene scheiding tot stand te brengen, mislukten. Zeer dikwijls werden die gevallen in de litteratuur niet vermeld en riep men, waar de eene actieve stof geen resultaat gaf, eene andere te hulp, om het vooropgestelde doel te bereiken. Naar de oorzaak, waardoor de splitsing mislukte, werd niet gezocht; of, zoo dit al gedaan werd, was eene verkeerde verklaring dikwijls het gevolg daarvan.

Zoo vermeldt Pictet [1]), dat eene poging, om het appelzuur, uit fumaarzuur verkregen, door middel van cinchonine in zijne optische antipoden te scheiden, niet gelukte. Door gefractioneerde kristallisatie was het niet mogelijk, twee verschillende lichamen te krijgen. De waterige oplossing gaf bij indampen, steeds hetzelfde cinchoninezout, dat een standvastig smelttraject (135°—140°) bezat en bij ontleding met ammonia telkens inactief zuur terugleverde.

Pictet trachtte dit verschijnsel te verklaren, door aan te nemen, dat dit appelzuur een analogon zou zijn van het antiwijnsteenzuur. Het is duidelijk, dat de structuur van het appelzuur een dergelijke isomeer niet toelaat.

Bremer [2]) constateerde hetzelfde verschijnsel toen hij zijn, door reductie uit druivenzuur verkregen, appelzuur met cinchonine wilde splitsen. Het gelukte hem ten slotte toch eene scheiding tot stand

¹) Ber. d. d. chem. Ges. **14**, 2648 (1881).
²) Ibid. **13**, 351 (1880).

te brengen, door de oplossing met *l*-appelzuur-cinchonine te enten. Hoogst merkwaardig is het, dat daarbij niet *l*-appelzuur-cinchonine uitkristalliseerde. Integendeel, *d*-appelzuur-cinchonine zette zich uit de oplossing af.

Intusschen heeft KRANNICH [1]) later opnieuw het gedrag van het appelzuur ten opzichte van cinchonine bestudeerd en is daarbij tot eene conclusie gekomen, die afwijkt van de gevolgtrekking, welke uit PICTET's proeven is te maken. Hij bracht eene heete oplossing van zuur-appelzuur-cinchonine op het waterbad tot kristallisatie door deze oplossing met kristallen van *d*-appelzuur-cinchonine te enten. Terstond scheidde zich eene groote hoeveelheid in kristallen af. De uitgekristalliseerde verbinding bleek bij ontleding met ammonia en verwijdering van het neergeslagen cinchonine, een zout van het rechtsdraaiende appelzuur te zijn geweest, want de aldus verkregen oplossingen draaiden het polarisatievlak naar rechts. Hieruit meende KRANNICH het besluit te mogen trekken, dat *i*-appelzuur-cinchonine niet bestaat.

Ik meen tegen deze conclusie te mogen aanvoeren, dat KRANNICH op deze wijze mogelijk de oplossing door te enten heeft gedwongen, de verbinding van het *d*-zuur af te zetten, terwijl toch misschien het partieel-racemische lichaam onder omstandigheden van evenwicht het lichaam is, dat bij de temperatuur van het waterbad de naast de oplossing stabiele phase is.

Voor het mislukken der scheiding zonder enting, geeft hij echter geene verklaring. Het is nu de groote verdienste van LADENBURG geweest, dat hij dit proces nader heeft bestudeerd, en de oorzaak heeft kunnen aanwijzen waardoor splitsingen langs dezen weg vaak een ongunstig resultaat kunnen opleveren.

Toen hij het β-pipecoline door middel van het bitartraat dezer base trachtte te splitsen [2]) in zijne beide actieve componenten, liet hij dit zout op het waterbad uitkristalliseeren. Uit de afgescheiden kristallen werd de base vrij gemaakt; zij bleek echter inactief te zijn. Hij kwam nu op het denkbeeld, het bitartraat zich bij lagere temperatuur te laten afscheiden en zegt hierover [3]): „Bei den ersten Versuchen erwies sich die Base als optisch inaktiv, weil die Kristallisation auf dem Wasserbade geschah, und diese Temperatur zu hoch war, d. h. die Umwandlungstemperatur niedriger liegt, als diejenige, bei welcher die Ausscheidung der Kristalle erfolgte.”

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[1]) Inaug. Diss. Breslau, 1901.
[2]) Ber. d. d. Chem. Ges. **27**, 75 (1894).
[3]) l. c.

Hij verklaart dus het negatieve resultaat zijner scheiding door aan te nemen, dat zich bij hoogere temperatuur uit de oplossing eene verbinding van het inactieve pipecoline met *d*-wijnsteenzuur afzet, eene verbinding derhalve van twee lichamen, die in configuratie slechts voor een deel elkaars spiegelbeeld zijn nl. van *d*-β-pipecoline-*d*-bitartraat met de overeenkomstige verbinding van *l*-β-pipecoline.

Eene dergelijke verbinding is, indien zij zich bij hoogere of bij lagere temperatuur naast hare verzadigde oplossing in hare zooeven genoemde bestanddeelen splitst, een volledig analogon van de door Van 't Hoff en zijne medewerkers bestudeerde anorganische dubbelzouten. Zij bezit dan naast oplossing eene overgangstemperatuur, zooals die in Hoofdstuk II besproken is, een overgangstraject en ten slotte een temperatuursgebied, waarbij zij naast hare verzadigde oplossing onontleed kan bestaan. Volgens deze theorie ontstaat er een nauw verband tusschen de spontane splitsing door kristallisatie (de natrium-ammoniumtartraten) en die met behulp van optisch actieve (zoutvormende) stoffen. Het gelukken van beide hangt af, of men beneden het overgangspunt, resp. het overgangstraject, de kristallisatie tot stand doet komen; dit traject is, (zooals we in Hoofdstuk II gezien hebben) voor de eigenlijke racematen nul, daar bij hen beide componenten gelijke oplosbaarheid bezitten; vandaar, dat hier werken beneden het overgangs*punt* voldoende is. De ongelijkheid van alle physische eigenschappen van lichamen met gedeeltelijke gelijke spiegelbeeld-configuratië, dus ook van hunne oplosbaarheid, is de oorzaak, dat bij hen een overgangs*traject* ontstaat.

De verschillende oplosbaarheid der actieve componenten biedt hier echter juist het groote voordeel, dat in de methode is gelegen. Terwijl immers bij de methode der spontane kristallisatie de beide actieve splitsingsproducten in gelijke hoeveelheden zijn uitgekristalliseerd en men de kristallen nauwlettend op grond van hunne verschillend gelegen hemiëdrische vlakken moet uitzoeken, treft men nu het groote voordeel aan, dat de minst oplosbare component zich het eerst en het rijkelijkst afzet, zoodat uit de eerste fractie der kristallisatie meestal het eene der beide bestanddeelen van de oorspronkelijke inactieve verbinding zich laat afzonderen.

Nu kunnen zich bij partiëele racemie twee gevallen voordoen:

1°. de dubbelverbinding (p r) strekt haar bestaansgebied naast oplossing naar *hoogere* temperaturen uit, zoodat het overgangspunt de *laagste* temperatuur is, waarbij zij naast oplossing kan bestaan;

2° het bestaansgebied breidt zich naast oplossing naar *lagere* temperaturen uit, in welk geval het overgangspunt de *maximum*

temperatuur aangeeft, waarbij de dubbelverbinding naast oplossing op den bodem kan voorkomen. [1])

Het eerste geval doet zich voor in het *i*-β-pipecoline-bitartraat [2]) en het *i*-tetrahydrochinaldine-bitartraat. [3])

Het tweede geval heeft tot voorbeeld druivenzure-strychnine [4]) en brucine-biracemaat [5]). Al deze voorbeelden zijn door LADENBURG en zijne medewerkers gevonden.

Reeds vóór LADENBURG's onderzoekingen in het gebied der partiëele racemie, had EMIL FISCHER [6]) nagegaan, of lichamen, wier structuurformules gedeeltelijk elkaars spiegelbeeld zijn, zooals bijv.

d-gluconzuur		en *l*-mannonzuur	
	COOH		*COOH*
	HCOH		*HCOH*
	OHCH		*HCOH*
	HCOH		*HOCH*
	HCOH		*HOCH*
	COOH		*COOH*

neiging vertoonen, om zich met elkaar tot verbindingen te vereenigen. Hij vond, dat dit nòch bij de zooeven genoemde zuren nòch bij de calciumzouten van *l*-mannon- en *d*-gluconzuur het geval was, want uit een mengsel van gelijke moleculaire hoeveelheden der genoemde stoffen, in water opgelost, scheidde zich in beide gevallen slechts één der stoffen uit de oplossing af.

Ook ASCHAN [7]) vond, dat kamferzuur en iso-kamferzuur zich niet vereenigen en ditzelfde constateerde LIEBERMANN [8]) bij kaneelzuurdibromide en allo-kaneelzuur-dibromide.

FISCHER besluit daaruit, dat er derhalve geen neiging bestaat tot vorming van dubbelverbindingen, waarin de beide samenstellende bestanddeelen in configuratie gedeeltelijk elkaars spiegelbeelden zijn. Hij voerde voor dergelijke lichamen den naam „partieel-racemische verbindingen" in.

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[1]) De verschijnselen worden hier bij één bepaalden druk beschouwd.
[2]) Ber. d. d. Chem. Ges. **36**, 1649 (1903) Inaug. Diss. Breslau 1903.
[3]) Ibid. **41**, 966 (1908) „ „ „ 1908.
[4]) Ibid. **31**, 1969 (1898) en **32**, 50 (1899) Inaug. Diss. Breslau 1899.
[5]) Ibid. **40**, 2279 (1907) Inaug. Diss. Breslau 1905.
[6]) Ibid. **27**, 3225 e. v. (1894).
[7]) Ibid. **27**, 2001 (1894).
[8]) Ibid. **27**, 2045 (1894).

Deze naam is door LADENBURG [1]) overgenomen voor de verbindingen, die hij verkregen had uit een racemisch zuur met eene actieve base resp. eene racemische base met een optisch actief zuur, en die dus feitelijk tot eene geheel andere klasse van lichamen dan de door FISCHER oorspronkelijk met dien naam aangeduide verbindingen behooren. Sprekende over het chininezout van het racemische pyrowijnsteenzuur, zegt hij: [2]) „Es ist also selbst eine racemische Verbindung, und zwar findet hier, da in den Componenten die Base jeweils dieselbe ist, die Säuren aber Spiegelbilder sind, das statt, was man zweckmässig als „partielle Racemie” bezeichnen kann; d. h. es ist eine Verbindung zweier Körper, die nur theilweise Spiegelbilder sind, sodass also durch die Verbindung eine theilweise Aufhebung der optischen Activität stattfindet, und der racemische Körper noch optische Activität besitzt”.

Definieert men de partiëele racemie als het verschijnsel, waarbij twee stoffen, die in configuratie gedeeltelijk elkaars spiegelbeeld zijn, zich verbinden, dan is tegen deze benaming in vergelijking met de eigenlijke racemie, die de vorming van verbindingen met volkomen spiegelbeeld-configuratie bedoelt, niet veel te zeggen. Toch heb ik van den aanvang af tegen deze benaming een bezwaar gevoeld. Wie toch met de partiëele racemie niet bekend is, zal meenen, dat eene partieel-racemische verbinding een lichaam is, dat bestaat uit een inactief racemisch deel, het zuur, en een deel, dat optische activiteit bezit (de base); zoo zal men zich, krachtens de benaming, het molecuul druivenzure-strychnine denken opgebouwd uit druivenzuur en strychnine, terwijl toch het gedrag van dit lichaam naast zijne waterige oplossing er op wijst, dat het is opgebouwd uit *d*-wijnsteenzure-strychnine + *l*-wijnsteenzure-strychnine. Stelt men het molecuul aldus voor $\begin{bmatrix} d\text{-wijnsteenzuur-} \\ l\text{-wijnsteenzuur-} \end{bmatrix} \begin{matrix} \text{strychnine} \\ \text{strychnine} \end{matrix}$, dan komt blijkbaar de binding in het dubbelzout niet tot stand tusschen actieve base en racemisch zuur, maar wel tusschen de beide biactieve zouten, i. c strychnine-*d*-tartraat en strychnine-*l*-tartraat. Daar echter deze naam voor de verbindingen in quaestie langzamerhand burgerrecht heeft verkregen, komt het mij niet wenschelijk voor er eene andere benaming tegenover te stellen.

Het leek mij evenwel niet ongewenscht, deze objectie tegen de gebruikelijk geworden nomenclatuur te maken.

Alvorens er toe over te gaan, de afzonderlijke gevallen, waarbij partiëele racemie met zekerheid geconstateerd is, te bespreken, wil ik

[1]) Ber. d. d. Chem. Ges. **31**, 938 (1898).

[2]) l. c.

nog een enkel woord zeggen over den strijd, die jaren lang tusschen LADENBURG en EMIL FISCHER geduurd heeft over het bestaan der partiëele racemie, en die eerst in den allerlaatsten tijd beslecht is.

Ik heb reeds vermeld, dat de term „partiëele racemie" van E. FISCHER afkomstig is en dat deze de uitdrukking wenschte toe te passen op mogelijke verbindingen, als bijv. van *l*-mannonzuur met *d*-gluconzuur.

Het is nu aan FISCHER niet mogen gelukken, verbindingen van dergelijke lichamen af te scheiden en zoo komt hij er toe, in zijn uitvoerige mededeeling betreffende zijne onderzoekingen over aminozuren, polypeptiden en proteïnen [1]) op pagina 572 te zeggen, dat het in de meeste gevallen niet gelukt, om isomeren, die gedeeltelijk elkaars antipoden zijn, door eenvoudige kristallisatie te scheiden. Het niet-gelukken dier scheiding meent hij daaraan te moeten toeschrijven, dat in de meeste gevallen de isomeren wegens hunne groote gelijkenis *mengkristallen* zullen vormen en dan schijnt hem geen bezwaar tegen het gebruiken van zijn term „partiëele racemie" voor mengkristallen, die 50 % van beide half-antipodische isomeren bevatten.

Een jaar later is E. FISCHER opnieuw op dit onderwerp teruggekomen [2]) en heeft op de verwarring, die door het gebruik van zijne uitdrukking „partiëele racemie" voor de door LADENBURG ontdekte dubbelzouten, ontstaan is, een helder licht laten vallen. Het is nl. nooit zijne bedoeling geweest, de dubbelzouten van LADENBURG voor mengkristallen te houden. Deze beschouwt hij evenzeer als echte verbindingen als alle andere, die goed gedefinieerd zijn. Maar in vele gevallen, waarbij „partiëele racemie" in de beteekenis van FISCHER, (verbinding tusschen bijv. de beide isomere broomisocapronyl-*l*-asparaginen) zou kunnen optreden, bleek mengkristal-vorming op te treden en *niet* het ontstaan van verbindingen tusschen de bedoelde isomeren. Nochtans is ook hier de mogelijkheid, dat „verbinding" tusschen de isomeren optreedt, volgens hem geenszins uitgesloten.

FISCHER wijst echter ten slotte op de weinig scherpe definitie, die LADENBURG van de partiëele racemie geeft. Volgens zijne meening is ook een lichaam van het type als de door PASTEUR ontdekte verbinding van *d*-ammoniumbitartraat met *l*-ammoniumbimalaat een partiëel-racemaat. [3])

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[1]) Ber. d. d. Chem. Ges. **39**, 530 (1906).

[2]) Ibid. **40**, 943 (1907) noot 4.

[3]) Het schijnt dat er eenige onzekerheid bestaat, of ook dit lichaam inderdaad eene verbinding dan wel een isomorph mengsel is. Verg. W. MEYERHOFFER „Gleichgewichte der Stereomeren" Leipzig 1906 p. 62.

Hiermede is de strijd Fischer-Ladenburg tot een eind gekomen en blijkt hij dus louter te berusten op eene ongeoorloofde toepassing van Fischer's uitdrukking „partiëele racemie" door Ladenburg.

Wij kunnen nu de gevallen, waarin het verschijnsel, dat ons bezighoudt, geconstateerd is, kortelings nagaan en de methoden mededeelen, volgens welke in al die gevallen met beslistheid kon worden bewezen, dat men inderdaad met *verbindingen* te doen had. Tevens zullen de voorbeelden, waarin het overgangspunt van deze verbindingen werd vastgesteld, worden opgegeven en tevens zal worden uiteengezet, langs welken weg die punten werden bepaald.

Ladenburg heeft tot nu toe over de partiëele racemie een achttal mededeelingen gepubliceerd, waarvan de laatste, tijdens mijn onderzoek verschenen, eene samenvatting geeft van de resultaten, die in de voorafgegane onderzoekingen zijn verkregen.

In zijne eerste mededeeling [1]) vermeldt Ladenburg, dat het inactieve pyrowijnsteenzuur niet met behulp van chinine en het inactieve β-oxyboterzuur niet met strychnine te splitsen is; de zouten die in beide gevallen uit de oplossingen uitkristalliseeren leveren bij verwijdering van de actieve base het inactieve zuur in den vorm van het ammoniumzout weer terug.

Er moest nu worden aangetoond, dat dit chinine- en dit strychninezout werkelijke verbindingen waren en niet beide een aequimoleculair mengsel van *d*-pyrowijnsteenzure-chinine met *l*-pyrowijnsteenzure-chinine resp. de overeenkomstige β-oxyboterzure-strychninezouten waren.

Daarom werd eerst aangetoond, dat, wanneer bij verschillende temperaturen de kristallisatie plaats vond, steeds een zout ontstond van een zuur, dat optisch inactief was. Het isoleeren van het zuur uit de afgescheiden kristalmassa had steeds op dezelfde wijze plaats en wel zóó, dat de kans op auto-racemisatie ervan, uitgesloten mocht worden geacht.

Met oplossingen in absoluten alcohol kreeg men bij 0°, 18°, 30° steeds dezelfde verbinding. Dit feit pleitte krachtig voor de werkelijke „verbindings"-natuur van het lichaam; immers, mocht het al toevallig mogelijk zijn, dat bij een der genoemde temperaturen de oplosbaarheid der beide half-antipoden gelijk was, onaannemelijk is het, dat bij twee zoo totaal verschillende lichamen als dergelijke isomeren plegen te zijn, de oplosbaarheid van beide over een temperatuurstraject van 0°—30° steeds gelijk zou blijven, waardoor zij

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[1]) Ladenburg en Herz, Ber. d. d. Chem. Ges. **31**, 524 (1898).

uit oplossingen, waarin zij in aequimoleculaire verhouding voorkomen, zich ook in diezelfde verhouding kristallijn zouden kunnen afzetten.

Nu werden de chininezouten van *d*-, *l*- en *i*-pyrowijnsteenzuur bereid. Het zout van het *l*-zuur kon niet zuiver verkregen worden, daar dit zuur slechts te bereiden was met een draaiingsvermogen van $^2/_5$ van het overeenkomstige *d*-zuur.

Het *d*-pyrowijnsteenzure-chinine smolt bij 169°—171°, het zout uit het *i*-zuur bij 174°—175°, dus eenige graden hooger, een feit dat LADENBURG ten gunste van het bestaan eener dubbelverbinding aanvoert. Het is echter duidelijk, dat eene dergelijke dubbelverbinding evengoed lager als hooger dan hare componenten kan smelten.

De oplosbaarheid van het *d*-zure zout bleek grooter dan die van het *i*-zure zout. Dit levert een argument voor eene verbinding, immers, wanneer het zout, dat *i*-zuur blijkt te bevatten, een mengsel was, dan zou krachtens de onderzoekingen van RÜDORFF [1]) de oplosbaarheid van dit mengsel grooter moeten zijn dan van eene enkele component afzonderlijk. In de 2e mededeeling [2]) wordt vermeld, dat men het *l*-zure zout zuiverder heeft verkregen en de oplosbaarheid ervan in alcohol heeft bepaald. Deze is zeer veel grooter dan die van *d*-zuur zout. Hiermede had LADENBURG de dubbelzoutnatuur van het *i*-pyrowijnsteenzure-chinine bewezen en het eerste geval van partiëele racemie met zekerheid aangetoond.

De volgende verhandelingen leveren nieuwe gevallen en tegelijk wordt hierin telkens gepoogd de overgangstemperatuur der nieuwontdekte partiëel racemische verbindingen vast te stellen.

Zoo bestudeert LADENBURG in zijne derde verhandeling [3]) met DOCTOR het neutrale strychnineracemaat.

Hier zijn de componenten *d*-wijnsteenzure-strychnine, *l*-wijnsteenzure-strychnine en druivenzure-strychnine gemakkelijk in zuiveren toestand te bereiden.

De smeltpunten, het kristalwatergehalte en de dichtheden worden van de 3 zouten bepaald en van allen verschillend gevonden. Tegelijk worden bij twee temperaturen (20° en 40°), later bij meer temperaturen, de oplosbaarheden bepaald en steeds verschillend gevonden. Het verschil in smeltpunt van een aequimoleculair mengsel en van het dubbelzout wordt vastgesteld, en men vindt, dat bij toevoeging van de voor de vorming van het dubbelzout

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[1]) Pogg. Ann. **148**, 456 en 555.

[2]) Zie pag. 27.

[3]) l. c.

uit zijne componenten vereischte hoeveelheid water, het aanvankelijk vochtige mengsel, na eenige dagen in een afgesloten fleschje bewaard te zijn, kurkdroog is geworden.

De stof vertoont dan het smeltpunt van de dubbelverbinding. Hiermede was bewezen, dat het lichaam geen mengsel was en verder werd getracht de overgangstemperatuur in dit geval te bepalen.

Dilatometerproeven voerden hierbij niet tot de gewenschte uitkomst, daarentegen wordt in de 4ᵉ mededeeling bericht, dat oplosbaarheidsbepalingen het overgangspunt op 30° deden vinden en deze temperatuur wordt later door tensimeterbepalingen op 29.5° gecorrigeerd.

Ik zal op dit onderzoek hier niet verder ingaan, daar er in het experimenteele gedeelte van dit proefschrift, gelegenheid genoeg zal bestaan om LADENBURG en DOCTOR's uitkomsten kritisch te bespreken en ze met de door mij gedane waarnemingen over deze zelfde zouten te vergelijken.

LADENBURG's 5ᵉ mededeeling brengt als nieuw geval van partiëele racemie het β-pipecoline-bitartraat. Deze stof heeft, zooals wij op pag. 2 zagen, historisch belang, daar bij haar het eerste geval van partiëele racemie werd opgemerkt. De eigenschappen en de bestaansvoorwaarden van het racemaat naast zijne waterige oplossing zijn hier echter nader bestudeerd.

Het racemaat blijkt ook hier weer, in zijne physische eigenschappen, geheel te verschillen van een aequimoleculair $(d + l)$ mengsel. Daartoe zijn de smeltpunten, dichtheden en kristalwatergehalten der drie stoffen in kwestie bepaald.

Waar we hier stoffen hebben, die misschien normale smeltverschijnselen vertoonen, (in tegenstelling met de brucine- en strychninetartraten, die alle onder ontleding smelten) lijkt het mij hier niet ongepast, op te merken, dat het bewijs, dat het partiëele racemaat werkelijk een chemisch individu en geen aequimoleculair mengsel van zijne samenstellende bestanddeelen is, ook kan geleverd worden (en dan met alle zekerheid) door de bepaling van de smeltlijn van het binaire stelsel d-β-pipecoline-d-tartraat en l-β-pipecoline-d-tartraat. Is het racemaat nl. eene werkelijke verbinding, dan zal zich dit vertoonen in den vorm der smeltlijn en hare gedaante ongeveer zijn als in fig. 9*bis*.

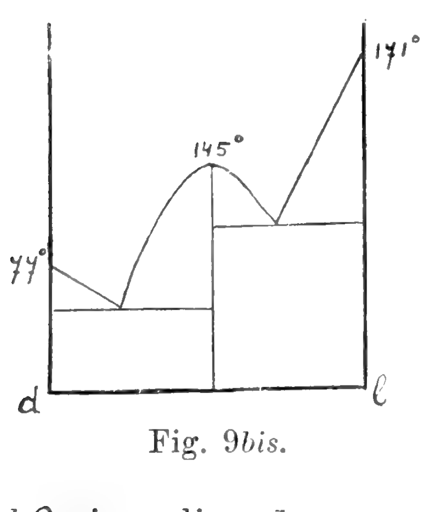

Fig. 9*bis*.

Op dergelijke wijze is, zooals ik kort voor de samenstelling van

dit hoofdstuk heb bemerkt, door FINDLAY en HICKMANS [1]) het karakter der *r*-amandelzure-*l*-mentholester als dubbelverbinding met volledige klaarheid blootgelegd. Tegelijk blijkt uit hunne onderzoekingen, dat deze ester van het racemische amandelzuur in zijne smelt tot een aanmerkelijk bedrag is gedissocieerd in zijne beide componenten.

Ook de overgangstemperatuur liet zich met behulp van den tensimeter volgens BREMER-FROWEIN vaststellen op 39.5°. Ik heb reeds vroeger vermeld, dat men hier een partiëel racemaat heeft, welks bestaansgebied zich naar hoogere temperaturen uitstrekt. Nog een geval van partiëele racemie vonden LADENBURG en FISCHER (6^e mededeeling) [2]) in het zure druivenzure brucine. Deze onderzoeking wijkt, ook in hare fouten, geenszins af van de vorige. Door oplosbaarheidsbepalingen werd de overgangstemperatuur op 44° bepaald. Langs tensimetrischen weg kon dit punt niet nader gecontroleerd worden, daar de omzetting zonder waterafsplitsing volgens LADENBURG en FISCHER plaats heeft.

Het laatste [3]) door LADENBURG en zijne leerlingen bestudeerde geval betreft het tetra-hydrochinaldine-bitartraat. Deze base is reeds door LADENBURG in 1894 [4]) met behulp van het gewone wijnsteenzuur gesplitst en op die wijze is de rechtsdraaiende base in zuiveren toestand bereid. Bij herhaling van deze proeven verkreeg men een base, die een kleiner draaiingsvermogen bezat dan het aanvankelijk geisoleerde *d*-tetrahydrochinaldine. Dit deed een geval van partiëele racemie vermoeden. Niet zonder moeite gelukte het nu, een zout te bereiden, welks base geheel vrij van optische activiteit was, nl. door vermenging van zeer geconcentreerde wijnsteenzuur oplossing, die op 60°—63° verhit was, met racemisch hydrochinaldine van diezelfde temperatuur. Onder deze omstandigheden kreeg men eene kristallijne brij, waarvan de kristallen bleken racemische base te bevatten.

Waarschijnlijk waren de grootere moeilijkheden van de bereiding van het partiëele racemaat hier gelegen in het lage smeltpunt van het zout nl. 72°—73° (*d* 90°—91°, *l* 62°—63°) en de zeer gemakkelijke hydrolytische splitsing ervan.

Deze laatste eigenschap maakte eene vaststelling van de overgangstemperatuur door oplosbaarheidsbepalingen onmogelijk; niettemin

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[1]) Journ. Chem. Soc. 91^1, 905 (1907).

[2]) Zie pag. 27.

[3]) 7^e meded. zie pag. 27.

[4]) Ber. d. d. chem. Ges. **27**, p. 76, 1894.

gelukte het, haar op ca 59° vast te stellen langs den weg der spanningsevenwichten in den tensimeter.

Ook hier is het racemaat naar hoogere temperaturen stabiel naast zijne oplossing.

Ladenburg stelt zich ten slotte de vraag, hoe eigenlijk hier het proces plaats grijpt en zegt: [1])

„Man kann allerdings annehmen, dass beim Zusammenbringen von *d*-Weinsäure mit *α*-Hydrochinaldine zunächst das partielle racemische Salz entsteht, welches sich aber, da es sich jenseits der Umwandlungstemperatur befindet, alsbald in die Einzelsalze zerlegt. Diese Zerlegung wird um so langsamer vor sich gehen, je mehr man sich der Umwandlungstemperatur nähert, und es wäre daher immerhin möglich, dass das geringe Drehungsvermögen der bei höherer Temperatur auskrystallisierenden Salze in dieser Weise erklärt werden könnte. Wahrscheinlich finde ich es aber nicht.

Plausibler ist die Annahme, dass die Löslichkeiten der beiden Einzelsalze bei höherer Temperatur einander näher kommen, so dass das auskrystallisierende Salz bei höherer Temperatur mehr von der leichter löslichen Komponente enthält und dadurch weniger aktiv ist. Leider haben die Eigenschaften der Salze nicht gestattet, Löslichkeitsbestimmungen zu machen und eine Entscheidung zwischen beiden Möglichkeiten zu treffen."

Daar bij deze zouten hydrolytische splitsing kan optreden, is hier moeilijk te beoordeelen, wat er bij de reactie tusschen *d*-wijnsteenzuur en het racemische hydrochinaldine gebeurt. Zien we echter voor een oogenblik hier van af, dan meen ik, dat fig. 5 blz. 14 hoofdstuk II in staat is, de verklaring te geven.

Men gaat uit van de racemische base en *d*-wijnsteenzuur in waterige oplossing. De samenstelling wordt dus aangegeven door een zeker punt der lijn *OC*. Heeft nu de kristallisatie bij een constante temperatuur plaats en ligt deze onder het overgangspunt i. c. beneden het bestaansgebied der partiëel racemische verbinding, naast oplossing van hare samenstelling, dan behoeven we slechts na te gaan, waar de lijn *OC* de isotherm van de temperatuur, waarbij het uitkristalliseeren plaats heeft, snijdt. Dit snijpunt is natuurlijk gelegen op de oplosbaarheidslijn der minst oplosbare componente [2]). Bij afkoeling tot de kristallisatie-temperatuur zal zich dus eerst deze afscheiden, hierbij verandert de oplossing in samenstelling en wel

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[1]) Ber. d. d. chem. Ges. **41**, p. 969 (1908).

[2]) Indien tenminste de fig. 9*ter* de verhoudingen weergeeft, wat in alle normale gevallen inderdaad zoo is.

zóó, dat ze rijker wordt aan de meest oplosbare componente. De afscheiding van de eerste gaat voort, tot men gekomen is in punt *P* (fig. 9*ter*), waar zich nu beide componenten naast elkaar uit de waterige oplossing afzetten. De ligging van dit laatste punt *P* is nu beslissend voor de activiteit der uit de kristallen te isoleeren base. Ligt *P* dicht bij de lijn *OC*, dan is de afgescheiden base zeer zwak draaiend, is echter *DP* groot, wat samenhangt met het verschil in oplosbaarheid der beide componenten bij de kristallisatie-temperatuur, dan zal de base grooter draaiend vermogen vertoonen. Oplosbaarheidsbepalingen zijn hier echter niet verricht en dus laat zich hier de theorie niet met het experiment vergelijken.

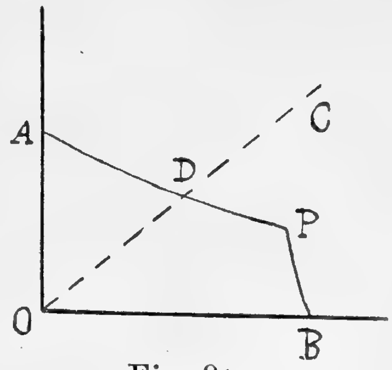

Fig. 9*ter*.

Zeer onlangs [1]) heeft Ladenburg in een samenvattend artikel zijne onderzoekingen over de partiëele racemie bijeengevoegd. Behalve, dat eenige experimenteele bizonderheden worden medegedeeld, die in het volgende hoofdstuk ter sprake zullen moeten komen, levert dit artikel niets nieuws op.

Van belang is echter het slotwoord, waar hij in korte trekken het proces der splitsing van racematen door zoutvorming nogmaals schetst. Ik wil hieruit het volgende citeeren: „Früher hatte man geglaubt, dass, wenn man zu einem Racemkörper eine optisch active Substanz hinzufüge, die mit jenem eine Verbindung bilden könne, vorher eine Spaltung des Racemkörpers stattfinde. Eine solche Annahme ist aber durch Nichts zu begründen, widerspricht im Gegentheil einer Reihe von bekannten analogen Thatsachen. So bildet die Traubensäure mit den Alkalien und vielen anderen Metallen wohl definirte Salze, aus denen man die Traubensäure auch wieder regeneriren kann. Warum sollte nun durch Zusatz von z. B. Cinchonin plötzlich *d*- und *l*-weinsaures Salz entstehen? Dazu müsste eine Einwirkung eines optisch-activen Körpers auf einem Racemkörper angenommen werden, die durchaus unverständlich und unphysikalisch wäre. [2])

Durch den Nachweis der Existenz partiell-racemischer Salze wird nun Alles klar. Es bildet sich zunächst stets die Verbindung des Racemkörpers mit der aktiven Substanz, welche eventuell, wenn innerhalb ihres Existenzgebietes gearbeitet wird, isolirt werden kann,

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[1]) Lieb. Ann. **364** p. 227—271 ('09).

[2]) Die Auffassung von Landolt (Optisches Drehungsvermögen 2 Aufl. S. 60, 86 u. s. w.) ist nur zu verstehen, wenn man annimmt, dass Racemkörper in Lösung nicht existiren, was aber zweifellos irrthümlich ist.

im anderen Fall aber in die beiden Einzelsalze zerfällt, die, weil sie nur teilweise Spiegelbilder sind, ganz verschiedene Eigenschaften haben, zur Trennung und Isolirung der activen Componenten mit grossem Vortheil benutzt werden können."

Ik kan mij met dezen gedachtengang niet vereenigen en stel mij de verschijnselen aldus voor: Wanneer men eene druivenzuur-oplossing heeft bereid, dan bevindt zich in die oplossing niet het druivenzuur als zoodanig; maar er bestaat in die oplossing, omdat zij eene homogene phase is, wier innerlijke toestand beheerscht wordt door de wet van de massawerking, een evenwicht tusschen inactieve druivenzuur-moleculen en moleculen van *d*- en *l*-wijnsteenzuur. Uit de bepalingen van de vriespuntsverlaging en kookpuntsverhooging van druivenzuuroplossingen is gebleken, dat hier dit evenwicht zeer naar de zijde der actieve componenten is verschoven. [1])

Neutraliseert men nu eene dergelijke oplossing met bijv. strychnine, dan ontstaan in de oplossing de strychninezouten, zoowel van *d*- en *l*-wijnsteenzuur als van druivenzuur, en weer zal zich volgens de wet van Guldberg en Waage een evenwicht instellen; ditmaal echter tusschen *d*-tartraat, *l*-tartraat en partiëel racemaat. Bij afkoeling zal dit evenwicht verschuiven en wel in den zin van partiëele racemaatvorming, indien de dissociatie van dit lichaam in zijne componenten een endotherm proces is.

Komt men nu op een temperatuur, waarbij het partiëele racemaat naast eene oplossing van zijne samenstelling kan bestaan en is aan de voorwaarden voor de kristallisatie voldaan, dan zal [2]) zich ook dit zout uit de oplossing afzetten en niets anders dan dit zout. Ligt de temperatuur van kristallisatie niet in het bestaansgebied van het partiëele racemaat, dan gebeurt wat op pag. 16 e. v. is geschetst.

Strekt het partiëele racemaat zijn bestaansgebied naast oplossing naar hoogere temperaturen uit, dan is wat gebeuren zal, gemakkelijk uit fig. 6 pag. 15 te begrijpen.

Twijfel aan het bestaan van partiëele racematen en van eigenlijke racematen in oplossing of in vloeibaren toestand behoeft niet te bestaan. Er is steeds een evenwicht tusschen racemaat resp. partiëel racemaat en zijne splitsingsproducten.

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[1]) Prof. Schreinemakers maakte mij er attent op dat hij [Zeitschr. f. phys. Chem. **33**, 74 (1900)] tot eenzelfde conclusie is gekomen op grond van het gedrag van het stelsel water-phenol-wijnsteenzuur (resp. druivenzuur). Kruyt.

[2]) Men bedenke echter, dat hier, waar het verschijnselen uit de organische chemie geldt, groote kans is op vertragingen, zoodat wel het racemaat boven zijne overgangstemperatuur uit de oplossing kan kristalliseeren en anderzijds ($d + l$) zich in het racemaatgebied niet tot racemaat verbinden.

Ik wil thans nog kort de aandacht vestigen op de overige gevallen van partiëele racemie, die men in de litteratuur verspreid vindt en zal deze in chronologische orde vermelden.

G. GOLDSCHMIDT [1]) geeft op, dat het hem niet gelukt is, het tetrahydropapaverine, dat hij trachtte in het bitartraat om te zetten, om aldus een splitsing ervan te bewerkstelligen, in zijne componenten uiteen te doen vallen. Hij verkreeg met wijnsteenzuur steeds het neutrale zout daarvan, dat steeds inactieve basis bleek te bevatten en schrijft nu de mislukking der splitsing toe aan de omstandigheid, dat hier het bitartraat niet kon worden geisoleerd. Na hetgeen voorafgegaan is, eischt het wel geen betoog, dat hij *d*-wijnsteenzure *r*-tetrahydropapaverine in handen heeft gehad. Ook POPE en PEACHEY [2]) konden in dit geval geen zuur tartraat van genoemde basis bereiden. Zij zagen echter terstond in, dat men hier met een geval van partiëele racemie te doen heeft en verwijzen naar LADENBURG's beschouwingen daaromtrent.

KIPPING [3]) bericht zelfs de vondst van isomere partiëel-racemische verbindingen in het geval van *r*-hydrindamine met broomcamphersulfonzuur, chloorcamphersulfonzuur en cis-π-camphaanzuur, welke isomerie, hoe interessant ook, wij niet nader zullen beschouwen.

Ongeveer terzelfder tijd deelde BACH [4]) mede, dat het wijnsteenzure zout van phenyl-α-picolylalkine bij regeneratie van de base, deze in haren racemischen vorm blijkt te bevatten, wat dus als een nieuw geval van partiëele racemie mag worden beschouwd.

Van meer belang zijn de beschouwingen ontwikkeld door KIPPING en HUNTER [5]) naar aanleiding van gevallen van partiëele racemie bij de tartraten van pheno-α-aminocycloheptaan en van hydrindamine. In de bedoelde publicatie vinden we naar aanleiding van de mogelijke zouten, die gevormd kunnen worden uit *d*-wijnsteenzuur en (*d*, *l*) pheno-α-aminocycloheptaan o. a. genoemd het zout $dA\left\{\begin{matrix} dB \\ lB \end{matrix}\right.$; ook dit zout zou men partiëel racemisch kunnen noemen, immers het is gevormd uit een tweebasisch zuur, welks eene zuurfunctie is geneutraliseerd door de *d*-base, de andere door de *l*-base. Zulk een zout zou niet eene partiëel racemische verbinding in den gewonen zin zijn, het is n.l. geen verbinding van twee lichamen, die in configuratie gedeeltelijk elkaars spiegelbeeld zijn. Dat is het

[1]) G. GOLDSCHMIDT, Monatshefte **19**, 321 ('98).
[2]) POPE en PEACHEY, J. Chem. Soc. **73**, 902 ('98) en Zeitschr. f. Krist. **31**, ('99).
[3]) KIPPING, J. Chem. Soc. **77**, 861 ('00).
[4]) BACH, Ber. d. d. chem. Ges. **34**, 2237 ('01).
[5]) KIPPING en HUNTER J. Chem. Soc. **81**, 576 ('02).

echter weer wel, als het zout bimoleculair $\left(dA\begin{cases}dB\\lB\end{cases}\right)_2$ is, het kan dan nl. eene verbinding zijn van $dA\begin{cases}dB\\dB\end{cases}+dA\begin{cases}lB\\lB\end{cases}$; of nu de normale tartraten van de *d*-base en van de *l*-base zich misschien vereenigen tot eene verbinding, die identiek is met die, welke men verkrijgt door achtereenvolgens op *d*-wijnsteenzuur eerst één molecuul *d*-base te laten inwerken en daarna één molecuul *l*-base, is echter niet onderzocht.

Partiëele racemie werd geconstateerd bij het (*d-l*) hydrindamine bitartraat, daarentegen niet bij het zure wijnsteenzure zout van pheno-α-aminocycloheptaan, van welke base de constitutie zeer veel overeenkomst met die van α-hydrindamine (d. i. pheno-α-aminocyclo-*pen*taan) vertoont. Wat echter in deze publicatie ons het meest interesseert, zijn de beschouwingen, die Kipping aangaande partiëele racemie ontwikkelt en die vrijwel lijnrecht staan tegenover die van Ladenburg.

Hij bespreekt allereerst Ladenburg's definitie van partiëel racemische verbindingen en wijst er op, dat deze auteur hen soms als verbindingen van gedeeltelijke antipoden beschouwt, soms als actieve zouten van een werkelijk racemisch zuur of racemische base. Dit is dus hetzelfde bezwaar, dat boven reeds door mij tegen den term „partiëele racemie" is aangevoerd. Nu schijnt Kipping verder racematen uitsluitend als „kristallijne vereenigingen van *d*- en *l*-isomeren te beschouwen, die geen ander bestaan dan in dien vorm" bezitten, terwijl juist Ladenburg een voorvechter is voor de vloeibare racematen en de racematen in oplossing.

Kipping denkt zich dus bij het smelten of oplossen van een racemaat totale dissociatie; Ladenburg daarentegen absoluut geene. Het zal wel niet noodig zijn te betoogen, dat de waarheid in het midden ligt en er in oplossing of smelt steeds racemaat-moleculen aanwezig moeten zijn in eene hoeveelheid, afhankelijk van het evenwicht, dat zich in die homogene phase volgens de massawerkingswet moet instellen. Hoe toch zouden anders racematen zich uit oplossing of smelt in vasten toestand kunnen afzetten, indien hunne moleculen niet reeds daarin aanwezig waren?

Bij het smeltpunt van een racemaat of in diens verzadigde oplossing zijn toch de moleculaire thermodynamische potentialen in het eerste geval van de moleculen van het vaste racemaat en die van het vloeibare aan elkaar gelijk, in het tweede geval die van de moleculen van het vaste racemaat en van die in oplossing.

In beide gevallen heeft μ_{vast} een bepaalde waarde, dus moet

ook $\mu_{vl.}$ of $\mu_{solv.}$ die zelfde waarde bezitten. Hieruit vloeit de noodzakelijkheid van het bestaan der racemaat-moleculen in smelt en oplossing voort.

Kipping gaat ten slotte nog na, welke gevallen zich kunnen voordoen, indien een uitwendig gecompenseerd tweebasisch zuur $(d, l)\,A$ geneutraliseerd wordt door een actieve basis dB.

In oplossing kunnen dan gevormd worden 2 zouten nl. $dA\begin{cases} dB \\ dB \end{cases}$ en $lA\begin{cases} dB \\ dB \end{cases}$, of als het zuur één basisch is $dA\ dB$ en $lA\ dB$.

Nu zijn de volgende 4 gevallen mogelijk: 1°. $dA\begin{cases} dB \\ dB \end{cases}$ en $lA\begin{cases} dB \\ dB \end{cases}$ verschillen in physische eigenschappen en zijn te scheiden door gefractioneerde kristallatie, wat vaak, hoewel niet altijd, het geval is. 2°. Zij kunnen zich bij de afscheiding uit de vloeistof (Kipping's standpunt) vereenigen in kristallografischen zin en eene stof leveren, die in kristalvorm en in overige physische eigenschappen verschilt van beide componenten. 3°. Zij kunnen mogelijkerwijze in gelijke hoeveelheden „naast elkaar" uit de oplossing afgezet worden als zuiver mengsel. 4°. Zij kunnen mengkristallen geven, in zekeren zin vergelijkbaar met de pseudoracemie, zooals die door Kipping en Pope is ontdekt, indien daartoe bij de componenten voldoende kristallografische verwantschap bestaat.

Evenzoo kan een actief tweebasisch zuur met een racemische base in oplossing de volgende zouten doen ontstaan: $^{a})\ dA\begin{cases} lB \\ dB \end{cases}$, $^{b})\ dA\begin{cases} dB \\ dB \end{cases}$ en $^{c})\ dA\begin{cases} lB \\ lB \end{cases}$.

Mogelijkerwijze kan zich het zout $^{a})$ als bepaalde verbinding uit de oplossing afzetten, indien dat niet zoo is, dan kunnen de onder $^{b})$ en $^{c})$ genoemde zouten zich gedragen op één der vier reeds aangegeven wijzen.

Kipping zegt nu, dat in alle genoemde gevallen, behalve in dat van gefractioneerde kristallisatie (geval 1°) en in het onder $^{a})$ geclassificeerde geval het afgescheiden zout overeenkomt met Ladenburg's definitie van partieel-racemische verbinding, „unless the meaning of „Verbindung" [1]) be interpreted as a crystallographic union, the result of which is to give a product differing from at least one of its components in crystalline form, and consequently in other properties; if this limitation be not made the term „partially racemie" would include a number of salts of different types

[1]) Cf. Ladenburg's verhandelingen op pag. 27 en 28 besproken.

in much the same way, as did at one time the term ‚racemie' (Kipping and Pope Trans. 1897 **71**, 989).

Hier is slechts aan toe te voegen, dat Ladenburg, hoewel niet altijd even consequent in zijne terminologie, met partiëele racematen niet anders bedoeld heeft dan dubbelzouten of dubbelverbindingen van zouten of andere lichamen, die slechts gedeeltelijk elkaars optische antipoden zijn. Dus wel „crystallographic unions" maar in stoechiometrische verhouding, lichamen dus, die als dubbelverbindingen, zoowel in vasten als in vloeibaren vorm (hoewel daar gedeeltelijk geplitst) voorkomen.

Zeer juist is aan het slot van dit artikel de opmerking, dat het karakter eener partieel-racemische verbinding als zoodanig kan uitgemaakt worden volgens dezelfde methode, die leidt tot identificatie van eigenlijke racematen nl. door bij gegeven temperatuur oplossingen te bereiden van het vermeende partiëele racemaat alleen, van hem zelf met de eene zijner splitsingsproducten en van hem met de andere zijner componenten. Zijn die oplossingen verschillend, dan is het lichaam eene werkelijke dubbelverbinding en geen aequimoleculair mengsel, zijn die drie oplossingen daarentegen onderling gelijk, dan is het lichaam een mengsel zijner beide componenten. Ware dit feit aan Ladenburg en zijne leerlingen, die met hem de partiëele racemie hebben bestudeerd, bekend geweest, dan zouden zij zich vele pogingen, om bewijzen aan te brengen voor het ware karakter als dubbelverbinding hunner partiëele racematen, hebben kunnen besparen.

Niet onvermeld mag in dit overzicht blijven, dat Meyerhoffer [1]) in zijne „Stereochemische Notizen" nog eens ten overvloede op de analogie tusschen partiëele racematen en anorganische dubbelzouten de aandacht heeft gevestigd. Het schijnt intusschen, dat noch het uitnemende artikel van Bakhuis Roozeboom [2]), noch deze publicatie tot degenen, die zich met de studie van ons onderwerp hebben beziggehouden, behoorlijk is doorgedrongen. De latere publicaties uit Ladenburg's school toch geven nog evenzeer blijk van een onvolledig inzicht, als de vroegere van Ladenburg en Doctor.

Veel uitvoeriger nog wordt de aandacht geschonken aan de partiëele racemie in Meijerhoffer's zeer lezenswaardig boekje „Die Gleichgewichte der Stereomeren" waarin ook tevens op de fouten en onwaarschijnlijkheden in de uitkomsten van vorige onderzoekers wordt gewezen. In het bizonder wordt daarin ook de partiëele racemie van het druivenzure strychnine ter sprake gebracht.

1) Ber. d. d. chem. Ges. **37**, 2604 (1904).
2) l. c. blz. 18.

Ook in verschillende leer- en handboeken heeft thans het begrip partiëele racemie ingang gevonden. Zoo wordt zoowel het „Lehrbuch der Stereochemie" van A. WERNER [1]) als in den 2[en] druk van het „Lehrbuch der Organischen Chemie" van VICTOR MEYER en P. JACOBSON [2]) de vorming van dubbelverbindingen bij gedeeltelijke optische antipoden besproken en men vindt vooral in het laatste boek een vrij volledige litteratuuropgave. De opmerking echter, dat de „tot nu toe vastgestelde gevallen van partiëele racemie uitsluitend betrekking hebben op zouten" is niet meer als „up to date" te beschouwen.

Een merkwaardig geval van partiëele racemie is door MC. KENZIE [3]) gevonden in den *l* menthol-ester van het racemische amandelzuur. Deze is, voor zoover mij bekend, het eerste voorbeeld van verbindingen, die in configuratie gedeeltelijk elkaars spiegelbeeld zijn, bij esters.

Zijn bestaan als „verbinding" is door A. FINDLAY en mej. HICKMANS [4]) bewezen door een bepaling der volledige smeltlijn van het binaire stelsel *d*-amandelzure *l*-menthol-ester + *l*-amandelzure- *l*-menthol-ester. Hierin komt de partiëel racemische ester als ware verbinding door het bezit eener eigen smeltlijn te voorschijn, wel is waar met een vlakken top, wat op hoogen graad van dissociatie in zijne beide biactieve bestanddeelen in de vloed wijst.

Een tweede publicatie over dezen zelfden merkwaardigen ester verscheen zeer korten tijd geleden van deze zelfde auteurs, waarin de stabiliteitsverschijnselen naast zijne verzadigde oplossing in 80 % alcohol bij verschillende temperaturen is bestudeerd. De aard van dit onderzoek is in sterke mate vergelijkbaar met dat van LADENBURG en DOCTOR, al moet erkend worden, dat de Engelsche onderzoekers een beter inzicht in de zaak hebben. Gemeten zijn oplosbaarheden van L, D, R, $(R + D)$ en $(R + L)$ bij 3 temperaturen nl. 35°, 25° en 10° in het genoemde oplosmiddel.

De oplosbaarheid wordt in een oplosbaarheidsdiagram, dat betrokken is op een constante hoeveelheid oplosmiddel met tot assen een D en L as, waarop de hoeveelheden opgelost D en L worden afgezet. Om in dit diagram ook de oplosbaarheid van $(R + D)$ en $(R + L)$ te kunnen aangeven, (die van R ligt natuurlijk

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[1]) A. WERNER, Lehrbuch der Stereochemie, p. 81. Jena 1904.

[2]) VICTOR MEYER und P. JACOBSON, Lehrbuch der Organischen Chemie 1[er] Band 2[te] Auflage, p. 105. Leipzig 1907.

[3]) J. ch. Soc. Trans **85**, 383 (1904).

[4]) Idem. **91**, 909 (1907).

[5]) Idem. **95**, 1386 (1909).

steeds op de lijn van gelijke samenstellingen D en L) was eene analyse dier oplossingen noodig, welke FINDLAY en mej. HICKMANS hebben bewerkstelligd op dezelfde wijze als LADENBURG en DOCTOR hunne ($D + L$) oplossingen hebben geanalyseerd nl. door den draaiingshoek der afdampingsresidu's hunner oplossingen te meten en uit dezen hoek in verband met het bekende draaiingsvermogen van L en D, het gehalte aan elk van deze in de oplossingen uit te rekenen. De juistheid dezer methode is aan bedenking onderhevig, zooals boven (cf. pag. 21 e. v.) reeds besproken is. Aangestipt zij nog, dat de door FINDLAY en HICKMANS telkens uit de vijf door hen bepaalde punten geconstrueerde isothermen een onaannemelijk verloop vertoonen, en daarmee eene sterke indicatie leveren, dat de gevolgde analyse-methode onbetrouwbaar is. Niet onvermeld mag worden gelaten, dat dit onderzoek met de menthol-amandelzure esters tot onderwerp, sterke verwantschap vertoont met mijne onderzoekingen over de strychnine tartraten, waardoor ik mij genoodzaakt heb gezien tot het doen eener voorloopige mededeeling [1]) om de gelijktijdigheid van mijne experimenten met die der Engelsche onderzoekers ter kennis te brengen.

Ik meen hiermede de gevallen van partiëele racemie, voor zoover ik die in de litteratuur heb kunnen vinden, voldoende te hebben besproken en zal thans overgaan tot het experimenteele gedeelte dezer verhandeling, waarin ik verslag zal doen van mijn experimenteele werk, en tevens mijne waarnemingen en de door mij gevolgde methode van onderzoek aan die der overige onderzoekers in dit gebied zal toetsen.

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[1]) Versl. Kon. Academie **18**[1], 329 (1909).

HOOFDSTUK IV.

EXPERIMENTEELE ONDERZOEKINGEN OVER DE STRYCHNINE TARTRATEN.

I. Inleiding.

a. Bereiding der uitgangsprodukten.

Als grondstoffen werden gebruikt:

d-wijnsteenzuur, uit den handel;

druivenzuur, afkomstig van Kahlbaum en strychnine van Zimmer & C°. en van Merck.

Het *l*-wijnsteenzuur werd bereid uit het druivenzuur door middel van cinchonine.

De drie stoffen voor het onderzoek vereischt: het strychnine *l*- resp. *d*-tartraat en het strychnine *r*-tartraat werden verkregen volgens het voorschrift van Doctor. [1])

b. Het kristalwater.

Het kristalwatergehalte werd op de gebruikelijke wijze bepaald door een afgewogen hoeveelheid in een droogstoof op 110° tot constant gewicht te verwarmen. De volgende uitkomsten worden daarbij verkregen:

TABEL 1.

	l-tartraat.		*d*-tartraat.		*r*-tartraat.	
hydraat	0.4789	1.1516	0.6928	1.0102	0.9330	1.5926
na verwarming	0.4407	1.0596	0.5932	0.8649	0.8314	1.4236
dus water	0.0382	0.0920	0.0996	0.1453	0.1016	0.1690
d. i. % water	7.998	7.98	14.376	14.384	10.89	10.61
d. i.	3.95*aq*	3.95*aq*	7.6	7.6	5.5	5.5
dus	4*aq*		$7^1/_2 aq$		$5^1/_2 aq$	

[1]) Ik moet hier helaas zeer onvolledig zijn. Omtrent deze (uit den aard der zaak dagelijks weerkeerende) bereidingen liet Dutilh geen enkele aanteekening achter. Kleine, door hem ingevoerde wijzigingen, blijven mogelijk. Kruyt.

Vergelijkt men deze uitkomsten met die van andere onderzoekers, dan komt men tot de volgende tabel (alle gemiddelde der series).

TABEL 2.

	l-tartraat.		*d*-tartraat.		*r*-tartraat.	
	°/₀ H_2O	mol. *aq.*	°/₀ H_2O	mol. *aq.*	°/₀ H_2O	mol. *aq.*
DOCTOR [1])	7.20	3½	13.37	7	12.47	6½
PASTEUR [2])	7.8	4	14.45	7½		
DUTILH...........	7.99	4	14.38	7½	10.75	5½

Men ziet dat voor *d*- en *l*-tartraat de uitkomsten van PASTEUR en de onze overeenstemmen, die van DOCTOR afwijken.

c. Het specifiek gewicht.

Specifiek gewicht werd zoowel van de anhydrische als van de gehydrateerde zouten bepaald. Het geschiedde door middel van gewone fleschjes-pycnometers met nauwkeurig ingeslepen stop. Als vloeistof werd watervrije toluol gebruikt, terwijl de lucht verwijderd werd door evacuatie aan de luchtpomp gedurende eenige minuten. De bepalingen geschiedden bij 25°.0.

De uitkomsten zijn nu:

TABEL 3.

anhydrisch *d*-tartraat	anhydrisch *l*-tartraat	anhydrisch racemaat [4])
1.430 1.428 1.428	1.382 1.381	1.384 1.386
gemidd. 1.429	*gemidd.* 1.382	*gemidd.* 1.385
DOCTOR [3]) 1.43218	DOCTOR 1.34050	DOCTOR 1.36653

[1]) l. c.

[2]) Ann. d. Ch. et d. Ph. [3] 38.

[3]) De bepalingen van DOCTOR zijn bij 20° geschied.

[4]) Deze getallen ontbraken in DUTILH's journaal. Daar juist hunne herbepaling van gewicht is, werden deze spec. gew. bepalingen op mijn verzoek door een leerling van Prof. VAN ROMBURGH, den heer C. F. VAN DUIN, chem. stud., verricht. Hem zij daarvoor hier dank gebracht. KRUYT.

gehydrateerd *d*-tartraat	gehydrateerd *l*-tartraat	gehydrateerd racemaat
1.391 1.390	1.386 1.388	1.373 1.370
gemidd. 1.391	*gemidd.* 1.387	*gemidd.* 1.372
DOCTOR 1.543	DOCTOR 1.60802	DOCTOR 1.46968

Ook hier bestaat dus een vrij groot verschil met DOCTOR's cijfers, vooral voor de gehydrateerde zouten.

d. De Smeltpunten.

DOCTOR geeft l. c. de volgende smeltpunten op

racemaat........ 222°
d-tartraat........ 228°
l- „ .... 242°

terwijl een aequimoleculair mengsel van *d*- en *l*-tartraat bij 233—236° smolt.

Er rees nu de vraag: zijn dit de smeltpunten der hydraten of der anhydrische stoffen? A priori is het laatste wel het waarschijnlijkst, daar bij deze hooge temperatuur de waterdampspanning natuurlijk een zeer aanzienlijke is en daardoor de bepaling van het smeltpunt (als dat al realiseerbaar is) groote voorzorgen eischt.

Bij de eerste bepalingen met het racemaat-hydraat in open capillairen bleek bij gewoon opwarmen nu al terstond, dat de hydraten zich ontleden, dat men bij onscherp te bepalen temperaturen gedeeltelijke smelting te zien kreeg van de kristallen, die door waterverlies dof werden. Bovendien had bij deze langzame verhitting bruinkleuring plaats. Brengt men de capillairen echter in H_2SO_4, dat reeds hoog verhit is (± 220°), dan heeft terstond een uitkoken van het kristalwater plaats en vervolgens een smelten. Bij de hydraten van de enkel-tartraten werd het volgende waargenomen. Brengt men het *d*-hydraat in een bad van 220° dan heeft eerst een moment geheel smelting plaats, onmiddelijk gevolgd door koken en hernieuwde stolling, die bij 227 à 228° door blijvende smelting wordt gevolgd. Daar er dus smelting van het hydraat was opgetreden, werd door toegesmolten capillairen in een telkens hooger verhit bad te dompelen, dit smeltpunt opgezocht. Geen smelting had plaats in baden van 120°, 140° resp. 160°, *wel* in een van 180°, niet bij 175°, wel bij 177.5° eveneens bij 174.5°. Het verschijnsel komt dus niet geheel regelmatig voor. Er werden nu capillairen gebruikt, die vlak boven de vaste stof waren toegesmolten; daarbij bleek dat totale

smelting zelfs bij 170° mogelijk is, bij 155° gedeeltelijke smelting tot een troebele vloeistof plaats had en bij temperaturen beneden 150° practisch geen vervloeiïng plaats had. Wij zullen de zaak dus zoo moeten opvatten: het hydraat heeft een smeltpunt omstreeks 177°, maar slechts metastabiel, daar de stabiele toestand daar anhydride-vloeistof en damp is. Een nadere bestudeering der monovariante evenwichten wordt echter onmogelijk gemaakt door de betrekkelijk spoedig optredende ontleding, die uit geelkleuring der smelt blijkt.

Bij racemaat en *l*-tartraat werden in toegesmolten buisjes analoge verschijnselen waargenomen. Het was intusschen genoegzaam gebleken, dat de smeltpunten boven 200° aan de anhydrische stoffen toekomen. Bij de bepaling van de smeltpunten der gebruikte praeparaten werd nu gevonden

TABEL 4.

	DUTILH	DOCTOR
anhydrisch *d*-tartraat.	227°—228°	228°
,, *l* ,, ...	246°	242°
,, *r* ,, ...	223°—224°	222°

een zeer bevredigende overeenstemming met de bepalingen uit LADENBURG's laboratorium.

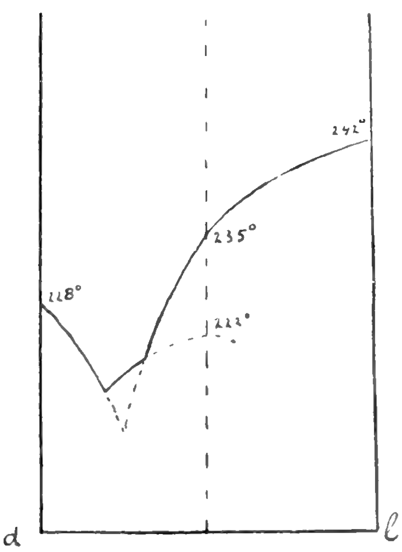

Fig. 10.

Er moet nu intusschen op een hoogst belangrijk feit de aandacht gevestigd worden, een feit waarvan de beteekenis aan LADENBURG c.s. ontsnapt schijnt te zijn. Een aequimoleculair mengsel der anhydrische tartraten smelt bij 233°—236°, het anhydrische racemaat daarentegen bij 222°. *Het anhydrische racemaat is dus geen mengsel der tartraten maar een chemisch individu.* Deze conclusie wordt bevestigd door de bepalingen der s. g. in onze vorige § besproken: *het s. g. van het anhydrische racemaat is niet hetzelfde als dat hetwelk men voor een mengsel der enkeltartraten berekent.*

Wij kunnen ons derhalve het smeltdiagram der anhydrische zouten voorstellen, zooals in fig. 10 geschetst. Wellicht ook is het anhydrysche racemaat *nimmer* stabiel naast vloeistof.

De beteekenis dezer conclusie zal ons in de volgende § blijken.

e. De onderzoekingen met den tensimeter van Bremer-Frowein.

Doctor heeft het overgangspunt in het hier besproken systeem ook tensimetrisch trachten te bepalen. Hij bracht daartoe in den eenen bol van den tensimeter racemaat, waaraan eenig water onttrokken was. Aannemend nu dat deze reactie zich afspeelt:

$$2\,[(r\text{-tartraat})\,6\tfrac{1}{2}\,aq] \rightleftarrows (d\text{-tartraat})\,7\,aq + (l\text{-tartraat})\,3\,^1/_2\,aq + 2\,^1/_2\,H_2O$$

zal dus aan deze zijde van den tensimeter een vier phasen systeem, bestaande uit *r*, *d* en *l*-tartraathydraat en gas ontstaan; de druk van dit monovariante systeem zal nu bij de temperatuur van het overgangspunt gelijk moeten zijn aan dien van het monovariante systeem, dat in de andere bol van den tensimeter zich bevindt nl. *d*- en *l*-tartraat, met hun verzadigde vloeistof en damp. M. a. w. de temperatuur, waar het spanningsverschil 0 is, is die van het overgangspunt, omdat in het overgangspunt de dampdruk-curven der monovariante systemen elkaar snijden. (Zie Hoofdstuk II).

Zulke verhoudingen blijken nu intusschen in het onderzochte systeem absoluut niet op te treden. Onttrekt men aan *r*-tartraat hydraat water dan splitst het zich niet in de twee enkeltartraat hydraten maar er ontstaat *r*-tartraat anhydride! In het eerste bolletje is dus heelemaal geen monovariant, maar een bivariant evenwicht, de dampspanning is door de temperatuur niet bepaald, van een 0 worden in het overgangspunt als criterium is dus geen sprake.

Onbegrijpelijkerwijze heeft Doctor echter een precies kloppende uitkomst gekregen, hetgeen niet slechts op bovenstaande theoretischen grond verwonderlijk is, maar ook om een experimenteeltechnische reden. Zijn geheele onderzoek is nl. op een middag tusschen 2 uur en 5.45 verricht, alle dampspannings-evenwichten stelden zich bij temperatuursveranderingen binnen enkele minuten in, hetgeen volkomen in strijd is met onze ondervinding en die van anderen, die ons over hun ervaring met den differentiaal tensimeter inlichtten.

Wij hebben dan ook maanden er aan besteed deze proef van Doctor te reproduceeren, maar steeds *zonder eenig succes.* Vreezend, dat zulks aan eigen onbekwaamheid of gebrek in de instrumenten zou liggen, bepaalden wij het overgangspunt van $Na_2\,SO_4$ 10 aq, vonden dit evenwel met zeer normaal functioneerden tensimeter tusschen 32° en 33°. De conclusie waartoe wij komen is derhalve deze: de bepaling van het overgangspunt is onmogelijk langs den door Doctor beschreven tensimetrischen weg.

II. De Oplosbaarheidsbepalingen.

Wij hebben er boven op gewezen, dat de conclusies waar Ladenburg en Doctor toe gekomen zijn, in strijd zijn met de theoretische verwachtingen en wij hebben dan ook reeds op fouten in hun proefmethoden gewezen (Cf. pag. 19 e. v.).

Het kwam er dus op aan vast te stellen op een wijze, die aan geen bedenking onderhevig is, hoe de zaken zich in dit systeem verhouden en daarom werd besloten de oplosbaarheids-isothermen bij verschillende temperaturen te bepalen, door na te gaan hoe de oplosbaarheid van D resp. L verandert als men L resp. D aan de oplossing toevoegt.

De uitvoering dezer bepalingen geschiedde aldus:

Voor de bepaling der oplosbaarheid der enkele stoffen in water werd in een fleschje de vaste stof met water geschud, door het fleschje aan een horizontale, roteerende as, in een thermostaat aangebracht, rond te draaien. Na eenigen tijd werd in een pipet, die van onder met een buisje was verbonden, waarin zich een watten prop in bevond, ongeveer 5 c.c.m. heldere vloeistof opgevangen en overgebracht in een klein glaasje met vlakgeslepen rand, waarop een vlakgeslepen deksel-plaatje werd gelegd. Door weging werd de hoeveelheid oplossing bepaald, waarna het glaasje ongedekt in een droogstoof werd gebracht en op 110° de inhoud tot constant gewicht ingedampt. Door eenige uren later eenzelfde proef te herhalen werd vastgesteld na hoeveel tijd de oplossing verzadigd was. Op grond daarvan werd steeds minstens 2 dagen geschud.

De bepaling der oplosbaarheid van de eene stof in onverzadigde oplossing van de andere werd als volgt uitgevoerd. Wil men b.v. de oplosbaarheid van *d*-tartraat bepalen in oplossingen, die *l*-tartraat bevatten, dan bereidt men zich eerst een hoeveelheid verzadigde *l*-tartraat-oplossing en zuigt die door watten van het vaste zout af. Brengt men nu in een fleschje, waarin zich reeds vast *d*-tartraat bevindt, een bekende hoeveelheid water en een bekende hoeveelheid der verzadigde *l*-oplossing toe. Pipeteert men nu, nadat de verzadiging aan *d*-tartraat bewerkstelligd is, wat heldere oplossing af, dan is uit te rekenen, hoeveel *l*-tartraat zich in die hoeveelheid bevindt en uit het gewicht der droge stof, die na indampen terugblijft (en natuurlijk *d*- + *l*-tartraat is) vindt men gemakkelijk de oplosbaarheid van het *d*-tartraat.

Heeft men op deze wijze de verschillende takken der isotherm bepaald, dan kan men de ligging der snijpunten, zooals die door graphische interpolatie gevonden zijn nog verifieeren door water

met twee Bodenkörper te schudden. Men vindt dan door indampen de som der beide oplosbaarheden. De meetkundige plaats van alle oplossingen, die een zekere totaal oplosbaarheid van de twee bestanddeelen representeeren, is een rechte lijn die de punten verbindt, welke op elk der consentratie-assen die totaal oplosbaarheid aangeven. Het door interpolatie gevonden snijpunt moet dus op deze lijn liggen.

Uit Tabel 5 ziet men de uitkomsten voor de isotherm van 40.0°. [1])

Hierbij zij het volgende opgemerkt. Als concentratie is steeds opgegeven het aantal m.G. op 5000 m.G. water. Uit proef 17 blijkt, dat het racemaat, hoewel niet stabiel naast oplossing, toch vrij bestendig is: de 1e en 2e bepaling geschiedden na 2 dagen, de 3e en 4e na 3 dagen, de 5e en 6e na 12 uur schudden. Voorts ziet men in de graphische weergave dezer uitkomsten (fig. 11) dat het punt voor vloeistof verzadigd aan *d*- en *l*-tartraat gevonden door graphische intra- (resp. extra-) polatie bevredigend overeenstemt met de zooeven genoemde meetkundige plaats. Bij proef 9 is het voorts gelukt een punt op het metastabiele verlengde der *d*-verzadigingslijn te bepalen.

[1]) Alle temperaturen zijn gecontroleerd met een door de Phys. Techn. Reichsanst. te Charlottenburg geijkte thermometer, verdeeld in 0.1°.

TABEL 5.

Isotherm van 40°.0.

N°.	Vaste phase.	Concentratie oplosmiddel.	Bij indampen van Gr. oplossing.	Gr. droogrest.	Samenstelling der opl. in m.G. *l*	*d*	Gemiddeld *l*	*d*
1	*d*	Water	4.9722	0.1708	0	177.8	**0**	**175.9**
			5.0132	0.1694	0	174.9		
			5.0038	0.1692	0	175.0		
			5.0044	0.1690	0	174.8		
			5.0016	0.1694	0	175.3		
			5.0028	0.1716	0	177.6		
2	*l*	id.	4.9872	0.1114	114.2	0	**114.5**	**0**
			5.0046	0.1120	114.5	0		
			4.9932	0.1118	114.5	0		
			4.9810	0.1118	114.8	0		
3	*l*	35.4 *d*	4.9986	0.1308	99.0	35.4	**98.8**	**35.4**
			5.0050	0.1306	98.6	35.4		
16	*l*	59.3 *d*	5.0168	0.1474	92.1	59.3	**92.5**	**59.3**
			5.0198	0.1482	92.8	59.3		
10	*l*	71.4 *d*	5.0120	0.1560	89.2	71.4	**89.6**	**71.4**
			5.0148	0.1568	89.9	71.4		
12	*l*	89.5 *d*	5.0134	0.1694	85.4	89.5	**85.5**	**89.5**
			5.0126	0.1696	85.6	89.5		
13	*l*	108.0 *d*	5.0216	0.1838	82.0	108.0	**82.8**	**108.0**
			5.0308	0.1852	83.1	108.0		
			5.0190	0.1846	82.9	108.0		
			5.0046	0.1844	83.3	108.0		
5	*d*	23.0 *l*	5.0118	0.1716	23.0	154.3	**23.0**	**154.4**
			5.0188	0.1720	23.0	154.4		
7	*d*	46.2 *l*	5.0196	0.1770	46.2	136.6	**46.2**	**136.5**
			5.0108	0.1766	46.2	136.4		
6	*d*	57.9 *l*	5.0174	0.1796	57.9	127.7	**57.9**	**127.4**
			5.0190	0.1790	57.9	127.0		
9	*d*	87.3 *l*	5.0104	0.1904	87.3	110.2	**87.3**	**110.7**
			4.9850	0.1900	87.3	111.1		
14	*d* + *l*	Water	5.0246	0.1920	196.9		**199.2**	
			5.0174	0.1930	200.0			
			5.0306	0.1930	199.5			
			5.0280	0.1938	200.4			
17	*r*	id.	5.0192	0.1874	193.9 *r*		**192.1**	
			5.0228	0.1876	194.0 *r*			
			5.0140	0.1843	190.8 *r*			
			5.0200	0.1836	189.8 *r*			
			5.0070	0.1853	192.2 *r*			
			5.0210	0.1854	191.7 *r*			

TABEL 6.

Isotherm van 25°.0.

N°.	Vaste phase.	Concentratie oplosmiddel.	Bij indampen van Gr. oplossing.	Bij indampen van Gr. droogrest.	Samenstelling der opl. in m.G. *l*	Samenstelling der opl. in m.G. *d*	Gemiddeld *l*	Gemiddeld *d*
1	*d*	Water	5.0123	0.1123	0	114.6	**0**	**114.5**
			5.0100	0.1126	0	114.9		
			10.0170	0.2234	0	114.1		
			10.0150	0.2239	0	114.4		
2	*l*	id.	4.9934	0.0712	72.3	0	**72.6**	**0**
			4.9921	0.0711	72.2	0		
			10.0080	0.1439	73.0	0		
			10.0012	0.1434	72.7	0		
7	*l*	23.0 *d*	5.0061	0.0857	64.1	23.0	**63.0**	**23.0**
			5.0000	0.0847	63.2	23.0		
			4.9997	0.0838	62.2	23.0		
			4.9917	0.0839	62.5	23.0		
6	*l*	46.2 *d*	5.0056	0.0996	55.3	46.2	**55.4**	**46.2**
			5.0162	0.1005	55.8	46.2		
			5.0046	0.0996	55.3	46.2		
			5.0093	0.0995	55.1	46.2		
8	*l*	69.7 *d*	5.0120	0.1152	47.9	69.7	**48.0**	**69.7**
			5.0040	0.1150	47.9	69.7		
			5.0152	0.1156	48.3	69.7		
			5.0167	0.1155	48.1	69.7		
3	*d*	14.6 *l*	5.0120	0.1157	14.6	103.5	**14.6**	**103.2**
			5.0002	0.1148	14.6	102.9		
5	*d*	29.2 *l*	5.0027	0.1176	29.2	91.2	**29.2**	**91.6**
			5.0100	0.1186	29.2	92.0		
4	*d*	43.9 *l*	5.0130	0.1213	43.9	80.1	**43.9**	**80.0**
			5.0165	0.1213	43.9	80.0		
			5.0130	0.1210	43.9	79.8		
			5.0160	0.1214	43.9	80.1		
9	*d* + *l*	Water	5.0303	0.1229	125.2		**125.2**	
			5.0107	0.1225	125.3			
			5.0124	0.1222	124.9			
			5.0031	0.1223	125.3			
					r + *l*		*r* + *l*	
10	*r* + *l*	id.	5.0140	0.1176	120.1		**120.1**	
			5.0040	0.1172	119.9			
			5.0100	0.1176	120.2			
					r + *d*		*r* + *d*	
11	*r* + *d*	id.	5.0024	0.1180	120.8		**120.5**	
			5.0200	0.1180	120.4			
			5.0030	0.1176	120.4			
					r		*r*	
13	*r*	id.	5.0060	0.1209	123.7		**123.5**	
			5.0136	0.1213	124.0			
			5.0130	0.1208	123.5			
			5.0140	0.1204	123.0			

In tabel 6 vindt men de uitkomsten der geheel overeenkomstig uitgevoerde bepaling voor de isotherm van 25°.

Men ziet hoe door de bepaling 1, 2, 7, 6, 8, 3, 5 en 4 de takken voor de oplosbaarheid van *l*- en *d*-tartraat zijn bepaald en door bepaling 9 de ligging van het snijpunt bevestigd is; dat vervolgens door de oplosbaarheid van $r + l$ te bepalen is vastgesteld in welk punt der *l*-tak het snijpunt met de racemaattak gelegen is; en dat vervolgens eenzelfde punt op de *d*-tak is gevonden voor $r + d$. Bij deze bepalingen 9, 10 en 11 is natuurlijk weer van diezelfde meetkundige-plaats-methode gebruik gemaakt als bij de isotherm van 40° is aangegeven. Eindelijk is de oplosbaarheid van racemaat in water bepaald. Het daardoor gevonden punt moet natuurlijk op het metastabiele gedeelte der *r*-oplosbaarheidstak liggen.

Ziet men de graphische voorstelling (fig. 11) aan, dan blijkt terstond dat wij *bij* 25° *in het overgangstraject zijn.* De tak der *r*-oplosbaarheid snijdt de lijn, die de assenhoek middendoor deelt nog niet, het racemaat is naast een oplossing, die *l* en *d* aequimoleculair bevat, niet bestendig. Ten overvloede is door een proef dit ook nog eens direct bewezen.

Een hoeveelheid der afgepipeteerde heldere vloeistof, die aan racemaat verzadigd was (van proef 13 afkomstig), werd gedurende eenige dagen met een weinig *l*-tartraat geschud; uit onze graphische voorstelling kunnen wij aflezen wat er gebeuren moet: aangezien de oplossing oververzadigd is t. o. v. *l*-tartraat, zal dit uitkristalliseeren en de vloeistofsamenstelling zich dus bewegen in de richting der met een pijl voorziene loodlijn. Inderdaad bleek de droogrest van 123.5 teruggegaan te zijn op 116.4, zie onderstaande Tabel. De proeven werden gedaan na resp. 1, 2, 3 en 8 dagen schudden.

TABEL 7.

N°.	Gr. Oplossing	Droogrest	Samenstelling	Gemiddeld
15	5.0102	0.1140	116.4	116.4
	5.0086	0.1140	116.4	
	5.0056	0.1132	115.6	
	4.9964	0.1144	117.1	

Ten einde eveneens te bewijzen, dat de aan $d + l$ verzadigde vloeistof (N°. 9) in labielen toestand was t.o.v. het racemaat, werd als proef N°. 14 deze vloeistof twee dagen met *r* geschud, waarbij eveneens een teruglopen werd geconstateerd. Zie Tabel 8.

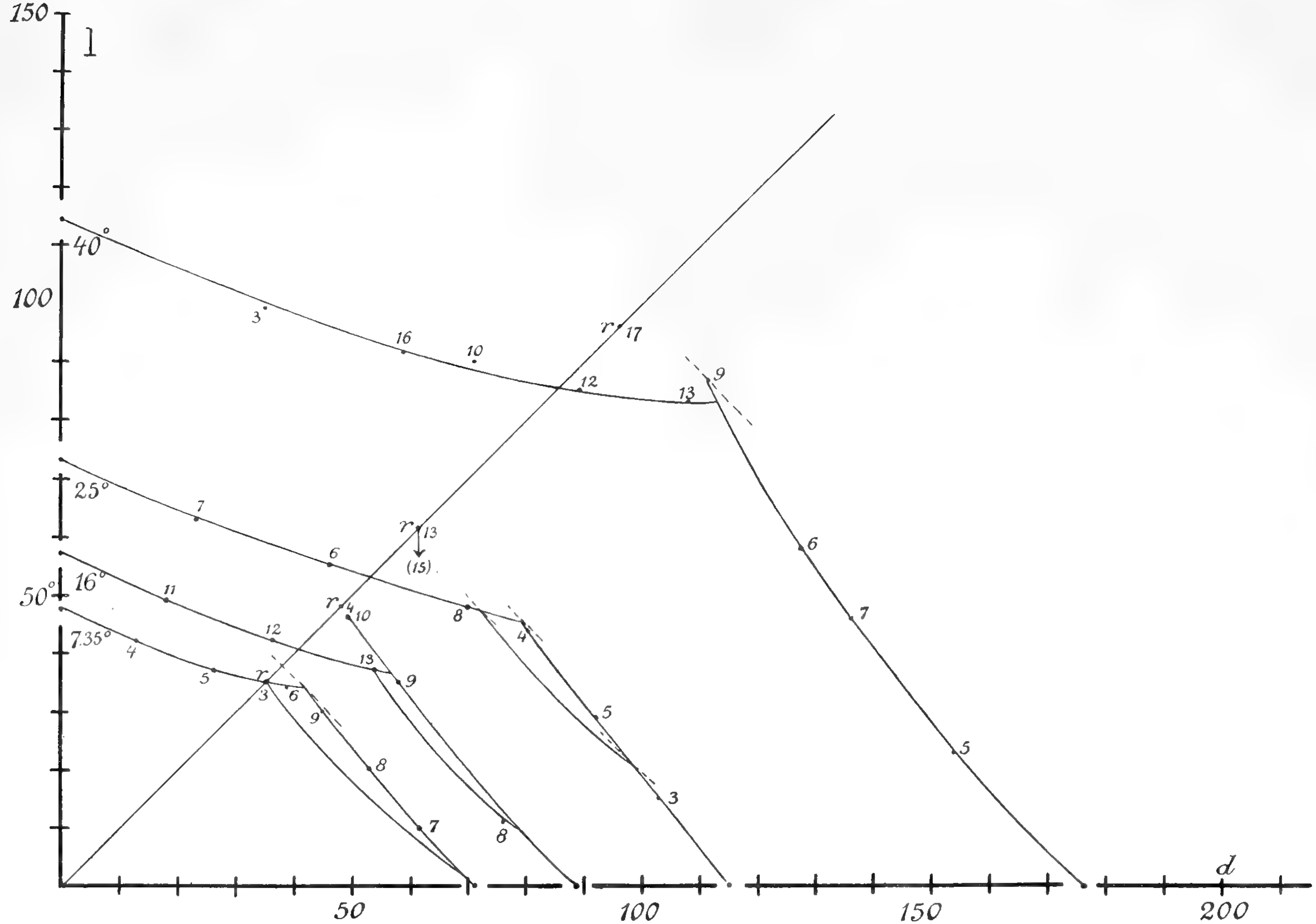

Fig. 11.

TABEL 8.

N°.	Gr. Oplossing	Droogrest	Samenstelling	Gemiddeld
14	5.0120	0.1212	123.9	123.6
	5.0080	0.1214	124.2	
	5.0146	0.1202	122.8	

Ten einde vast te stellen bij welke temperatuur het overgangstraject zijn laagste begrenzing had, werd nu een isotherm bij aanzienlijk lagere temperatuur bepaald nl. bij 16°. Tabel 9 geeft een overzicht van de uitkomsten, die in fig. 11 weer graphisch zijn weergegeven. Men ziet nu terstond in dat wij hier bij 16° *nog steeds in het overgangstraject zijn;* de r-oplossing is nog steeds labiel.

Bij deze tabel dient het volgende aangeteekend te worden. Het principe is natuurlijk weer hetzelfde als bij de isotherm van 25°: eerst de d- en l-oplosbaarheidstakken bepalen en hun snijpunt controleeren, dan de snijpunten bepalen met de racemaat-oplosbaarheidslijn ($r + l$ en $r + d$). Dat bleek nu echter minder eenvoudig te gaan. Schudt men nl. racemaat en d met water, dan heeft men aanvankelijk en nog gedurende een betrekkelijk langen tijd (tengevolge van het langzaam voortschrijdende oplossen) racemaat naast water. Aangezien wij nog in het overgangstraject zijn is het rac. onder deze omstandigheden labiel en zal zich in d en l splitsen. Het aanwezige vaste d-zout zal dat proces, bijwijze van entingsmateriaal, bevorderen. Zoodoende zal al spoedig zooveel r gesplitst zijn, dat er vast l op den bodem ligt. Met zuiver water als oplossingsvloeistof werden dan ook zeer onregelmatige uitkomsten verkregen. Ten einde nu deze racemaatsplitsing te voorkomen werden geheel of gedeeltelijk verzadigde oplossingen der enkeltartraten als schud-vloeistoffen gebruikt. Met zulke oplosmiddelen zijn de proeven 5 en 6 uitgevoerd, die, zooals men ziet, nu zeer regelmatige uitkomsten gegeven hebben. Nochtans lijken ons de uitkomsten voor de oplosbaarheid van $r + d$ niet geheel vertrouwbaar en schijnt het ons onmogelijk een volkomen correcte methode te vinden voor de bepaling van de oplosbaarheid van het racemaat naast zijn meest oplosbaar splitsingsproduct. Gelukkig is het echter juist de oplosbaarheid van racemaat met het minst oplosbare splitsingsproduct waaruit wij het einde van het overgangstraject moeten leeren kennen.

TABEL 9.

Isotherm van 16°.

N°.	Vaste phase.	Concentratie oplosmiddel.	Bij indampen van Gr. oplossing.	Bij indampen van Gr. droogrest.	Samenstelling der opl. in m.G. *l*	Samenstelling der opl. in m.G. *d*	Gemiddeld *l*	Gemiddeld *d*
1	*d*	Water	5.008	0.0868	0	88.2	**0**	**88.6**
			5.012	0.0873	0	88.6		
			5.002	0.0876	0	89.1		
			5.003	0.0870	0	88.5		
2	*l*	id.	4.9886	0.0563	57.1	0	**57.5**	**0**
			4.9804	0.0564	57.2	0		
			4.9500	0.0566	57.8	0		
			5.0000	0.0572	57.9	0		
11	*l*	17.8 *d*	4.996	0.0654	48.5	17.8	**48.8**	**17.8**
			4.998	0.0660	49.1	17.8		
12	*l*	35.7 *d*	4.997	0.0768	42.3	35.7	**42.5**	**35.7**
			4.996	0.0768	42.6	35.7		
			5.002	0.0771	42.5	35.7		
			5.001	0.0772	42.5	35.7		
13	*l*	53.7 *d*	4.962	0.0880	36.6	53.7	**37.0**	**53.7**
			4.961	0.0881	36.6	53.7		
			4.958	0.0887	37.4	53.7		
			4.963	0.0886	37.2	53.7		
8	*d*	11.5 *l*	5.007	0.0862	11.5	76.1	**11.5**	**76.5**
			5.000	0.0868	11.5	76.8		
9	*d*	34.7 *l*	5.000	0.0904	34.7	57.4	**34.7**	**57.7**
			5.007	0.0902	34.7	57.2		
			5.002	0.0906	34.7	57.7		
			5.007	0.0914	34.7	58.5		
10	*d*	46.4 *l*	4.965	0.0937	46.4	49.8	**46.4**	**49.4**
			4.950	0.0929	46.4	49.2		
			4.950	0.0930	46.4	49.2		
			4.966	0.0933	46.4	49.3		
3	*d* + *l*	Water	5.006	0.0960	97.7		**98.1**	
			5.011	0.0966	98.3			
			5.010	0.0962	98.3			
					r + *d*		*r* + *d*	
5	*r* + *d*	Verg. in de tekst.	5.009	0.0885	89.9		**89.9**	
			5.004	0.0882	89.7			
			4.996	0.0882	89.9			
			4.935	0.0874	90.1			
					r + *l*		*r* + *l*	
6	*r* + *l*	id.	5.006	0.0912	92.8		**92.9**	
			5.005	0.0910	92.8			
			5.004	0.0915	92.8			
			4.999	0.0917	93.4			
					r		*r*	
4	*r*	Water	4.962	0.0928	95.3		**95.6**	
			4.952	0.0934	95.4			
			4.961	0.0936	96.1			
			4.963	0.0932	95.6			
			5.004	0.0928	94.5			
			4.005	0.0930	94.5			

TABEL 10.

Isortherm van 7°.35.

N°.	Vaste phase.	Concentratie oplosmiddel.	Bij indampen van Gr. oplossing.	Gr. droogrest.	Samenstelling der opl. in m.G. *l*	*d*	Gemiddeld *l*	*d*
2	*d*	Water	3.9882	0.0550	0	69.9	**0**	**70.7**
			4.4532	0.0617	0	70.2		
			4.9483	0.0697	0	71.4		
			4.9644	0.0698	0	71.3		
1	*l*	id.	4.9731	0.0467	47.4	0	**47.4**	**0**
			4.9613	0.0465	47.3	0		
			4.9278	0.0465	47.6	0		
			4.9544	0.0465	47.4	0		
5	*l*	26.3 *d*	5.0248	0.0628	37.0	26.3	**37.1**	**26.3**
			5.0080	0 0628	37.2	26.3		
5	*l*	39.5 *d*	5.0021	0.0724	33.9	39.5	**34.0**	**39.5**
			4.9938	0.0714	34.0	39.5		
4	*l*	13.1 *d*	5.0474	0.0558	42.6	13.1	**42.2**	**13.1**
			4.8934	0.0532	41.8	13.1		
7	*d*	10.0 *l*	4.9863	0.0708	10.0	62.0	**10.0**	**62.1**
			4.9850	0.0709	10.0	62.1		
8	*d*	20.0 *l*	4.9538	0.0714	20.0	53.1	**20.0**	**53.3**
			4.9430	0.0716	20.0	53.5		
9	*d*	30.1 *l*	4.9640	0.0746	30.1	46.2	**30.1**	**45.9**
			4.9612	0.0740	30.1	45.6		
14	*d* + *l*	Water	4.9617	0.0745	76.2		**75.5**	
			4.9808	0.0744	75.8			
14*bis*			4.975	0.0736	75.1			
			4.996	0.0738	75.0			
					r + *d*		*r* + *d*	
11	*r* + *d*	id.	5.0164	0.0743	75.2		**75.0**	
			5.0270	0.0742	74.9			
					r + *l*		*r* + *l*	
12	*r* + *l*	id.	4.8998	0.0701	72.6		**70.0**	
			4.9582	0.0699	71.5			
12*bis*		id.	4.950	0.0688	70.4			
			4.968	0.0685	69.0			
12*ter*		id.	5.032	0.0684	68.9			
			4.948	0.0674	69.0			
			5.022	0.0682	68.8			
			4.983	0.0679	69.1			
					r		*r*	
3*bis*	*r*	id.	4.942	0.0675	69.2		**70.1**	
			4.946	0.0678	69.5			
3*ter*		id.	4.940	0.0688	70.6			
			4.959	0.0691	70.7			
			4.927	0.0686	70.6			
			4.948	0.0689	70.6			

Om het einde van het overgangstraject te vinden moest dus naar nog lagere temperatuur gegaan worden, welke proeven in een koude Januarimaand werden uitgevoerd. De thermostaat werd daarom op een binnenplaats in de open lucht geplaatst en was door die plaatsing buiten den wind (die het vlammetje zou uitblazen) en het zonlicht. Zoodoende kon een reeks proeven gedaan worden bij 7°.35, waarvan de uitkomsten in Tabel 10 zijn aangegeven.

Men zal uit de tabel en uit fig. 11, die de uitkomsten weergeeft, zien dat wij thans inderdaad het einde van het overgangstraject bereikt hebben. De oplosbaarheid van het racemaat (70.1) is binnen de proeffouten gelijk geworden aan die van racemaat + l tartraat (70.0); de lijn, die de assenhoek midden doordeelt, gaat juist door het snijpunt der l en r oplosbaarheidtakken.

Toch diende nog gecontroleerd of inderdaad bij een iets hoogere temperatuur de twee juist genoemde oplosbaarheden genoegzaam verschilden om met eenige zekerheid te kunnen zeggen, dat hier inderdaad het einde van het overgangstraject gevonden is. Daartoe werd nog bij 8.°9 de oplosbaarheid van r en van $r + l$ bepaald. Tabel 11, die deze uitkomsten opsomt, toont aan dat daar inderdaad een verschil valt te constateeren, dat de proeffouten verre te boven gaat.

TABEL 11.

Oplosbaarheden bij 8°.9.

N°.	Vaste phase.	Oplossings middel.	Bij indampen van		Oplosbaarheid	Gemiddeld
			Gr. Oploss.	Gr. droogr.		
1	r	water	5.002	0.0737	74.8	**75.2**
			4.766	0.0710	75.6	
			4.979	0.0735	74.9	
			4.950	0.0736	75.5	
2	$r + l$		4.958	0.0712	72.8	**72.3**
			4.961	0.0714	73.0	
			5.093	0.0714	71.1	
			4.994	0.0714	72.5	

Wij mogen derhalve als conclusie zeggen, dat *het overgangstraject zich naar lagere temperaturen tot* (afgerond) $7\frac{1}{2}°$ *uitstrekt.*

Door enkele proeven wenschten wij nog te constateeren of het door Ladenburg en Doctor (l. c.) aangegeven punt van ongeveer 30° als overgangspunt, d. i. bovenste grens van het overgangstraject, inderdaad juist was. Het zekerst was dat na te gaan, wanneer men althans niet weer geheele isothermen wilde bepalen, uit het gelijkworden der oplosbaarheden van $r + l$ en $d + l$. Bij 25° waren deze waarden resp. 120.1 en 125.2 en waren wij dus beslist beneden de overgangstemperatuur. Bij 30° vonden wij uitkomsten, weergegeven in Tabel 12.

TABEL 12.

Oplosbaarheden bij 30°.

N°.	Vaste phase	Bij indampen van		Oplosbaarheid	Gemiddeld
		Gr. oplossing	Gr. droogrest		
1	$r + l$	5.011	0.1362	139.7	**140.6**
		5.010	0.1371	140.7	
		5.010	0.1372	140.7	
		5.009	0.1378	141.4	
2	$d + l$	5.018	0.1372	140.5	**141.0**
		5.018	0.1372	140.5	
		4.964	0.1368	141.7	
		5.008	0.1376	141.3	

Daar dus de waarde voor $d + l$ niet meer hooger is dan die voor $r + l$ schijnen wij juist even boven het overgangspunt te zijn; daar de afwijking niet veel van de proeffout verschilt mogen wij dit punt slechts op **29°—30°** vaststellen.

Ter contrôle werden intusschen nog de in Tabel 13 aangegeven bepalingen bij 32° verricht.

TABEL 13.

Oplosbaarheden bij 32°.

N°.	Vaste phase.	Bij indampen van		Oplosbaarheid	Gemiddeld
		Gr. oplossing	Gr. droogrest		
1	$r+l$	5.009	0.1234	126.3	**126.4**
		5.007	0.1234	126.3	
		5.006	0.1238	126.8	
		5.004	0.1235	126.5	
2	$d+l$	5.011	0.1486	152.8	**152.8**
		5.020	0.1484	152.3	
		5.013	0.1492	153.4	

Deze uitkomsten bevestigen het voorafgaande, zooals zonder meer duidelijk zal zijn.

Een opmerking zij gemaakt over de waarde der door LADENBURG en DOCTOR gegeven cijfers voor de oplosbaarheid. Op het eerste gezicht ziet men al dat aan hun methode deze twee fouten kleven: 1°. werd na 3 uur reeds tot analyseeren der oplossing overgegaan; 2°. werd daarbij niet van een watten filter gebruik gemaakt, terwijl men toch hier nimmer een volkomen bezinken der vaste stof bewerkstelligen kan. Is het feit dat L. en D. tot theoretisch onmogelijke uitkomsten gekomen zijn reeds een bewijs tegen hun methode, Tabel 14, die eenige uitkomsten bevat van die onderzoekers en de onze bij zelfde temperaturen, toont genoegzaam, dat zij nimmer de eindevenwichten bereikt hebben. Eenige malen vindt men niet gelijke temperaturen vergeleken, maar dan zal men zien, dat de door L. en D. gevonden oplosbaarheden, hoewel bij een *hoogere* temperatuur bepaald toch nog een *geringere* waarde aanwijzen dan bij onze proeven gevonden werd.

TABEL 14.

Temp.	Vaste phase.	Oplosbaarheden volgens		Bemerkingen.
		LADENBURG en DOCTOR	DUTILH	
42°	$d+l$	187.4	—	De r oplosbaarheid bij 42° volgens L. en D. is aan die volgens D. ongeveer gelijk.
40°	id.	—	199.2	
30°	$d+l$	128.8	141.0	
»	$r+l$	128.7	140.6	
7°	$d+l$	74.0	75.5	De bepaling v. D. geschiedden bij 7.35° doch die van L. en D. feitelijk bij 7° ± 0.5°.
	r	69.4	70.1	
19°	$d+l$	97.5	—	Men merke op dat D's bepalingen hier 3° lager werden uitgevoerd dan bij L. en D.
16°	id.	—	98.1	
19°	r	94.9	—	
16°	id.	—	95.6	

Zooals men ziet zijn de waarnemingen van LADENBURG en DOCTOR aanzienlijk veel te laag, behalve bij 7°, waar het verschil inderdaad gering is.

Het spreekt vanzelf dat men dan ook beter doet uit deze cijfers van LADENBURG en DOCTOR geen conclusies te trekken.

MEYERHOFFER heeft op eenige afwijkingen in die cijfers opmerkzaam gemaakt [1]), welke op omzettingen in de bestanddeelen zouden wijzen. Wij vermoedden, dat deze afwijkingen, gezien het voorafgaande, wel aan de onjuistheid der cijfers zouden zijn toe te schrijven. „Doch müssen diese Verhältnisse noch eingehender geprüft werden", schrijft MEYERHOFFER, die dus terecht ook niet op dit getallenmateriaal wil afgaan. Wij hebben nu één dier afwijkingen nagegaan: MEYERHOFFER zegt van de cijfers van L. en D.: „Auch weist die Löslichkeit des l-Tartrats zwischen 27 und 30° eine Unregelmässigkeit und hernach eine Richtungsänderung auf, in dem $\frac{d\text{-Löslichkeit}}{d\,t}$ merkwürdigerweise oberhalb 30° grösser ist als unterhalb 27°. Dies deutet auf eine Veränderung des l-Tartrats zwischen 27 und 30° hin . . . . ."

Wij hebben nu de oplosbaarheidslijn van het l-tartraat van 25° tot 30° bepaald. Tabel 15 geeft daarvan de uitkomsten weer.

[1]) Gleichgewichte der Stereoisomeren (l. c.) pag. 49 voetnoot.

Men ziet daaruit dat van een dergelijke bizonderheid geen sprake is, dat, binnen de proeffout $\frac{d\text{-Löslichkeit''}}{\text{,,}\quad dt}$ constant is, dat dus de oplosbaarheid lineair met de temperatuur verandert.

TABEL 15.

Oplosbaarheidscurve van het *l*-tartraat van 25° bis 30°.

Temp.	Bij indampen van		Oplosbaar-heid	Gemiddeld	$\frac{d\text{-Löslichk.''}}{\text{,,}\quad dt}$
	Gr. oplossing	Gr. droogrest			
25°	4.993	0.0712	72.3		
	4.992	0.0711	72.2	**72.6**	
	10.008	0.1439	73.0		
	10.001	0.1434	72.7		2.6
26°	4.997	0.0738	75.0	**75.2**	
	4.982	0.0740	75.4		2.8
27°	5.005	0.0768	77.9	**78.0**	
	4.997	0.0768	78.0		
28°	4.997	0.0792	80.5		2.2
	4.999	0.0788	80.1	**80.2**	
	4.998	0.0782	79.5		
	4.992	0.0792	80.6		2.1
29°	4.951	0.0800	82.1	**82.3**	
	4.954	0.0804	82.5		
30°	4.950	0.0827	85.0	**85.1**	2.8
	4.949	0.0829	85.2		

Ten einde ook op dat gebied een inzicht te krijgen omtrent de waarde van L. en D's. cijfers, hebben wij nog eenige bepalingen van het optisch draaiingsvermogen aan verzadigde oplossingen verricht.

In een polarimeter (volgens LANDOLT, driedeelig gezichtsveld, fabr. SCHMIDT & HAENSCH) werd eerst onderzocht de bij 30° aan *d* en *l* verzadigde oplossing, van welke wij (zie Tabel 12²) gevonden hadden, dat de concentratie 141.0 per 5000 m.G. oplosmiddel was d. i. 2.82 gr. per 100 gr. Gevonden draaiing — **2.50°**.

Nu kan men uit de cijfers van L. en D. terugrekenen welke draaiing zij afgelezen hebben, indien zij hunne bij 30° aan *d* en *l* verzadigde oplossingen gebezigd hebben.

Zij geven nl. op:

Gevonden conc. 2.575 Gr. op 100 Gr. water.
51.002 % *d* d. i. 1.313 Gr. op 100 Gr. water.
48.998 % *l* d. i. 1.262 „ „ „ „ „

Dit is geconcludeerd uit

$$[\alpha_D]_d = -20.6073 + 0.9367 \times 1.313 \text{ dus } = -19.38^\circ$$
$$[\alpha_D]_l = -31.3634 + 1.3564 \times 1.262 \text{ dus } = -29.65^\circ$$
$$[\alpha_D]_{d+l} \; -49.03^\circ$$

zoodat de afgelezen draaiing α (indien eenzelfde buislengte is gebruikt) aldus is te vinden

$$\alpha_D = \frac{100\alpha}{l \times c}$$
$$-49.03 = \frac{100\,\alpha}{2 \times 2.575} \; {}^{1})$$

waaruit volgt $\alpha = -$ **2.52**°

merkwaardigerwijze (practisch) hetzelfde getal als wij vonden.

Hieruit (aangenomen dat de fout niet in de polarimeterbepalingen gelegen is) zou men moeten concludeeren, dat L. en D. en wij dezelfde verzadigde vloeistoffen in handen gehad hebben en de fout bij hen in hun analyse-methode gelegen is, waarover wij boven al gesproken hebben.

Een toetsing der bovengebruikte Ladenburg-Doctorsche-formule van het *l*-zout leverde een zeer onbevredigend resultaat. De oplossing, die bij 25° aan *l*-zout verzadigd was, en bij analyse bleek te bevatten 72.6 m.G. per 5000 m.G. water en dus **1.452** Gr. per 100 Gr. water, leverde een draaiing α van -0.93°. Volgens bedoelde formule was hare concentratie negatief! Ook andere bepalingen leverden geen overeenstemming.

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[1]) De door L. en D. gegeven concentratie.

Samenvatting.

Overzien wij de onderzoekingen boven beschreven omtrent de strychninetartraten dan komen wij tot de volgende conclusies:

Uit de onderzoekingen aangaande smeltpunt en spec. gew. der strychninetartraten blijkt, dat het druivenzuurstrychnine anhydrisch bestaanbaar is en dat dientengevolge de tensimeterbepaling van het overgangspunt zooals door Ladenburg en Doctor uitgevoerd onmogelijk tot juiste resultaten kan voeren.

Door een reeks oplosbaarheidsbepalingen, die zoo ingericht waren, dat de hoeveelheden der componenten rechtstreeks bekend waren, werden isothermen bepaald en vastgesteld, dat er geheel overeenkomstig de door Bakhuis Roozeboom gestelde verwachtingen, een overgangstraject in dit systeem optreedt en wel tusschen $7\frac{1}{2}$ en 30°.

Met proeven gedocumenteerde bezwaren werden tegen alle onderzoekingsmethoden van Ladenburg en Doctor aangevoerd. Ook een door Meyerhoffer uit hun data afgeleid vermoeden werd door experimenten opgeheven.

HOOFDSTUK V.

EXPERIMENTEELE ONDERZOEKINGEN OVER DE ZURE BRUCINE-TARTRATEN.

I. Inleiding.

a. Door de in het vorige hoofdstuk beschreven onderzoekingen is feitelijk de juistheid van ROOZEBOOM's critiek op de onderzoekingen van LADENBURG geheel gerechtvaardigd. Het nader nagaan van een tweede systeem behoefde dus niet zóó uitgevoerd te worden, dat ook hier een volledig isothermen-net werd bepaald. Slechts kwam het er op aan na te gaan of in dit systeem, waar het stabiliteitsgebied van het partiëele-racemaat zich naar *hoogere* temperaturen uitstrekt (cf. fig. 6), ook een overgangstraject zou zijn aan te toonen en daarmede de LADENBURGsche voorstelling ook in dit systeem afgewezen zou kunnen worden. De uitkomsten zullen echter in nog veel hooger mate de onvoldoendheid van het onderzoek van LADENBURG en FISCHL [1]) aanwijzen.

b. Bereidingswijze der uitgangsproducten.

Omtrent de wijnsteen- en druivenzuur praeparaten zij naar het vorige hoofdstuk verwezen. De gebruikte *brucine* was een praeparaat afkomstig van ZIMMER & C°.

b. Bereiding van het zure d-wijnsteenzure brucinezout.

Er werd afgewogen 1.000 gr. wijnsteenzuur, die, na oplossing, met brucine werd geneutraliseerd tot lakmoespapier niet meer rood werd; daarna werd de overmatige brucine uit de heete oplossing

[1]) Ber. der deutsch. chem. Ges. **40**, 2281 (1907).

afgefiltreerd en wederom 1.000 gr. wijnsteenzuur toegevoegd. Uit de aldus verkregen ± 150 ccM. kristalliseerde eerst in koud water, dan in ijs het zure zout als fijne naaldjes (langer dan bij het overeenkomstige *d*-zout) uit. Ze werden op den zuigtrechter van de moederloog gescheiden.

Bereiding van het zure l-wijnsteenzure brucine.

De bereiding van dit zout geschiedde op overeenkomstige wijze: het zout zette zich als kleine, straalsgewijze zich tot wratten samenhoopende kristallen op bodem en wanden van het bekerglas af. Het afzuigen werd bemoeilijkt door de brijige consistentie der moederloog, daarom werd deze verwijderd door de kristallen tusschen filtreerpapier zooveel mogelijk te drogen. Daarna bleven ze eenige dagen aan de lucht liggen.

Bereiding van het zure druivenzure brucine.

Ook hier geschiedde de bereiding op gelijke wijze. Slechts werd bij verschillende bereidingen de kristallisatie bij verschillende temperaturen uitgevoerd. Eenmaal werd ook afgewogen: 2.000 gr. druivenzuur en 9.390 brucine, welke hoeveelheden elkander juist moesten neutraliseeren indien de brucine anhydrisch was. Dat bleek ook hier weer inderdaad het geval.

c. Analyse der proefstoffen.

Daar een eerste reeks oplosbaarheidsbepalingen voor ons onverklaarbare uitkomsten had gegeven, hebben wij, ter vaststelling dat geen onregelmatigheden in onze proefstoffen oorzaak daarvan waren, deze aan verschillende analysemethoden onderworpen.

TABEL 16.

		Elementair Analyse		Stikstofbepaling (Dumas)	
		gev.	ber.	gev.	ber.
d-zout	C	59.77	59.56	**1.** 5.43	5.14
	H	6.15	5.88	**2.** 5.44	5.14
l-zout	C	59.81	59.56	5.5	5.14
	H	6.00	5.88		
r-zout		niet verricht		5.5	5.14

Allereerst werden elementair analyses en stikstofbepalingen gedaan. (Zie Tabel 16). Deze gaven normale uitkomsten, maar vormen geen criteria ervoor of we nu inderdaad de zure zouten zuiver in handen hebben. De procentische elementairsamenstelling der *neutrale* en *zure* zouten verschilt te weinig, dat een verontreiniging van eenige procenten der eerste in de laatste aanleiding zou geven tot grootere afwijkingen dan de normale proeffouten dezer methoden zijn.

Een titrimetrische analyse bleek intusschen de gewenschte zekerheid te verschaffen. Titreert men nl. de zure zouten met ongeveer 0.01 n. *KOH* en rosolzuur als indicator, dan heeft de kleuromslag plaats als het neutrale brucine-kaliumzout is gevormd. De vloeistof blijft dan ook helder tot den kleuromslag, terwijl bij gebruik van b.v. phenolphtaleïne de troebeling door brucine, door *KOH* afgescheiden, reeds optreedt vóór nog kleuromslag heeft plaats gevonden.

Een koolzuurvrije *KOH*-oplossing werd gesteld op barnsteenzuur en bleek 0.01163 n. 1 c.c.m. dezer loog correspondeert dus met $\frac{1163}{1000} \times 5.445$ mgr. $= 633$ mgr. wijnsteenzuurzout. Bij de series B. en C. werd een andere oplossing gebruikt, waarbij 1 c.c.m. met 649 mgr. correspondeerde. De tabellen 17 bevatten de uitkomsten:

TABEL 17.

A. *Titratie zuur d-brucine tartraat.*

Afgewogen 0.5291 Gr. anhydr. zout, opgelost in 100 c.c.m. water.

c.c.m. opl.	c.c.m. *KOH*		mgr. zout		
			gevonden	berekend	
25.00	20.97	gem. 20.92	**132.4**	**132.3**	Het (eenmaal omgekristalliseerde) zout is dus als zuiver te beschouwen.
25.00	20.88				
25.00	20.91				

B. *Titratie zuur l-brucine tartraat.*

Afgewogen 0.6070 gehydr. zout, opgelost tot 100 c.c.m.

c.c.m. opl.	c.c.m. KOH	mgr. zout		
		gevonden	berekend	
10.00	8.02			
10.00	7.98 8.00 per 10 c.c.m. gem.	**51.92**	**52.5**	Het (eenmaal omgekristalliseerde) zout is dus als zuiver te beschouwen.
20.00	16.00			

C. *Titratie zuur druivenzuur brucine.*

Afgewogen 0.5324 anhydr. zout, opgelost tot 100 c.c.m.

c.c.m. opl.	c.c.m. KOH	mgr. zout		
		gevonden	berekend	
10.00	8.04			
10.00	8.06 8.05	**52.2**	**53.2**	Ook het racemaat is dus het zuivere zure zout.
20.00	16.20			
20.00	16.20 gem. 16.20	**105.1**	**106.4**	

Ten slotte werd nog nagegaan of het zure bestanddeel van het racemaat inderdaad optisch inactief was. Daartoe werd uit een zouthoeveelheid (die volgens straks te vermelden proeven 11.8 °/₀ kristalwater bevatte) met ammonia de brucine gepraecipiteerd. Daar de oplossing echter met HNO_3 zich nog rood kleurde en dus niettegenstaande de overmaat ammonia nog brucine bevatte, werd zij tot droog toe ingedampt, het residu weer opgelost, gefiltreerd en met loodacetaat gepraecipiteerd. Het loodracemaat werd afgezogen, uitgewasschen en gesuspendeerd in lauw water, waarna in de op het waterbad geplaatste oplossing zwavelwaterstof werd geleid, terwijl voor herhaald omroeren werd zorggedragen. Loodsulfide zette zich grof af, het werd afgefiltreerd, weer met water aangeslibd en H_2S werd

weer ingeleid tot het filtraat niet meer zuur reageerde. Zoodoende werd het zuur zuiver en voor polarimetrisch onderzoek geschikt verkregen.

Zoowel een 10 °/₀ oplossing van het zuur als die van de er uit bereide ammoniumzouten was volkomen inactief.

d. Het kristalwater.

De bepalingen van het kristalwater geschiedden als bij de strychnine-tartraten beschreven is. De uitkomsten vindt men in Tabel 18.

TABEL 18.

	d-tartraat.		*l*-tartraat.				racemaat.			
hydraat	0.7945	0.8532	0.4598	0.7624	0.9849	1.1611	1.6069	1.0370	1.1587	1.1408
na verwarming	0.7925	0.8518	0.3964	0.6566	0.8510	1.0050	1.4758	0.9134	1.0213	1.0056
dus water	0.0020	0.0024	0.0634	0.1058	0.1339	0.1561	0.1311	0.1236	0.1374	0.1352
d.i. °/₀ water	0	0	13.79	13.85	13.59	13.44	11.84	11.89	11.86	11.85
d.i.	anhydr.	anhydr.	4.84*aq*	4.86*aq*	4.74*aq*	4.70*aq*	4.06*aq*	4.08*aq*	4.07*aq*	4.06*aq*
dus	*anhydr.*		5 *aq*				4 *aq*			

Ladenburg en Fischl (l. c.) vonden voor de enkel tartraten dezelfde waarden, voor het rac. daarentegen $2\frac{1}{2}$ *aq*. Wanneer men nu bedenkt, dat hierboven vermeld is hoe ons zout een optisch inactief zuur bevat (juist dit 11.8 °/₀ kristalwater bevattende zout is voor bepaling gebruikt!) en dat een inactief mengsel tot $2\frac{1}{2}$ *aq* zou voeren, dan rijst er wel zeer ernstige bedenking tegen het praeparaat waarvan L. en F. het kristalwatergehalte bepaalden. Aan het einde van dit hoofdstuk komen wij hier nog nader op terug.

e. Het specifiek gewicht.

Voor de bepalingsmethode van het sp. gew. wordt weder naar het vorige hoofdstuk verwezen.

De uitkomsten zijn samengevat in Tabel 19 en gelden (ook die van Fischl) voor 25°.

TABEL 19.

anhydrisch *d*-tartraat	anhydrisch *l*-tartraat	anhydrisch racemaat
1.492 1.492 1.493	1.455 1.452 1.457	1.422 1.421
gem. 1.492	*gem.* 1.455	*gem.* 1.422
Fischl 1.30967	Fischl 1.14972	Fischl 1.26029

gehydrateerd *d*-tartraat	gehydrateerd *l*-tartraat	gehydrateerd racemaat
bestaat niet	1.434 1.436 1.436 1.437	1.443 1.444 1.442
	gem. 1.436	*gem.* 1.443
	Fischl 1.74633	Fischl 1.63962

Terloops zij hier opgemerkt, dat het racemaat de eigenaardigheid vertoont zich met het spec. lichtere water tot een specifiek zwaarder hydraat te verbinden. Aangezien deze eigenaardigheid zeldzaam is en voorzoover ons bekend nimmer bij een stof met zoo hoog spec. gew. (1.4) geconstateerd, hebben wij het verschijnsel door een *dilatometerproef* gecontroleerd. In de peer van een dilatometer werd 3.5697 gr. anhydrisch racemaat gebracht, en de dilatometer op de gebruikelijke wijze met toluol gevuld door de 0.8 mM. wijde capillair.

Door middel van een zweepcapillair werd nu 0.3 c.c.m. water in de peer gebracht. Deze waterdruppel viel niet naar onderen maar bleef aan den wand hangen. Nadat de capillair van boven was dichtgetrokken werd de stand van den meniscus bij 25°.00 aangeteekend. Vervolgens werd de druppel met de anhydrische stof in aanraking gebracht. De meniscus bleek nu 5 c.m. gedaald, 18 uur later zelfs 7 c.m. De hydratatie had dus onder merkbare contractie plaats gehad.

e. De smeltpunten.

De algemeene opmerkingen over de smeltpunten der strychnine tartraten gemaakt gelden ook geheel voor de brucine zouten. Van

de hydraten waren weer geen scherpe smeltpunten te bepalen en de feitelijke bepalingen geschiedden dus weer aan de anhydrische zouten. Gevonden werd:

TABEL 20.

	DUTILH	FISCHL
d-zout	256°	246°
l- „	242°	228°
r- „	250°	240°

f. De tensimeterbepalingen.

FISCHL, meenende dat de omzetting in het overgangspunt aangegeven wordt door een vergelijking:

$$2 \text{ rac. } 2\tfrac{1}{2}\ aq. \rightleftarrows d + l\ 5\ aq.,$$

dat er dus geen vloeistofphase bij gevormd wordt, heeft er van afgezien tensimeterproeven te verrichten. Nu ons gebleken was, dat de reactie, die zich eventueel in een overgangspunt af zou moeten spelen,

$$2 \text{ rac. } 4\ aq. \rightleftarrows d + l\ 5\ aq. + 3\ H_2O$$

was, hebben wij getracht tensimetrisch dit punt op te sporen. Tevergeefs echter; nu zou dat aan dezelfde oorzaak kunnen worden toegeschreven als bij de strychninetartraten. Immers uit de Tabel 19 blijkt ten duidelijkste, dat het anhydrische racemaat geen mengsel der enkeltartraten is, dat het racemaat zich dus ook hier bij wateronttrekking niet noodzakelijk in de enkeltartraten behoeft te splitsen. Maar uit het hier volgende onderzoek der oplosbaarheden zal blijken, dat er nog een andere reden is en wel de meest klemmende, die maar mogelijk is, nl. dat er hoogstwaarschijnlijk heelemaal geen dergelijk overgangspunt in dit systeem optreedt.

II. De Oplosbaarheidsbepalingen.

Zooals boven gezegd lag het niet in onze bedoeling een volledig stel oplosbaarheidsbepalingen te verrichten van het isothermennet in dit systeem. Met het oog op de groote verschillen in oplosbaarheid der enkeltartraten bestond de hoop dat, wanneer men bepalingen op de Meyerhoffersche manier (Vgl. pag. 20 e.v.) verrichtte, de lijnen voor de evenwichten-met-twee-Bodenkörper voldoende ver uiteen zouden liggen om met scherpte conclusies daaruit te kunnen trekken. Daarom hebben wij bij verschillende temperaturen achtereenvolgens de totaal-oplosbaarheden bepaald van resp. *d*, *l*, *r*, *d* + *l*, *r* + *l* en *r* + *d*.

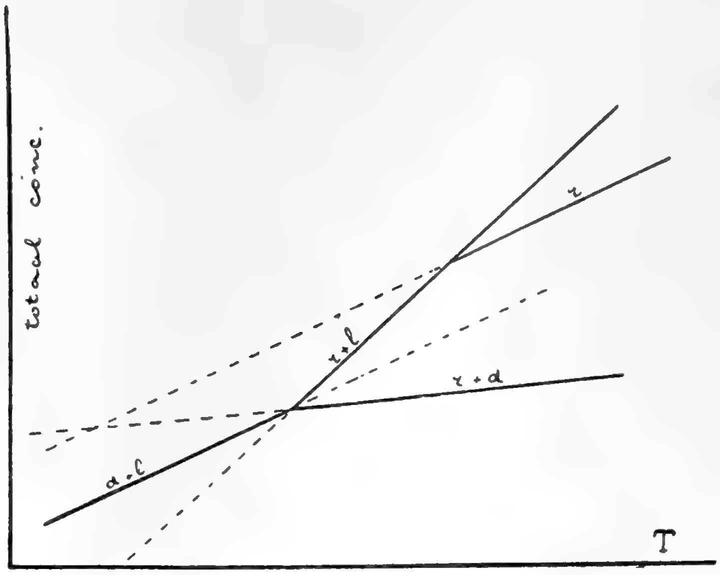

Fig. 12.

Wij herinneren nu (Vgl. fig. 12) dat in het overgangspunt de oplosbaarheden van *d* + *l*, *r* + *l* en *r* + *d* gelijk zullen moeten worden, terwijl in het einde van het overgangstraject de *r* + *l*-oplosbaarheid gelijk aan die van *r* wordt.

TABEL 21.

Oplosbaarheden bij 20°.

N°.	Vaste phase.	Bij indampen van			Concentratie.	Gemiddeld.	Opmerking.
		Aantal dagen geschud.	Gr. oplossing.	Gr. droogrest.			
47	*d* + *l*	1	4.997	0.0688	69.8	69.6	
			4.994	0.0684	69.4		
48	*r* + *l*	1	4.994	0.0794	80.8	80.6	
			5.003	0.0792	80.4		
49	*r* + *d*	1	5.052	0.0690	69.2	69.3	
			5.061	0.0694	69.5		
50	*r*	1	5.011	0.0682	69.0	69.0	
			5.010	0.0682	69.0		

TABEL 22.

Oplosbaarheden bij 25°.

N°.	Vaste phase.	Bij indampen van: Aantal dagen geschud.	Bij indampen van: Gr. oplossing.	Bij indampen van: Gr. droogrest.	Concentratie.	Gemiddeld.	Opmerking.
1	d	1	4.981	0.0494	50.1		
			4.974	0.0495	50.3		
			4.353	0.0436	50.6		
			4.9749	0.0496	50.4	50.4	
32		2	4.992	499	50.6		
			4.968	496	50.4		
2	l	2	4.999	0.0899	91.4		
			5.0025	0.0897	91.3		
		$2\frac{1}{2}$	4.7638	0.0852	91.1		
			4.9815	0.0894	91.4	92	
33		2	5.010	0.0914	92.9		
			5.001	912	92.9		
3	r	2	4.8816	913	95.3	94	
			4.9205	925	93.9		
		3	3.2923	560	86.5	87	
			4.9526	848	87.1		
		9	4.9508	946	97.4	98	
			4.9624	957	98.3		
19		1	5.026	895	90.7		De bij 35° verzadigde oplossing gebruikt.
			5.056	895	90.1		
			5.061	898	90.3		
			5.005	893	90.4		
31		2	5.010	879	89.3		
			5.004	879	89.4		
54		1	5.010	866	87.9		De bij 20° verzadigde oplossing gebruikt.
			5.018	864	87.6		
4	$d + l$	2	5.0076	859	87.3		
			4.9775	851	87.0		
		9	4.3077	741	87.4		
20		1	4.919	850	87.9	87.6	Zie 19.
			4.975	856	87.5		
		2	5.008	866	88.0		
			5.002	864	87.9		
6	$r + l$	1	4.9526	0.1068	110		
			4.9024	0.1058	110		
21		1	2.645	0.0535	103		Zie 19.
			4.964	0.0994	102		
		2	2.144	0.0438	104		
			1.041	0.0215	105		
52		1	4.999	0.1025	105		
			5.023	0.1026	105		
5	$r + d$	2	4.9490	858	88.2		
			4.9582	858	88.3		
			4.9666	854	87.5		
			4.9543	855	87.8		
22		1	5.004	886	90.1		Zie 19.
			4.224	755	90.9		
			5.008	907	92.2		
			4.924	888	91.8		
53		1	5 083	874	87.5		Zie 54.
			5.057	882	88.7		

TABEL 23.

Oplosbaarheid bij 35°.

N°.	Vaste phase.	Bij indampen van			Concentratie.	Gemiddeld.	Opmerking.
		Aantal dagen geschud.	Gr. oplossing.	Gr. droogrest.			
35	*d*	1	4.987	0.0625	63.5	63.6	
			4.968	0.0624	63.6		
36	*l*	1	5.085	0.1600	162	162	
			5.076	0.1600	163		
15	*r*	1	5.0727	0.1474	147		
			5.0708	0.1476	147		
		2	5.015	0.1468	151		
			5.018	0.1468	151		
34		1	5.022	0.1385	142		
			5.027	0.1389	142		
58		1	5.055	0.1372	139		
			5.038	0.1368	140		
16	*d* + *l*	1	5.019	0.1484	152		
			5.025	0.1486	152		
		2	5.065	0.1486	151		
			5.070	0.1483	151	152	
55		1	2.172	0.0642	152		
			4.855	0.1428	152		
			4.494	0.1326	152		
17	*r* + *l*	1	5.0270	0.1632	168		
			5.0226	0.1628	168		
		2	5.011	0.1610	166		
			4.496	0.1449	167		
56		1	1.709	0.0537	162		
			2.471	0.0782	163		
18	*r* + *d*	1	5.0260	0.1465	150		
			5.0200	0.1472	151		
		2	5.008	0.1458	150		
			5.003	0.1462	150		
57		1	4.370	0.1192	140		
			4.686	0.1283	141		

TABEL 24.

Oplosbaarheid bij 44°.

N°.	Vaste phase.	Bij indampen van Na schudtijd.	Gr. oplossing.	Gr. droogrest.	Concentratie.	Gemiddeld.	Opmerking.
38	*d*	1 D.	4.968	0.0778	79.5	79.5	
			4.959	0.0777	79.6		
39	*l*	1 D.	5.013	0.2251	233	232	
			3.302	0.1458	231		
37	*r*	18 Uur	5.037	0.2444	255		
			5.052	0.2451	255		
62		1 Uur	3.550	0.1494	220		
		2 Uur	5.068	0.2109	217		
		4 Uur	5.066	0.2030	209		
64		15 Min.	5.021	0.2113	220		
		33 „	5.042	0.2120	219		
		3 Uur	5.032	0.2339	244		
8	*d + l*	1 D.	5.060	0.2751	287		
			5.0552	0.2722	285		
		3 D.	5.031	0.2674	280	280	
			5.023	0.2656	279		
12		1 D.	5.022	0.2374	248		
			4.794	0.2270	248		
			4.501	0.2132	249		
			5.024	0.2376	248		
59		1 Uur	5.065	0.2453	254		
		2 Uur	5.046	0.2509	261		
		4 Uur	5.048	0.2539	268		
63		2 D.	5.028	0.2634	276		
			5.033	0.2635	276		
9	*r + l*	1 D.	5.048	0.2642	276		
			5.048	0.2663	278		
		3 D.	2.020	0.1062	277		
13		1 D.	5.0764	0.2689	280		
			5.0928	0.2686	279		
		2 D.	5.0254	0.2665	280		
60		1 Uur	5.061	0.2976	312		
66		15 Min.	5.037	0.2956	305		
		32 „	4.816	0.2825	311		
10	*r + d*	3 D.	5.043	0.1642	168		
			5.037	0.1648	169		
14		1 D.	5.033	0.1632	168		
			5.015	0.1622	167		
			5.027	0.1660	171		
			5.018	0.1657	171		
61		1 Uur	4.437	0.1791	210		
		2 Uur	4.750	0.1877	206		
		5 Uur	5.018	0.1964	204		
65		15 Min.	5.035	0.2169	225		
		32 „	5.071	0.2187	225		

TABEL 25.

Oplosbaarheid bij 50°.

N°.	Vaste phase.	Bij indampen van			Concentratie.	Gemiddeld.	Opmerking.
		Na schudtijd.	Gr. oplossing.	Gr. droogrest.			
42	*d*	1 D.	4.909	0.0910	92.7	92.7	
			4.989	0.0908	92.7		
43	*l*	1 D.	5.115	0.3147	328	328	
			5.068	0.3122	328		
23	*r*	3 D.	5.0798	0.2021	207		
			5.0512	0.2008	207		
		4 D.	5.0468	0.2016	208		
			5.0572	0.2018	208		
27		5 D.	5.0168	0.2010	209		Vervolg van 23.
			4.9963	0.2000	209		
40		1 D.	5.044	0.2469	257		
			5.032	0.2464	257		
41			5.001	0.2524	266		De oplossing gebruikt, die bij 44° verzadigd was.
			5.024	0.2537	266		
44		1 Uur	5.022	0.2400	247		
			5.035	0.2396	248		
		2 Uur	4.821	0.2276	251		
			4.213	0.1981	250		
		3 Uur	5.079	0.2383	246		
			5.072	0.2379	246		
45		1 Uur	5.054	0.2786	292		
			5.058	0.2769	289		
		2 Uur	5.021	0.2603	273		
			4.797	0.2476	272		
46		½ Uur	5.037	0.2891	305		
			5.041	0.2888	303		
67—68		20 Min.	5.037	0.2846	299		
			5.020	0.2844	300		
		38 „	3.493	0.1966	298		
			3.319	0.1866	298		
69	*d* + *l*	16 Uur	3.776	0.2669	380	378	
			2.223	0.1550	374		
		2 D.	3.643	0.2563	378		
			4.126	0.2912	380		
25	*r* + *l*	3 D.	5.058	0.3536	376		
			5.067	0.3529	374		
		4 D.	5.0595	0.3821	408		
			5.0338	0.3798	408		
29		5 D.	5.080	0.3453	364		Vervolg van 25.
			5.014	0.3407	364		
71		15 Min.	5.134	0.4336	461		
		30 „	5.062	0.4274	461		
		6 Uur	4.008	0.3246	441		
		20 Uur	4.108	0.2972	390		
			4.156	0.3007	390		
26	*r* + *d*	3 D.	4.970	0.2139	225		
			5.013	0.2156	225		
		4 D.	4.989	0.2139	224		
			4.981	0.2138	224		
30		5 D.	4.940	0.2126	225		Vervolg van 26.
			4.663	0.2002	224		
70		20 Min.	5.061	0.2631	274		
		30 „	5.032	0.2579	270		
		1 Uur	5.025	0.2530	264		

De Tabellen 21, 22, 23, 24 en 25 vatten de uitkomsten dezer bepalingen samen.

Men ziet, dat deze tabellen een geheel ander aanzien hebben dan die bij de strychnine tartraten. Terwijl daar na 1 à 2 $\times$ 24 uur de uitkomsten zóó standvastig waren, dat van het telkens herhaald opgeven der schudtijden is afgezien, veranderen hier de uitkomsten op zonderlinge en (men lette op paralelbepalingen, met zelfde schudtijden) onregelmatige wijze. Dat is tenminste het geval met die proeven waarin racemaat, hetzij alleen, hetzij met *d* of *l*, als Bodenkörper optreedt. De oplosbaarheden van *d*, *l* en vrijwel ook die van $d + l$ verloopen normaal. Volgens LADENBURG en FISCHL zou het racemaat boven 44° stabiel zijn, onze tabellen laten zien, dat hoe hooger de temperatuur wordt, des te onregelmatiger worden alle oplosbaarheidsbepalingen, waarbij met *r* geschud wordt; tot aan 50° krijgt men uit deze tabellen al zeer weinig den indruk, dat met een stabiel lichaam oplosbaarheidsbepalingen worden uitgevoerd. Integendeel, er blijkt een ontleding plaats te hebben, die met verhooging van temperatuur sneller voortschrijdt. Het zijn vooral de bepalingen, waar wij tenslotte toe gekomen zijn, die met volgnummers boven de 44, die deze meening bevestigen.

Gaat men b.v. eens na wat er gebeurt met een oplossing van *r* in water van 50°. Wij ontleenen dan aan Tabel 25 de volgende gemiddelden.

TABEL 26.

Oplosbaarheid van *r* bij 50°.

N°.	Na tijd.	Opl.	Andere Serie.
67—68	20 Min.	300	
	38 „	298	
45	1 Uur	291	247 (No. 44)
	2 Uur	273	250 (id.)
	3 Uur		246 (id.)
40	1 D.	257	
23	3 D.	207	
	4 D.	208	
	5 D.	209	

Men ziet daaruit, dat *na enkele minuten* een oplosbaarheid wordt bereikt, welke van af dat oogenblik voortdurend afneemt. Dit is toch zeker niet het gedrag van een stabiel lichaam. Nu zou men

deze proef nog kunnen verklaren met de veronderstelling, dat nochtans de opvatting van L. en F. van een overgangspunt bij 44° juist is, maar dat deze r oplosbaarheid bij 50° er een is binnen het overgangstraject, waarmede dus ons oorspronkelijk doelwit zou zijn getroffen. Wij gelooven echter niet, dat deze uitlegging juist is, de proeven pleiten er veelmeer voor, dat de verhoudingen in het stelsel der zure brucine tartraten met water een geheel andere is. Werpt men een blik in Tabel 24, dan ziet men, dat het bestaan van een overgangspunt bij 44° niet te rijmen is met onze oplosbaarheidsbepalingen bij die temperatuur. Welke cijfers men ook als de juiste wil nemen, de maxima na enkele minuten, dan wel de getallen, die men na dagen schudden vindt, geen enkele combinatie toont, dat bij die temperatuur

$$\text{Oplosbaarh.}_{d+l} = \text{Oplosbaarh.}_{r+l} = \text{Oplosbaarh.}_{r+d}.$$

De getallen van $d + l$ en r, die *niet* gelijk behoeven te worden, lijken nu en dan wel wat op elkander (Vgl. proef 37 en 8) en men begrijpt dan ook wel, dat FISCHL de foutieve praemisse hier met een onjuiste interpretatie van het experiment gecombineerd heeft. Dat hij voor $r + l$ ook een getal in die buurt vond is ook te begrijpen (Vgl. proef 13), maar hoe hij op kan geven, dat ook voor $r + d$ een waarde in die buurt optreedt (Vgl. de proeven 10, 14, 61 en 65) is weer even onbegrijpelijk als de tensimeterbepalingen van DOCTOR zijn!

Trouwens de eigenaardige gang in de waarden, die wij vonden voor $r + l$ en $r + d$ bij en boven 44°, duidt op de instabiliteit van het systeem.

Wanneer wij dus tot de conclusie komen, dat LADENBURG en FISCHL onjuiste conclusies omtrent het onderhavige systeem gesteld hebben op een uiterst zwak feitenmateriaal, dan dringt de vraag zich natuurlijk aan ons op, hoe wij ons de verhoudingen in dit systeem dan *wel* te denken hebben. Allereerst moeten wij dan vaststellen, dat blijkens in de inleiding van dit hoofdstuk vermelde proeven de zure druivenzure brucine ongetwijfeld een chemisch individu is, maar dat zij, althans beneden 50°, nimmer stabiel naast oplossingen is. Dat zij het alsnog bij hoogere temperaturen zou worden is natuurlijk niet uitgesloten, maar waarschijnlijk is dat niet. Zet men nl. de gevonden waarden voor oplosbaarheden, hoe ruw men daarbij ook middelen moet, uit (zie fig. 13), dan ziet men toch nog wel een tendenz in die lijnen, maar zeker geen

van $r + l$ en $r + d$ om elkaar te snijden naar hoogere temperaturen. Ons lijkt dus het meest waarschijnlijk, dat het zure druivenzure zout een zeer onbestendige verbinding is.

Hoe men zich nu het eigenaardige verloop der oplosbaarheden met den tijd moet verklaren, hoezeer wij onze aandacht daaraan gegeven hebben, durven wij daar geen positieve uitspraak over te doen. Een voor de handliggende onderstelling zou zijn, dat het onbestendige racemaat zich telkens in de enkeltartraten splitst; dan zou men echter steeds op $d + l$ oplosbaarheden moeten uitkomen, als

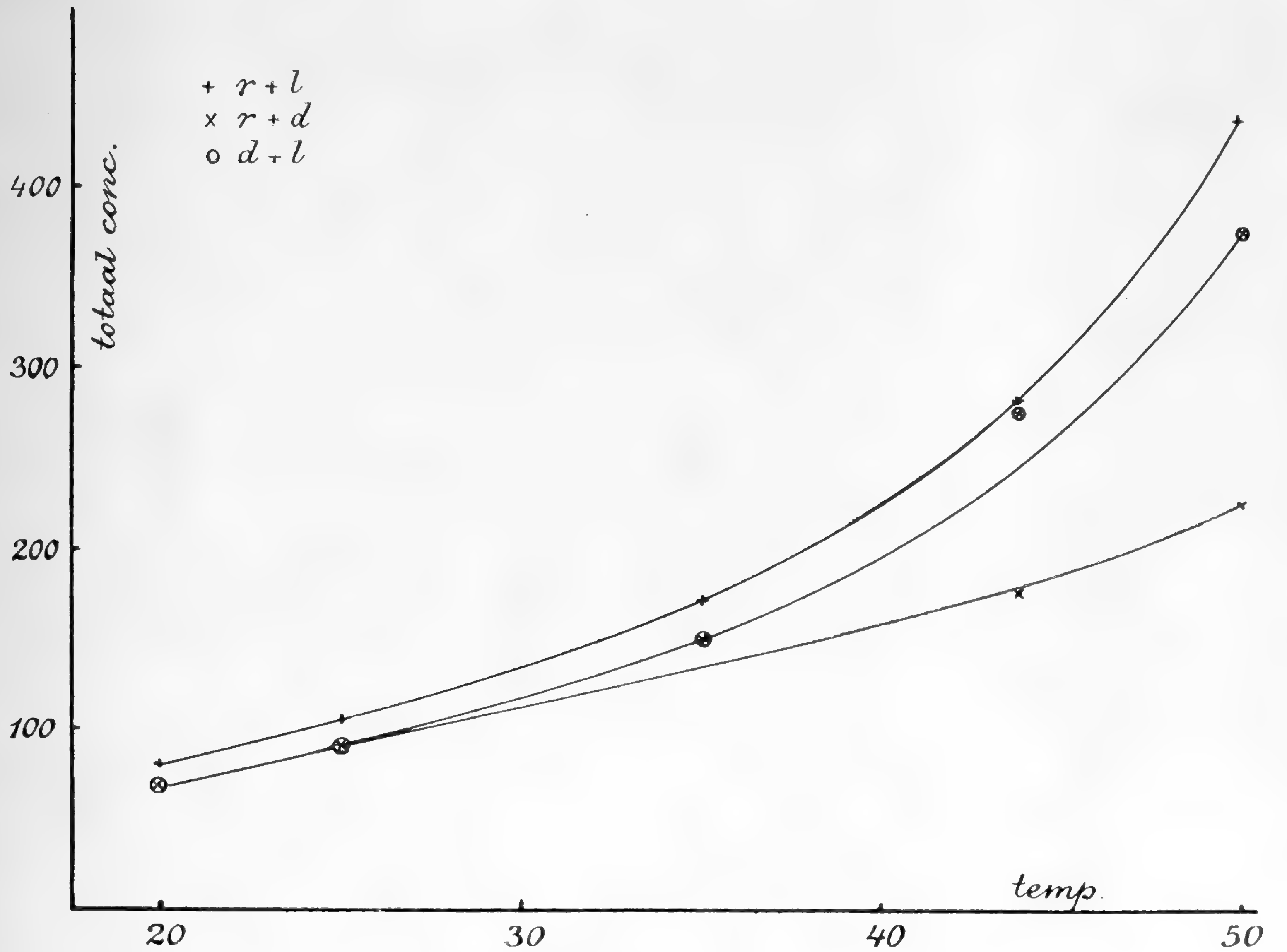

Fig. 13.

er tenminste voldoende Bodenkörper is, wat bij onze proeven inderdaad het geval was. Daar nu die waarden niet gevonden zijn, moet de verklaring elders gezocht worden en dan schijnt het ons zeer waarschijnlijk, dat het gecompliceerde verloop der oplosbaarheidsproeven daarin te zoeken is, dat de normale brucine tartraten hier tevens een rol spelen. Er is echter te weinig omtrent deze stoffen bekend (oplosbaarheid), dan dat men dit met zekerheid zou kunnen zeggen, maar hun bestaan opent zeker in dit systeem de mogelijkheid tot complicaties als wij in bovenstaande tabellen aantreffen.

Samenvatting.

Overzien wij de resultaten van de bovenbeschreven onderzoekingen omtrent de zure brucine tartraten, dan komen wij tot de volgende conclusies.

De physische constanten (spec. gew., smeltpunt) der zure brucine tartraten zijn anders, dan door LADENBURG en FISCHL opgegeven, eveneens het kristalwatergehalte der druivenzure verbinding. De bestaanbaarheid van een anhydrisch druivenzuur zout belet echter weer tensimetrisch onderzoek.

De oplosbaarheidsbepalingen door FISCHL verricht laten zich niet reproduceeren. Integendeel leeren nauwkeurige bepalingen, waarbij na verschillende tijden analyses zijn verricht, dat in de met racemaat geschudde oplossingen een ontleding plaats heeft, *ook boven* 44°. Bij die temperatuur ligt zeker niet het overgangspunt, want de oplosbaarheden van $r + d$ en $r + l$ worden er absoluut niet gelijk. Kortom, de verhoudingen in dit systeem zijn van totaal anderen aard dan door LADENBURG c.s. vermoed wordt. Naar alle waarschijnlijkheid is de partiëel racemische verbinding steeds een labiel lichaam en zijn slechts de enkeltartraten naast oplossing bestendig.

SLOTBESCHOUWING.

Zooals in de inleiding dezer verhandeling vooropgesteld werd, was de bedoeling dezer onderzoekingen na te gaan of ROOZEBOOM's beschouwingen over de systemen, waarin partieel racemische verbindingen optreden, juist zijn; een dergelijk onderzoek scheen gewenscht, omdat LADENBURG nog steeds de aangevochten voorstellingswijze gehandhaafd heeft.

Tegenover ROOZEBOOM's theorie stond al het experimenteele materiaal uit LADENBURG's laboratorium. Dat materiaal is nu aan een nauwgezette contrôle onderworpen in twee gevallen. Bij het nawerken van het strychnine-tartraten-systeem kwamen al dadelijk zoowel numerieke als principieele afwijkingen voor den dag, terwijl een breeder opgezet onderzoek der isothermen, o. i. de eenige geheel „einwandsfreie" onderzoekingsmethode, *absoluut de opvatting van Roozeboom bevestigde.* In het systeem der brucinetartraten zijn de verhoudingen gecompliceerder, onze onderzoekingen hieven echter in elk geval de waarde van dit Ladenburgsche materiaal geheel op, daar de verhoudingen in dit systeem zeker geheel anders zijn, dan ze door zijn leerling FISCHL bepaald zijn. Daarbij komt, dat wij op algemeene fouten in de onderzoekingsmethoden wezen, bezwaren, die ook gelden voor de overige, niet nagewerkte systemen.

Onze verwachting, aan de (trouwens theoretisch onaanvechtbare) opvatting van ROOZEBOOM een experimenteel bewijs terzijde te stellen en daarmede tegelijk het materiaal, dat ermede in tegenspraak scheen, te kunnen kritiseeren, mag dus naar wij meenen wel als vervuld beschouwd worden.

Analytical treatment of the polytopes regularly derived from the regular polytopes.

(Sections II, III, IV).

BY

P. H. SCHOUTE.

Verhandelingen der Koninklijke Akademie van Wetenschappen te Amsterdam.

(EERSTE SECTIE).

DEEL XI N°. 5.

(WITH ONE PLATE).

AMSTERDAM,

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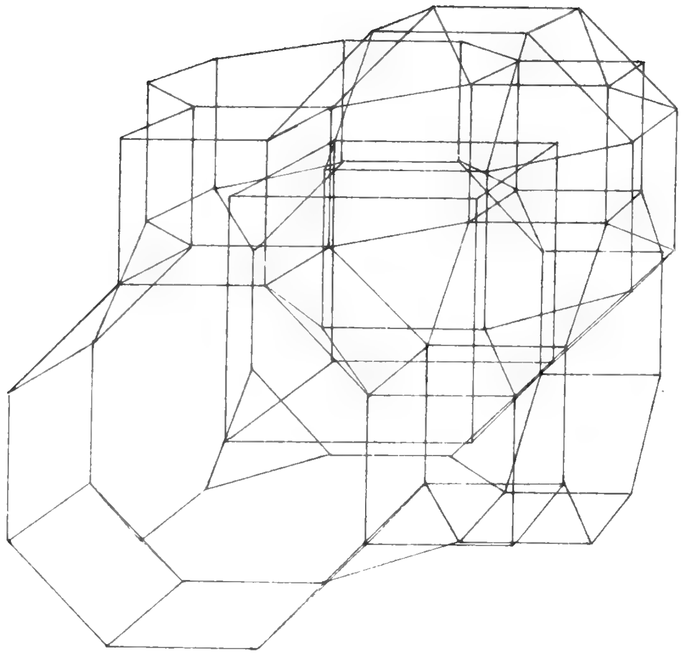
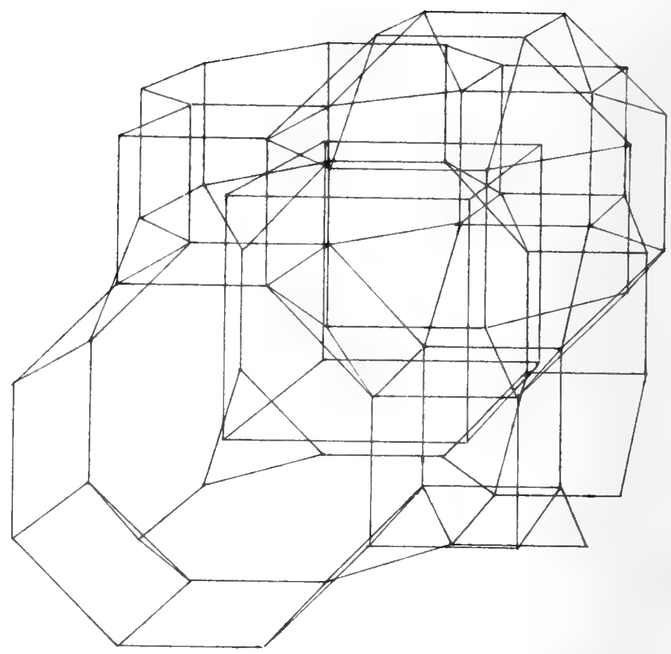
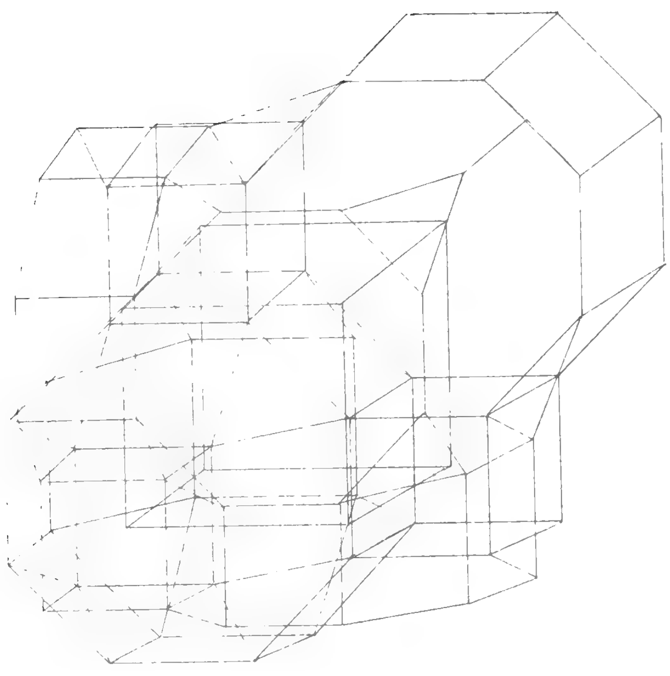


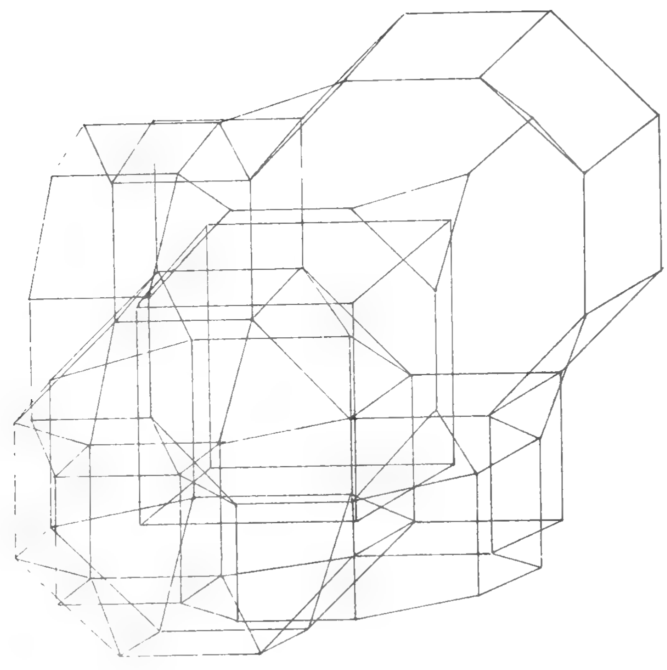


KONINKLIJKE AKADEMIE VAN WETENSCHAPPEN TE AMSTERDAM.

Verzoeke het stereoscoopplaatje, behoorende bij de verhandeling van den Heer P. H. SCHOUTE: *„Analytical treatment of the polytopes regularly described from the regular polytopes."* Section II, III, IV, (Verhandelingen 1e Sectie, Deel XI, N°. 5) te vervangen door het hierbijgaande.

Please replace the stereoscopic plate belonging to the memoir of Prof. P. H. SCHOUTE: *„Analytical treatment of the polytopes regularly described from the regular polytopes."* Section II, III, IV. (Verhandelingen 1e Sectie, Dl. XI N°. 5) by the enclosed one.





Analytical treatment of the polytopes regularly derived from the regular polytopes.

(Sections II, III, IV).

BY

P. H. SCHOUTE.

Verhandelingen der Koninklijke Akademie van Wetenschappen te Amsterdam.

(EERSTE SECTIE).

DEEL XI N°. 5.

(WITH ONE PLATE).

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Analytical treatment of the polytopes regularly derived from the regular polytopes.

Section II: Polytopes and nets derived from the measure polytope.

A. *The symbol of coordinates.*

46. The distance r between two points P, P', the ordinary rectangular coordinates of which are $\mu_1, \mu_2, \ldots, \mu_n$ and $\mu'_1, \mu'_2 \ldots, \mu'_n$ is represented by the formula

$$r^2 = \sum_{i=1}^{n} (\mu_i - \mu'_i)^2 \ldots \ldots \ldots \ldots \ldots \ldots \quad 2).$$

Now we repeat here the question of art. 1:

"Under what circumstances will the series of points obtained by giving to the set of coordinates $\mu_1, \mu_2, \ldots, \mu_n$ a determinate set of values taken in all possible permutations form the vertices of a polytope all the edges of which have the same length, say unity?"

The answer is nearly the same as that given in art. 1:

"If the n values $a_1, a_2, \ldots, a_n$ are arranged in decreasing order, so that we have

$$a_1 \geqq a_2 \geqq \ldots \geqq a_k \geqq a_{k+1} \geqq \ldots \geqq a_n,$$

the difference $a_k - a_{k+1}$ of any two adjacent values must be either $\frac{1}{2}\sqrt{2}$ or zero."

The proof runs on the same lines as that given in art. 1. The geometrical result can be stated in the following general form:

"Under the conditions stated, the polytope the vertices of which are represented by the symbol

$$(a_1, a_2, a_3, \ldots, a_n)$$

is the same as that obtained in the first section for $n - 1$ and $a_k - a_{k+1}$ either one or zero. It is a derivative of the regular simplex the vertices of which determine on the n axes OX_i of coordinates positive segments OA_i, $(i = 1, 2, \ldots, n)$, of the same length $b = \sum_1^n a_i$".

This simple result, in close connection with the new deduction of formula 1), shows us that we shall have to enlarge the scope of our symbol of coordinates in order to find something new.

47. We remember that the symbols $[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}]$ and $[\frac{1}{2}\sqrt{2}, 0, 0]$ represent the coordinates of the vertices of cube and octahedron with edge unity, if the *square* brackets indicate that all the permutations of the values they include must be taken, each value being affected successively either by the *positive* or by the *negative* sign. Moreovor $\frac{1}{2}[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}]$ and $-\frac{1}{2}[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}]$ can represent in the same way the two tetrahedra, the vertices of which form together the vertices of the cube $[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}]$, if by the coefficient $\frac{1}{2}$ we indicate the vertices with an even, by the coefficient $-\frac{1}{2}$ the vertices with an odd number of negative coordinates.

In connection with this we amplify the question of art. 1 as follows: "Under what circumstances will the symbols

$$[a_1, a_2, \ldots, a_n], \quad \pm\tfrac{1}{2}[a_1, a_2, \ldots, a_n]$$

represent the vertices of polytopes in S_n, all the edges of which have the same length, say unity?"

The answer to this question runs as follows:

Theorem XXVIII. "If the values $a_1, a_2, \ldots, a_n$ are arranged in decreasing order, a_p being the smallest non vanishing one, and if a_k, a_{k+1} represent any couple of adjacent unequal ones, we must have in the case of the first symbol $[a_1, a_2, \ldots, a_n]$

$$\left.\begin{array}{lll} \textit{either}\ p = n,\ a_n = \frac{1}{2} & , & a_k - a_{k+1} = \frac{1}{2}\sqrt{2} \\ \textit{or}\quad p < n,\ a_p = \frac{1}{2}\sqrt{2}, & & a_k - a_{k+1} = \frac{1}{2}\sqrt{2} \end{array}\right\},$$

in the case of the second symbol $\pm\frac{1}{2}[a_1, a_2, \ldots, a_n]$

$$p = n,\ a_{n-1} = a_n = \tfrac{1}{4}\sqrt{2},\ a_k - a_{k+1} = \tfrac{1}{2}\sqrt{2}."$$

Proof. The part of the proof concerned with the common value $\frac{1}{2}\sqrt{2}$ of the difference $a_k - a_{k+1}$ of two unequal adjacent digits is the same as that given in art. 1. So we have to add only a few words about the values of a_n in the case of the first and of a_{n-1} and a_n in the case of the second symbol.

Symbol $[a_1, a_2, \ldots, a_n]$. In the supposition $a_k - a_{k+1} = \frac{1}{2}\sqrt{2}$ the length of the edge of the polytope is unity. Therefore the distance $2a_n$ between the points

$$\begin{array}{l} P \ldots x_1 = \quad a_n,\ x_2 = a_1,\ x_3 = a_2, \ldots \\ Q \ldots x_1 = -a_n,\ x_2 = a_1,\ x_3 = a_2, \ldots \end{array}$$

which are transformed into each other by inverting the sign of

a_n, must be unity, which gives $a_n = \frac{1}{2}$, unless P and Q coincide which happens for $a_n = 0$. So in the case $p = n$ we have $a_n = \frac{1}{2}$.

In the case $p < n$ we consider the points

$$P \ldots x_1 = a_p, x_2 = 0\ , x_3 = a_1, x_4 = a_2, \ldots$$
$$Q \ldots x_1 = 0\ , x_2 = a_p, x_3 = a_1, x_4 = a_2, \ldots$$

passing into each other by interchanging x_1 and x_2. The distance $a_p \sqrt{2}$ between these points is unity for $a_p = \frac{1}{2}\sqrt{2}$.

Symbol $\pm \frac{1}{2}[a_1, a_2, \ldots, a_n]$. Here a_n differs from zero; for the supposition $a_n = 0$ is incompatible with the division of the vertices represented by the symbol $[a_1, a_2, \ldots, a_n]$ into the two groups $\pm \frac{1}{2}[a_1, a_2, \ldots a_n]$, the inversion of the sign of zero having no effect whatever.

Here the point

$$P \ldots x_1 = a_n\ , x_2 = a_{n-1}, x_3 = a_{n-2}, \ldots$$

must be considered in combination with the points

$$Q \ldots x_1 = a_{n-1}, x_2 = a_n\ , x_3 = a_{n-2}, \ldots$$
$$R \ldots x_1 = -a_n\ , x_2 = -a_{n-1}, x_3 = a_{n-2}, \ldots$$

corresponding with it as to the coordinates $x_3, x_4, \ldots x_{n+1}$, as these points Q and R are the nearest ones to P obtainable either by interchanging two digits or by inverting the signs of two digits. Now we have under these circumstances

$$PQ^2 = 2\,(a_n - a_{n-1})^2, \quad PR^2 = 4\,(a^2_n + a^2_{n-1}),$$

from which ensues $PQ < PR$. So we must have $PQ = 0$, $PR = 1$, giving $a_n = a_{n-1} = \frac{1}{4}\sqrt{2}$.

48. In the case of the first symbol $[a_1, a_2, \ldots, a_n]$ we are confronted with two possibilities, as we have to choose between $a_n = \frac{1}{2}$ and $a_n = 0$, i. e. between a group containing the measure polytope $[\frac{1}{2}, \frac{1}{2}, \ldots \frac{1}{2}]$ and an other group containing the cross polytope $[\frac{1}{2}\sqrt{2}, 0, \ldots 0]$. Do the two regions lying on different sides of the limiting demarcation line cover the same ground as the group of the measure polytope on one side and the group of the cross polytope on the other? The answer to this question depends on the manner of deduction of these two groups. If we follow closely the geometrical manner of deduction developed by M[rs]. STOTT the contraction forms derived from the measure polytope *do* possess coordinate symbols winding up in zero, whilst on the other hand the form derived from the cross polytope by means of a set of expansions under which e_{n-1} occurs are represented by coordinate symbols containing no zero. These two exceptional facts which

prove the close relationship between the progeniture of the two patriarchs, cube and octahedron, can be extended so as to make the two families quite *identical* with each other; to that end we have only to derive from each of the two, cube and octahedron, *all* the expansion and contraction forms, the number of which amounts in S_n to $2^n - 1$. This important fact, which will be proved later on, enables us to treat in the second and third sections the forms with the symbols $[a_1, a_2, \ldots, \frac{1}{2}]$ and $[a_1, a_2, \ldots, 0]$ successively, without being obliged to postpone the study of the corresponding nets built up by forms of both groups.

In order to avoid fractions we will multiply the digits by two in this section and the next one; under this circumstance the last digit is unity or zero, the difference $a_k - a_{k+1}$ of two unequal adjacent digits is $\sqrt{2}$ and the symbol represents a polytope with edge 2. Moreover in order to simplify the symbols we will write p' for $1 + p\sqrt{2}$ and put if possible $\sqrt{2}$ outside the brackets, substituting e. g. $[11100]\sqrt{2}$ for $[\sqrt{2}, \sqrt{2}, \sqrt{2}, 0, 0]$.

49. For $n = 2, 3, 4, 5$ we have successively in the symbols explained in the memoir of M^{rs}. Stott: [1])

$n = 2.$

$[1\,1] = p_4$	$[1'1] = e_1 p_4 = p_8$		$[10]\sqrt{2} = ce_1 p_4 = \overline{p_4}$

$n = 3.$

$[1\,1\,1] = C$	$[1'1\,1] = e_2 C = RCO$		$[100]\sqrt{2} = ce_2 C = O$
$[1'1'1] = e_1 C = tC$	$[2'1'1] = e_1 e_2 C = tCO$	$[110]\sqrt{2} = ce_1 C = CO$	$[210]\sqrt{2} = ce_1 e_2 C = tO$

$n = 4.$

$[1\,1\,1\,1] = C_8$	$[2'2'1'1] = e_1 e_2 C_8$		$[2110]\sqrt{2} = ce_1 e_2 C_8$
$[1'1'1'1] = e_1 C_8$	$[2'1'1'1] = e_1 e_3 C_8$	$[1110]\sqrt{2} = ce_1 C_8$	$[2110]\sqrt{2} = ce_1 e_3 C_8$
$[1'1'1\,1] = e_2 C_8$	$[2'1'1\,1] = e_2 e_3 C_8$	$[1100]\sqrt{2} = ce_2 C_8$	$[2100]\sqrt{2} = ce_2 e_3 C_8$
$[1'1\,1\,1] = e_3 C_8$	$[3'2'1'1] = e_1 e_2 e_3 C_8$	$[1000]\sqrt{2} = ce_3 C_8$	$[3210]\sqrt{2} = ce_1 e_2 e_3 C_8$

$n = 5.$

$[1\,1\,1\,1\,1] = C_{10}$	$[2'2'1'1\,1] = e_2 e_3 C_{10}$		$[22100]\sqrt{2} = ce_2 e_3 C_{10}$
$[1'1'1'1'1] = e_1 C_{10}$	$[2'1'1'1\,1] = e_2 e_4 C_{10}$	$[11110]\sqrt{2} = ce_1 C_{10}$	$[21100]\sqrt{2} = ce_2 e_4 C_{10}$
$[1'1'1'1\,1] = e_2 C_{10}$	$[2'1'1\,1\,1] = e_3 e_4 C_{10}$	$[11100]\sqrt{2} = ce_2 C_{10}$	$[21000]\sqrt{2} = ce_3 e_4 C_{10}$
$[1'1'1\,1\,1] = e_3 C_{10}$	$[3'3'2'1'1] = e_1 e_2 e_3 C_{10}$	$[11000]\sqrt{2} = ce_3 C_{10}$	$[33210]\sqrt{2} = ce_1 e_2 e_3 C_{10}$
$[1'1\,1\,1\,1] = e_4 C_{10}$	$[3'2'2'1'1] = e_1 e_2 e_4 C_{10}$	$[10000]\sqrt{2} = ce_4 C_{10}$	$[32210]\sqrt{2} = ce_1 e_2 e_4 C_{10}$
$[2'2'2'1'1] = e_1 e_2 C_{10}$	$[3'2'1'1'1] = e_1 e_3 e_4 C_{10}$	$[22210]\sqrt{2} = ce_1 e_2 C_{10}$	$[32110]\sqrt{2} = ce_1 e_3 e_4 C_{10}$
$[2'2'1'1'1] = e_1 e_3 C_{10}$	$[3'2'1'1\,1] = e_2 e_3 e_4 C_{10}$	$[22110]\sqrt{2} = ce_1 e_3 C_{10}$	$[32100]\sqrt{2} = ce_2 e_3 e_4 C_{10}$
$[2'1'1'1'1] = e_2 e_3 C_{10}$	$[4'3'2'1'1] = e_1 e_2 e_3 e_4 C_{10}$	$[21110]\sqrt{2} = ce_2 e_3 C_{10}$	$[43210]\sqrt{2} = ce_1 e_2 e_3 e_4 C_{10}$

[1]) For the deduction of the e and c symbols from the symbol of coordinates compare the part D of this section; here $\overline{p_4}$ means: p_4 turned 45° about the centre.

In Table IV added at the end of this memoir are put on record for $n = 3, 4, 5$, the different polyhedra and polytopes deduced from the measure polytope. Of this table the first column contains the symbols of deduction of the polytope from measure polytope and cross polytope — with the first of which we are concerned in this section only — and the third the symbol of coordinates. The second and the following columns will be explained farther on.

Here we have $[1100]\sqrt{2} = C^{(2)}{}_{24}$, $[1000]\sqrt{2} = C^{(2)}{}_{16}$, $[10000]\sqrt{2} = C^{(2)}{}_{32}$.

Remark. If we invert the sign of all the coordinates of a vertex V of the polytope we get the coordinates of an other vertex V' of that polytope for which the centre of the segment PP' is the origin of coordinates O. So, all the forms derived analytically from the measure polytope admit central symmetry, as the geometrical deduction by means of the operations e and c requires it.

B. *The characteristic numbers.*

50. In the case of the simplex the direct method for the determination of the characteristic numbers proceeding regularly from vertices to edges, from edges to faces, etc. was preceded by an easier method fulfilling the exigencies of the cases $n = 4$ and $n = 5$, working from both sides, the vertex side and the side of the limiting element of the highest number of dimensions; in this case of the measure polytope we will do likewise. [1])

Here also the number of vertices is easily found. If all the n digits of the symbol of coordinates are different it is $2^n \cdot n!$; of the two factors 2^n and $n!$ of this product the first is due to the power of choosing arbitrarily the signs of the n digits, whilst the second corresponds to the power of permutating them. This product must be divided by $2!$ for any two, by $3!$ for any three digits being equal, etc.

In order to be able to find the number of the limiting bodies ($n = 4$) and that of the limiting polytopes ($n = 5$) we have to prove here the

THEOREM XXIX. "The non vanishing coefficients c_i of the coordinates x_i in the equation $c_1 x_1 + c_2 x_2 + \ldots = p$ of a limiting space S_{n-1} of the polytope deduced from the measure polytope of S_n must all of them have the same *absolute* value."

The difference between this theorem and the corresponding one for the simplex (theorem II of art. 6) lies in the addition of the word "absolute", therefore printed in italics. This amplification is necessary here, in connection with the power of assigning to each of the n digits of the coordinate symbol either the positive or the negative sign. But the proof runs quite in the same lines. If in the case of the polytope $[1 + 2\sqrt{2},\ 1 + \sqrt{2},\ 1 + \sqrt{2},\ 1]$ we start from the equation $2x_1 - x_2 = p$ and try to determine the

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[1]) The treatment of the offspring of the measure polytope with which we are concerned now — and of that of the cross polytope which comes next — will be copied as much as possible from Section I.

vertices of the polytope for which the expression $2x_1 - x_2$ becomes either a maximum or a minimum we find the maximum $3 + 5\sqrt{2}$ for $x_1 = 1 + 2\sqrt{2}$, $x_2 = -(1 + \sqrt{2})$ and the minimum $-(3+5\sqrt{2})$ of the same absolute value for $x_1 = -(1 + 2\sqrt{2})$, $x_2 = 1 + \sqrt{2}$. So, for values of p between $3 + 5\sqrt{2}$ and $-(3 + 5\sqrt{2})$ the space $2x_1 - x_2 = p$ intersects the polytope, whilst it cannot contain a limiting body but at most a limiting face only for the extreme values $\pm(3 + 5\sqrt{2})$ of p, as each of the two couples of equations $x_1 = 1 + 2\sqrt{2}$, $x_2 = -(1 + \sqrt{2})$ and $x_1 = -(1 + 2\sqrt{2})$, $x_2 = 1 + \sqrt{2}$ determines a plane. Here too, as far as the vertices of the polytope are concerned, any linear equation $c_1 x_1 + c_2 x_2 + \ldots = p$ represents k different equations if the non vanishing coefficients c_i admit k different *absolute* values. Here too the theorem is not reversible. As to the theory of the determination of the number of faces ($n = 4$) and the number of limiting bodies ($n = 5$) compare the end of art. 6.

Remark. In accordance with the central symmetry of the polytope $[a_1, a_2, \ldots, a_n]$ any two parallel spaces S_{n-1}, represented by the equations $x_i + x_k + x_l + \ldots = \pm p$ and lying therefore on different sides at the same distance from the origin, bear either both or none of them a limit $(l)_{n-1}$ of the polytope. So, in the determination of the limits $(l)_{n-1}$ we can restrict ourselves here to the equations $x_i + x_k + x_l + \ldots =$ maximum.

51. We now treat at full length two examples, one in S_4 and one in S_5.

Example $[\mathbf{1 + 2\sqrt{2}, 1 + \sqrt{2}, 1 + \sqrt{2}, 1}]$ [1]).

The number of vertices is $2^4 . 4!$ divided by $2!$, i. e. $16 . 24 : 2 = 192$.

The number of the edges passing through each vertex is five. For the pattern vertex

$$1 + 2\sqrt{2}\,,\ 1 + \sqrt{2}\,,\ 1 + \sqrt{2}\,,\ 1$$

is adjacent to the five vertices

$$\left.\begin{array}{llll}
1 + \sqrt{2}, & 1 + 2\sqrt{2}, & 1 + \sqrt{2}, & 1 \\
1 + \sqrt{2}, & 1 + \sqrt{2}, & 1 + 2\sqrt{2}, & 1 \\
1 + 2\sqrt{2}, & 1, & 1 + \sqrt{2}, & 1 + \sqrt{2} \\
1 + 2\sqrt{2}, & 1 + \sqrt{2}, & 1, & 1 + \sqrt{2} \\
1 + 2\sqrt{2}, & 1 + \sqrt{2}, & 1 + \sqrt{2}, & -1
\end{array}\right\},$$

[1]) In vol. XI of the „Wiskundige Opgaven" we have recently treated the polytope $[1 + 3\sqrt{2}, 1 + 2\sqrt{2}, 1 + \sqrt{2}, 1]$ and its projections on its four kinds of axes (problem 78) and deduced the symbol of characteristic numbers of the polytope $[1 + (n-1)\sqrt{2}, 1 + (n-2)\sqrt{2}, \ldots, 1 + \sqrt{2}, 1]$ of S_n (problem 80). For the latter point compare also my paper „On the characteristic numbers of the polytopes $e_1 e_2 \ldots e_{n-2} e_{n-1} S(n+1)$ and $e_1 e_2 \ldots e_{n-2} e_{n-1} M_n$ of space S_n" (Mathematical congress, Cambridge, August 1912).

which may be indicated by the brackets and the negative sign after 1 in the symbol

$$1+2\sqrt{2}\ ,\ 1+\sqrt{2}\ ,\ 1+\sqrt{2}\ ,\ 1(-).$$

So the number of edges is $\dfrac{192\times 5}{2} = 480$.

In order to find spaces which may contain limiting bodies we have to consider the equations

$$\begin{array}{lll}
a) \ldots \pm x_1 & = 1 & + 2\sqrt{2}, \\
b) \ldots \pm x_1 \pm x_2 & = 2 & + 3\sqrt{2}, \\
c) \ldots \pm x_1 \pm x_2 \pm x_3 & = 3 & + 4\sqrt{2}, \\
d) \ldots \pm x_1 \pm x_2 \pm x_3 \pm x_4 & = 4(1 & + \sqrt{2}).
\end{array}$$

a). The equation $x_1 = 1 + 2\sqrt{2}$ gives us for the other coordinates the system represented by $x_2, x_3, x_4 = [1+\sqrt{2}, 1+\sqrt{2}, 1]$, i.e. an $e_1\,C$. This $t\,C$ presents itself 2.4 times, as in the equation $\pm x_i = 1 + 2\sqrt{2}$ the sign may be either positive or negative (factor 2), while the index i may be any of the four indices 1, 2, 3, 4 (factor 4).

b). The condition $x_1 + x_2 = 2 + 3\sqrt{2}$ gives $x_1, x_2 = (1 + 2\sqrt{2}, 1 + \sqrt{2})$ and $x_3, x_4 = [1 + \sqrt{2}, 1]$, i.e. we have for the coordinates in their natural order of succession

$$x_1, x_2, x_3, x_4 = (1+2\sqrt{2}, 1+\sqrt{2})\,[1+\sqrt{2}, 1]$$

representing an octagonal prism P_8 with end planes parallel to $O(X_3\,X_4)$ and edges normal to these planes parallel to the lines $x_1 + x_2 =$ constant in $O(X_1\,X_2)$; this P_8 occurs $2^2.6$ times, as we dispose in $\pm x_i \pm x_j = 2 + 3\sqrt{2}$ over two couples of signs (factor 2^2) and the pair of indices i, j stands for any of the combinations of the four indices by two (factor 6).

c) In the supposition $x_1 + x_2 + x_3 = 3 + 4\sqrt{2}$ we find in the same way

$$x_1, x_2, x_3, x_4 = (1+2\sqrt{2}, 1+\sqrt{2}, 1+\sqrt{2})\,[1],$$

i.e. a triangular prism P_3 occurring $2^3.4$ times.

d) Finally for $\Sigma x = 4(1+\sqrt{2})$ we get

$$x_1, x_2, x_3, x_4 = (1+2\sqrt{2}, 1+\sqrt{2}, 1+\sqrt{2}, 1),$$

which — compare the last result of art. 46 — is a $C\,O$, occurring 2^4 times.

So, all in all we have got the limiting bodies

$$8\,tC\ ,\ 24\,P_8\ ,\ 32\,P_3\ ,\ 16\ CO;$$

so their number is 80.

As the numbers of faces of tC, P_8, P_3, CO are respectively 14, 10, 5, 14, the total number of faces is

$$\tfrac{1}{2}\,(8\times 14 + 24\times 10 + 32\times 5 + 16\times 14) = 368.$$

So the final result is (192, 480, 368, 80), in accordance with the law of Euler.

Remark. In the case of the measure polytope C_8 of S_4 represented by [1, 1, 1, 1] the spaces represented by

$$\begin{array}{ll} a)\ \ldots\ldots & x_1 = 1 \\ b)\ \ldots\ldots & x_1 + x_2 = 2 \\ c)\ \ldots\ldots & x_1 + x_2 + x_3 = 3 \\ d)\ \ldots\ldots & x_1 + x_2 + x_3 + x_4 = 4 \end{array}$$

contain respectively a limiting cube, a face, an edge, a vertex of C_8. So we find here in the case of the chosen example

8 tC	of	body	import,
24 P_8	,,	face	,,
32 P_3	,,	edge	,,
16 CO	,,	vertex	,,

52. *Example* $[\mathbf{1+3\sqrt{2},\ 1+2\sqrt{2},\ 1+2\sqrt{2},\ 1+\sqrt{2},\ 1}]$.

The number of vertices is $2^5 . 5! : 2! = 32 . 120 : 2 = 1920$.

The number of edges passing through each vertex is six, as can be derived from the symbol

$$1+3\sqrt{2},\ 1+2\sqrt{2},\ 1+2\sqrt{2},\ 1+\sqrt{2},\ 1\,(-),$$

containing five brackets and the negative sign after 1. So the number of edges is $\dfrac{1920\times 6}{2} = 5760$.

In this case the limiting polytopes can only lie in spaces S_4 with equations of the form

$$\begin{array}{lll} a)\ \ldots & \pm x_1 & = 1+3\sqrt{2}, \\ b)\ \ldots & \pm x_1 \pm x_2 & = 2+5\sqrt{2}, \\ c)\ \ldots & \pm x_1 \pm x_2 \pm x_3 & = 3+7\sqrt{2}, \\ d)\ \ldots & \pm x_1 \pm x_2 \pm x_3 \pm x_4 & = 4+8\sqrt{2}, \\ e)\ \ldots & \pm x_1 \pm x_2 \pm x_3 \pm x_4 \pm x_5 & = 5+8\sqrt{2}, \end{array}$$

corresponding respectively to

a)..2 . 5 polytopes $(1+3\sqrt{2})[1+2\sqrt{2},\ 1+2\sqrt{2},\ 1+\sqrt{2},\ 1]$,
b)..2^2.10 „ $(1+3\sqrt{2},\ 1+2\sqrt{2})\,[1+2\sqrt{2},\ 1+\sqrt{2},\ 1]$,
c)..2^3.10 „ $(1+3\sqrt{2},\ 1+2\sqrt{2},\ 1+2\sqrt{2})\,[1+\sqrt{2},\ 1]$,
d)..2^4. 5 „ $(1+3\sqrt{2},\ 1+2\sqrt{2},\ 1+2\sqrt{2},\ 1+\sqrt{2})\,[1]$,
e)..2^5. „ $(1+3\sqrt{2},\ 1+2\sqrt{2},\ 1+2\sqrt{2},\ 1+\sqrt{2},\ 1)$,

Of these groups of polytopes the first, of polytope import, can be studied by itself; it proves to be a form with the characteristic numbers (192, 384, 248, 56), an $e_1 e_2 C_8$. The second group consists of prisms on $[1+2\sqrt{2},\ 1+\sqrt{2},\ 1] = tCO$ as base, the third group of prismotopes (3 ; 8), the fourth group of prisms on $(1+3\sqrt{2},\ 1+2\sqrt{2},\ 1+2\sqrt{2},\ 1+\sqrt{2}) = CO$ as base. According to art. 46 the fifth group, of vertex import, contains forms $e_1 e_3 S(5)$. So we find

$$10\, e_1 e_2 C_8 + 40 P_{tCO} + 80\,(8;3) + 80\, P_{CO} + 32\, e_1 e_3 S(5) = \\ = 242 \text{ polytopes},$$

and, as $e_1 e_2 C_8$, P_{tCO}, (8; 3), P_{CO}, $e_1 e_3 S(5)$ admit respectively 56, 28, 11, 16, 30 limiting bodies

$$\tfrac{1}{2}(10 \times 56 + 40 \times 28 + 80 \times 11 + 80 \times 16 + 32 \times 30) = \\ = 2400 \text{ polyhedra}.$$

So, according to the law of Euler, the number of faces is 6000, and the final result a (1920, 5760, 6000, 2400, 242). [1])

53. We pass now to the more direct method going straight on from vertices to limits with the highest number of dimensions, and apply it to the second example

$$[\mathbf{1+3\sqrt{2},\ 1+2\sqrt{2},\ 1+2\sqrt{2},\ 1+\sqrt{2},\ 1}]$$

of the preceding article. But in order to make the symbols less clumsy and thereby the method more manageable we represent once more $1+p\sqrt{2}$ by p'.

The number of vertices was and remains 1920.

According to the symbols the edges split up into four groups, viz. $(3'\,2')$, $(2'\,1')$, $(1'\,1)$, $[1]$. Here $(3'\,2')$ means that any determinate pair of coordinates each affected by a given sign take the interchangeable values $3'$ and $2'$, the other coordinates retaining the same values; whilst $[1]$ means that any determinate coordinate takes successively the values $+1$ and -1, the other coordinates remaining unaltered.

[1]) The fourth and the sixth column of Table IV contain the characteristic numbers and the limiting elements of the highest number of dimensions. The meaning of the second column, of the small subscripts in column four and of the fraction in column five, will be explained later on.

It is easy to calculate the numbers of edges of each group. Through the pattern point with the coordinates $3', 2'\, 2'\, 1', 1$ pass — on account of the two digits $2'$ — *two* edges $(3'\, 2)$ and $(2'\, 1')$, and *one* edge $(1'\, 1)$ and $[1]$. So there are in toto

1920 edges $(3'\, 2')$, 1920 edges $(2'\, 1')$, 960 edges $(1'\, 1)$, 960 edges $[1]$, i. e. 5760 edges.

Remark. We may notice that $[1]$ with one digit only is equivalent, as to the representation of edges, to $(3'\, 2')$, $(2'\, 1')$, $(1'\, 1)$ with two digits. This difference is explained by the different character of the symbols: the digits between square brackets have given *absolute* values, whilst the digits between round brackets satisfy a linear equation, the sum of the digits being constant. This difference will repeat itself throughout the whole section; so $[1'\, 1]$ is a face, an octagon, and $(3'\, 2'\, 2')$ is a face, a triangle, etc.

By applying the notions of "unextended" and "extended" symbols, of the "syllables" of these symbols, etc., given for the offspring of the simplex in art. 9, to the group of polytopes deduced from the measure polytope we easily extend this direct method to faces. According to the symbols the faces split up into eight groups, viz: the triangles $(3'\, 2'\, 2')$ and $(2'\, 2'\, 1')$, the squares $(3'\, 2')\, (2'\, 1)$, $(3'\, 2')\, (1'\, 1)$, $(3'\, 2')\, [1]$, $(2'\, 1')\, [1]$, the hexagon $(2'\, 1'\, 1)$ and the octagon $[1'\, 1]$. In the pattern vertex P concur one of each of the two groups of triangles, one octagon and — on account of the two digits $2'$ — two of each of the four groups of squares, two hexagons. So we find

$$1920\left(\frac{2 \text{ triangles}}{3} + \frac{8 \text{ squares}}{4} + \frac{2 \text{ hexagons}}{6} + \frac{1 \text{ octagon}}{8}\right)$$

$= 1280$ triangles $+$ 3840 squares $+$ 640 hexagons $+$ 240 octagons, i. e. 6000 faces.

According to the symbols the limiting bodies split up into nine groups:

$$(3'\, 2'\, 2'\, 1'),\ (3'\, 2'\, 2')(1'\, 1),\ (3'\, 2'\, 2')\, [1],\ (3'\, 2')\, (2'\, 1'\, 1),\ (3'\, 2')\, (2'\, 1')\, [1],$$
$$(3'\, 2')\, [1'\, 1],\ (2'\, 2'\, 1'\, 1),\ (2'\, 2'\, 1')\, [1],\ [2'\, 1'\, 1],$$

i. e. taken in the same order of succession, of

$$CO,\quad P_3,\quad P_3,\quad P_6,\quad C,$$
$$P_8,\quad tT,\quad P_3,\quad tCO.$$

So we find through P

$$CO + 3\, P_3 + 2\, P_6 + 2\, C + 2\, P_8 + tT + 2\, tCO$$

and therefore in toto

$$1920\left(\frac{CO}{12}+\frac{3\,P_3}{6}+\frac{2\,P_6}{12}+\frac{2\,C}{8}+\frac{2\,P_8}{16}+\frac{tT}{12}+\frac{2\,tCO}{48}\right)$$
$$=160\,CO+960\,P_3+320\,P_6+480\,C+240\,P_8+160\,tT+80\,tCO$$

i. e. 2400 limiting polyhedra.

According to the symbols the limiting polytopes split up into five groups viz. $(3'\,2'\,2'\,1'\,1)$, $(3'\,2'\,2'\,1')[1]$, $(3'\,2'\,2')[1'\,1]$, $(3'\,2')[2'\,1'\,1]$, $[2'\,2'\,1'\,1]$, i. e., taken in the same order of succession, of $e_2\,e_3\,S(5)$, P_{CO}, $(3\,;\,8)$, P_{tCO}, $e_1\,e_2\,C_8$.

So we find through P

$$e_2\,e_3\,S(5)+P_{CO}+(3\,;8)+2\,P_{tCO}+e_1\,e_2\,C_8$$

and therefore in toto

$$1920\left(\frac{e_2\,e_3\,S(5)}{60}+\frac{P_{CO}}{24}+\frac{(3\,;8)}{24}+\frac{2\,P_{tCO}}{96}+\frac{e_1\,e_2\,C_8}{192}\right)=$$
$$=32\,e_1\,e_3\,S(5)+80\,P_{CO}+80\,(3\,;\,8)+40\,P_{tCO}+10\,e_1\,e_2\,C_8,$$

i. e. the same 242 polytopes found in the preceding article.

54. If we exclude once more the "petrified" syllables (11), (111), etc. introduced in art. 9 we can state the:

Theorem XXX. "We obtain the extended symbols of all the groups of d-dimensional limits $(P)_d$ with different symbol of any given n-dimensional polytope $(P)_n$ derived from the measure polytope M_n of space S_n, if we split up the n digits of the pattern vertex in all possible ways, *either* into $n-d$ *or* into $n-d+1$ groups of adjacent digits, place all these groups with exception of the last one of the second case between round and this last one between square brackets, and consider these bracketed groups as the syllables of the extended symbol."

Proof. As in art. 10 we represent the $n-d$ different syllables in round brackets by $(\,.\,.\,)^{k_1}, (\,.\,.\,)^{k_2}, \ldots, (\,.\,.\,)^{k_{n-d}}$. So, in the first case we have the relation $k_1+k_2\ldots+k_{n-d}=n$, whilst addition of the syllable $[\,.\,.\,]^{k'}$ with k' digits leads in the second case to the condition $k_1+k_2+\ \ldots+k_{n-d}+k'=n$. In both cases we suppose in order to fix the ideas that to $(\,.\,.\,)^{k_1}$ correspond the coordinates $x_1, x_2, \ldots, x_{k_1}$, to $(\,.\,.\,)^{k_2}$ the coordinates $x_{k_1+1}, x_{k_1+2}, \ldots, x_{k_1+k_2}$, etc. and in the second case to $[\,.\,.\,]^{k'}$ the coordinates $x_{n-k'+1}$, $x_{n-k'+2}, \ldots, x_n$.

Here too the proof splits up into three parts. As the first case can be deduced from the second by supposing $k'=0$, we indicate the alterations which the three parts of the proof of art. 10 have to undergo for the second case only.

a). *The polytope obtained is a* $(P)_d$.

By the exclusion of petrified syllables we are sure here too that any syllable $(\,.\,.\,)^k$ with k digits allows the vertex, the coordinates of which are the n digits of the symbol of $(P)_n$, to coincide successively with all the vertices of a definite $k-1$-dimensional polytope $(P)_{k-1}$ situated in a space S_{k-1} determining equal segments on k of the n axes OX_i. Moreover the unique syllable $[\,.\,.\,]^{k'}$ with k' digits allows that vertex to coincide successively with all the vertices of a definite k'-dimensional polytope $(P)_{k'}$ situated in a space $S_{k'}$ parallel to the space of coordinates $S'_{k'}$ containing the k' axes OX_i, where i is successively $n-k'+1$, $n-k'+2, \ldots, n$. The spaces bearing these $n-d+1$ polytopes $(P)_k$, $(k = k_1, k_2, \ldots k_{n-d})$, and $(P)_{k'}$ are by two normal to each other. For $(P)_{k_1}$ lies in the space $S_{k_1} = O(X_1 X_2 \ldots X_{k_1})$, $(P)_{k_2}$ lies in the space $S_{k_2} = O(X_{k_1+1} X_{k_1+2} \ldots X_{k_1+k_2})$, etc. and now the spaces S_{k_1}, $S_{k_2}, \ldots$, $S_{k_{n-d}}$, $S_{k'}$ form a set of coordinate spaces containing together all the axes OX_i once, i. e. they are by two perfectly normal to each other. So, as two spaces lying in spaces perfectly normal to each other are themselves perfectly normal to each other, the spaces bearing the $n-d+1$ polytopes found above partake by two of that property. So the polytope under consideration is a prismotope with $n-d+1$ constituents and this prismotope is a $(P)_d$; for its number of dimensions is the sum of the numbers k_1-1, $k_2-1, \ldots, k_{n-d}-1$, k' of the dimensions of the constituents, i. e. the sum of the numbers $k_1, k_2, \ldots k_{n-d}$ diminished by $n-d$, i. e. n diminished by $n-d$, i. e. d.

b). *The* $(P)_d$ *obtained is a limit of* $(P)_n$.

According to the manner in which $(P)_d$ is obtained the coordinates of its vertices satisfy the $n-d$ mutually independent equations

$$x_1 + x_2 + \ldots + x_{k_1} = p_1,\ x_{k_1+1} + x_{k_1+2} + \ldots + x_{k_1+k_2} = p_2, \text{ etc.},$$

if p_1 is the sum of the first k_1 digits of the pattern vertex, p_2 the sum of the next k_2 digits, etc. As in art. 10 these equations can be written in the form

$$\sum_{i=1}^{k_1} x_i = p_1,\ \sum_{i=1}^{k_1+k_2} x_i = p_1 + p_2,\ \ldots,\ \sum_{i=1}^{k_1+k_2+\ldots+k_{n-d}} x_i = p_1 + p_2 + \ldots + p_{n-d},$$

representing $n-d$ limiting spaces S_{n-1} of $(P)_n$, as each of the right hand members is a maximum. For the rest of this part we refer to art. 10.

c) *By means of the theorem we obtain all the limits* $(P)_d$ of $(P)_n$.

For this part compare also art. 10.

55. We apply the notion of end digits and middle digits of the syllables, introduced in art. 12, to the syllables in round brackets occurring in the symbols of the polytopes deduced from the measure polytope, in order to be able to repeat theorem XXX, in a version connected with the more practical unextended symbols, in the following form:

THEOREM XXX′. "We obtain the unextended symbol of a polytope $(P)_d$ the vertices of which are vertices of the given $(P)_n$, if we put the lowest k digits of the pattern vertex between square brackets, where k takes successively one of the values $0, 1, 2, \ldots, d$, and place before it, of the $n - k$ remaining digits, between round brackets either one group of $d - k + 1$ interchangeable digits, or two groups containing together $d - k + 2$ interchangeable digits, or three groups containing together $d - k + 3$ interchangeable digits, etc., this process winding up where the total number of groups is $n - d + k$ for $n < 2d - k + 1$ and d for $n > 2d - k - 1$".

"This $(P)_d$ will be a limiting polytope of $(P)_n$, if the syllables between round brackets satisfy the two following conditions:

1⁰. each syllable with middle digits exhausts these digits of the symbol of $(P)_n$,

2⁰. no two syllables without middle digits have the same end digits".

The proof of this new version can be deduced fron the articles 10, 12 and 54.

By means of theorem XXX′ we deduce the limits $(P)_6$ of the polytope $(P)_{10}$ represented by the symbol $[\mathbf{5' 4' 4' 3' 3' 2' 2' 2' 1' 1}]$, of which — as is easily shown [1]) — the $(P)_9$ of art. 12 represented by (5443322210) is the limit g_0 of vertex import. If we put together the different $(P)_6$ for which the k has the same value we find for $k = 0$ the 58 polytopes given in art. 12 and for $k = 1, 2, \ldots, 6$ successively groups of 33, 11, 9, 6, 2, 1, i. e. in toto 120 polytopes. If for brevity the last syllable — between square brackets — is put at the head of each group, these are

[1]) In rectangular coordinates the polytope g_0 is (5′ 4′ 4′ 3′ 3′ 2′ 2′ 2′ 1′ 1) which may be simplified by passing to parallel axes with the point 1, 1, ..., 1 as origin, i. e. by subtracting a unit from all the coordinates. If we then bear in mind that according to art. 1 we have to divide the coordinate values by $\sqrt{2}$ if we pass to barycentric coordinates on account of the new unit of length, we find (5443322210).

From this relation between a polytope deduced from the measure polytope and its polytope of vertex import can be deduced generally that the number of these polytopes in S_n, the measure polytope itself included, is $C + 2N + 1$, where C and N represent the numbers of central symmetric and of non central symmetric polytopes in S_{n-1} of simplex extraction, the simplex itself included.

$k = 1$, last syllable [0]

(544332), — (54433) (21) — (5443) (322), (5443) (32) (21), (5443) (221), — (544) (3322), (544) (332) (21), (544) (3222), (544) (322) (21), (544) (32) (221), (544) (2221), — (54) (43322), (54) (4332) (21), (54) (433) (221), (54) (43) (3222), (54) (43) (332) (21), (54) (43) (32) (221), (54) (33222), (54) (3322) (21), (54) (332) (221), (54) (32221), — (443322), — (44332) (21), — (4433) (221), — (443) (3222), (443) (322) (21), (443) (32) (221), — (433222), — (43322) (21), — (4332) (221), — (433) (2221), — (43) (32221), — (332221)

$k = 2$, last syllable [10]

(54433), — (5443) (32), — (544) (322), (544) (322), — (54) (4332), (54) (43) (322), — (44332), — (443) (322), — (43322), — (43) (3222), — (33222)

$k = 3$, last syllable [210]

(5443), — (544) (32), — (54) (433), (54) (43) (32), — (4433), — (443) (32), — (4332), — (43) (322), — (3322)

$k = 4$, last syllable [2210]

(544), — (54) (43), — (443), — (433), — (43) (32), — (332)

$k = 5$, last syllable [22210]

(54), — (43)

$k = 6$, *only* syllable [322210].

We remark, that in general the k of the theorem indicates how many of the axes of the rectangular system of coordinates are parallel to the space S_6 bearing the $(P)_6$. For $d = n - 1$, i. e. if we determine the limits of the highest number of dimensions, the k is at the same time the index of the symbol g_k indicating the import. For comparison we put side by side in the next table the different g_k of the polytope $(P)_{10}$ just treated and those of its polytope of vertex import

(5443322210)	g_0		
(544332221)[0]	g_1	(544332221)	g_8
(54433222)[10]	g_2	(54433222)(10)	g_7
(5443322)[210]	g_3	(5443322)(210)	g_6
(544332)[2210]	g_4	(544332)(2210)	g_5
(54433)[22210]	g_5	(54433)(22210)	g_4
(5443)[322210]	g_6	(5443)(322210)	g_3
(544)[3322210]	g_7	(544)(3322210)	g_2
(54)[43322210]	g_8	(54)(43322210)	g_1
[443322210]	g_9	(443322210)	g_0

From the examples given in the art. 51 and 52 it is clear that in the enumeration of the limits of the highest number of dimensions we proceed from $k = n - 1$ to $k = 0$; this principle has been followed too in column five of Table IV.

C. *Extension number and truncation integers and fractions.*

56. Theorem XXXI. "The new polytopes, all with half edges of length unity, can be found by means of a regular extension of the measure polytope followed by a regular truncation, either at the vertices alone, or at the vertices and the edges, or at the vertices, edges and faces, etc."

This theorem is an immediate consequence of that given in art. 50 (theorem XXIX) about the equality of the absolute value of the non vanishing coefficients c_i of the coordinates x_i in the equation $\pm c_1 x_1 \pm c_2 x_2 \pm \ldots = p$ of a limiting space S_{n-1} of the polytope. As to the proof we can refer to art. 15.

The extension number is always equal to the largest digit of the symbol of coordinates. So, if in the case $[2' 1' 1]$ of tCO of three-dimensional space the cube $[1\,1\,1]$ with edge 2 is extended to the cube $[2' 2' 2']$ with edge $2(1 + 2\sqrt{2})$ it is precisely large enough to enable us to deduce $[2' 1' 1]$ from it by truncation; for the limit of face import lies in the space $\pm x_i = 2'$. Likewise in the case $[\sqrt{2}, \sqrt{2}, 0, 0]$ of C_{24} in S_4, which symbol winds up in zero, we have to extend the eightcell $[1\,1\,1\,1]$ to $[\sqrt{2}, \sqrt{2}, \sqrt{2}, \sqrt{2}]$ by multiplying its linear dimensions by $\sqrt{2}$, etc,

The manner in which the amount of truncation is measured most easily can be explained as follows. If the measure polytope $M_n^{(2)} = [\overbrace{11\ldots1}^{n}]$ of S_n with centre O is extended to $M_n^{(2\varepsilon)} = [\overbrace{\varepsilon\,\varepsilon\ldots\varepsilon}^{n}]$, ε being the extension number, and this extended $M_n^{(2\varepsilon)}$ is truncated at a k-dimensional limit $M_k^{(2\varepsilon)}$ with centre M by a space S_{n-1} normal to OM cutting in R any edge PQ of $M_n^{(2\varepsilon)}$ one end point P of which belongs to $M_k^{(2\varepsilon)}$, then $\dfrac{PR}{PQ}$ is considered as the "truncation fraction". Now, as we will prove immediately, PR is always a multiple of $\sqrt{2}$ with half the edge of $M_n^{(2)}$ as unit, whether the symbol of coordinates of the polytope deduced from $M_n^{(2\varepsilon)}$ by truncation terminates in unity or in zero; so, in the relation $PR = q\sqrt{2}$ the multiplicator q which is integer may be called the "truncation integer". So the truncation

fraction $\frac{q\sqrt{2}}{2\varepsilon}$ is irrational if the symbol of coordinates of the polytope winds up in 1 and rational if the last digit of that symbol is zero.

57. If we indicate the truncation numbers corresponding successivily to a truncation at a vertex, an edge, a face,... by $\tau_0, \tau_1 \tau_2, \ldots$ and p' stands once more for $1 + p\sqrt{2}$ we have:

THEOREM XXXII. "If $[m'_0, m'_1, m'_2, \ldots, m'_{n-1}]$ is the symbol of coordinates of a polytope deduced from the measure polytope $M_n^{(2)}$ of S_n — where m'_{n-1} stands for either 1 or 0 — the truncation numbers $\tau_0, \tau_1, \tau_2, \ldots$ are

$$\tau_0 = n\, m_0 - \sum_{i=0}^{n-1} m_i,\quad \tau_1 = (n-1)\, m_0 - \sum_{i=0}^{n-2} m_i,\quad \tau_2 = (n-2)\, m_0 - \sum_{i=0}^{n-3} m_i, \ldots \text{"}$$

Proof. Here m'_0 is the extension number. Now, if we wish to calculate τ_k and we take for the vertices P and Q of the extented measure polytope $[m'_0, m'_0, \ldots, m'_0]$ the points $m'_0, m'_0, \ldots, m'_0$ and $-m'_0, m'_0, \ldots, m'_0$ differing in the sign of x_1 only, we have to apply the theorem of page 27 (art. 17) with respect to the equation $x_1 + x_2 + \ldots + x_{n-k} = c_t$, $(t = 1, 2, 3)$, where c_t is determined by the condition that this space is to contain successively the points P, Q and the pattern vertex $m'_0, m'_1, m'_2, \ldots, m'_{n-1}$ of the polytope under consideration. So we find

$$c_1 = (n-k)\, m'_0,\quad c_2 = (n-k-2)\, m'_0,\quad c_3 = \sum_{i=0}^{n-k-1} m'_i$$

and therefore

$$\frac{PR}{PQ} = \frac{(n-k)\, m'_0 - \sum_{i=0}^{n-k-1} m'_i}{2\, m'_0}$$

But, as $2m'_0$ is PQ, the numerator is PR. As the rational part of $(n-k)\, m'_0$ is equal to that of $\sum_{i=0}^{n-k-1} m'_i$, viz. $n-k$ for $m'_{n-1} = 1$ and zero for $m'_{n-1} = 0$, this numerator is a multiple of $\sqrt{2}$, as we have stated at the end of the preceding article. So we find $\tau_k = (n-k)\, m_0 - \sum_{i=0}^{n-k-1} m_i$, as the theorem has it.

In the case of the polytope P_{10} represented by $[5'4'4'3'3'2'2'2'1'1]$ and in the case of $[5443322210]$ we get

$$\tau_0 = 24,\ \tau_1 = 19,\ \tau_2 = 15,\ \tau_3 = 12,\ \tau_4 = 9,\ \tau_5 = 6,\ \tau_6 = 4,$$
$$\tau_7 = 2,\ \tau_8 = 1.$$

But the extension number of the first polytope is $1+5\sqrt{2}$, that of the second is 5.

Remark. In the application of the method of measuring the amount of truncation introduced for the simplex to the measure polytope we experience that the truncation fraction may become an improper fraction. This means that the point of intersection R of the truncating space S_{n-1} with the edge PQ lies on PQ produced at the side of Q.

If we wish to avoid this inconvenience we can determine the amount of truncation in the following new way. If O is once more the centre of the polytope and M the centre of the limit $M_k^{(2\varepsilon)}$ of the extended measure polytope $M_n^{(2\varepsilon)}$ at which the truncation is to take place, whilst the truncating space S_{n-1} normal at OM cuts OM in P, we may consider $\dfrac{PM}{OM}$ as measure for the amount of truncation. Then we find

$$\frac{PM}{OM}=\frac{(n-k)\,m'_0-\sum\limits_{i=0}^{n-k-1} m'_i}{(n-k)\,m'_0},$$

from which it ensues that the new truncation fraction is deduced from the old one by multiplication by $\dfrac{2}{n-k}$.

But instead of altering our method of measuring the amount of truncation we prefer to put up with the inconvenience indicated. So in Table IV the truncation numbers are indicated, after the extension number where $q'=1+q\sqrt{2}$ and $q''=q\sqrt{2}$, according to the original system in column seven.

D. *Expansion and contraction symbols.*

58. We now prove the theorem:

Theorem XXXII. "The expansion e_k, $(k=1, 2, 3, \ldots, n-1)$, applied to the measure polytope $M_n^{(2)}$ of S_n changes the symbol of coordinates $[\overbrace{1, 1, \ldots, 1}^{n}]$ of that polytope into an other symbol which can be obtained by adding $\sqrt{2}$ to the first $n-k$ digits.

Proof. The operation of expansion e_k is performed by imparting to all the limits $M_k^{(2)}$ of $M_n^{(2)}$ a translational motion, to equal distances away from the centre O of $M_n^{(2)}$, each $M_k^{(2)}$ moving in the direction of the line OM joining O to its centre M, these M_k^2 remaining equipollent to their original position, the motion being

extended over such a distance that the two new positions of any vertex which was common to two adjacent $M_k^{(2)}$ shall be separated by the length 2 of an edge.

Now if we move the limit $M_k^{(2)}$ for which we have

$$x_1 = x_2 = \ldots = x_{n-k} = 1, \quad x_{n-k+1}, x_{n-k+2}, \ldots, x_n = [\overbrace{1, 1, \ldots, 1}^{k}]$$

in the manner described in the direction of the line joining O to its centre M, for which

$$x_1 = x_2 = \ldots = x_{n-k} = 1, \quad x_{n-k+1} = x_{n-k+2} = \ldots = x_n = 0,$$

to a λ times larger distance from O we get a new position of this $M_k^{(2)}$ characterized by

$$x_1 = x_2 = \ldots = x_{n-k} = \lambda, \quad x_{n-k+1}, x_{n-k+2}, \ldots, x_n = [\overbrace{11 \ldots 1}^{k}],$$

in which it is a limit $M_k^{(2)}$ of the new polytope $[\overbrace{\lambda\lambda \ldots \lambda}^{n-k} \overbrace{11 \ldots 1}^{k}]$ and according to the last ten lines of art. 48 this polytope belongs to the progeniture of $M_n^{(2)}$ if we have $\lambda = 1 + \sqrt{2}$. So the result is $[\overbrace{1'1' \ldots 1'}^{n-k} \overbrace{11 \ldots 1}^{k}]$, which proves the theorem, and we find by the way:

Theorem XXXIII. "In the expansion e_k the limits $M_k^{(2)}$ of $M_n^{(2)}$ are moved away from the centre to a distance always equal to $1 + \sqrt{2}$ times the original distance."

This comes true, for $1 + \sqrt{2}$ is the first digit of the symbol of coordinates of the new polytope and, as we found in art. 56, this first digit represents the extension number.

As the distance OM was $\sqrt{(n-k)}$ it becomes $(1 + \sqrt{2})\sqrt{(n-k)}$.

Remark. We may express the influence of the operation e_k on the symbol $[\overbrace{11 \ldots 1}^{k}]$ without interval between the digits by saying that it creates an interval $\sqrt{2}$ between the $n + k^{th}$ and the $n + k + 1^{st}$ digit.

59. Theorem XXXIV. "The influence of any number of expansions $e_k, e_l, e_m, \ldots$ of $M_n^{(2)}$ on its symbol $[\overbrace{11 \ldots 1}^{n}]$ is found by adding together the influences of each of the expansions taken separately."

Proof. We begin by combining two expansions only.

In the succession of two expansions the subject of the second is to be what its original subject has become under the influence of the first. So in the case $e_2 e_1 C$ of the cube C (fig. 13^a) the

original subject of e_2 (the square) is transformed by e_1 into an octagon (fig. 13[b]) and now the octagon is moved out, in the case $e_1 e_2 C$ the linear subject of e_1 (the edge) is transformed by e_2 into a square (fig. 13[c]) and now this square is moved out; in both cases the result (fig. 13[d]) is the same, a tCO. In general, for $k > l$, in the case $e_k e_l M_n^{(2)}$ the subject $M_k^{(2}$ of e_k is transformed by e_l into an $M_k^{(2')}$, while in the case $e_l e_k M_n^{(2)}$ the subject $M_l^{(2)}$ of e_l is transformed by e_k into an $n-1$-dimensional limit g_l of the import l. Here also the geometrical condition "that the two new positions of any vertex shall be separated by the length of an edge" makes the distance over which the second motion of any of these pairs has to take place equal to the distance described in the first motion of the other pair; i. e. if $M_l^{(2)}$ is a limit of the limit $M_k^{(2)}$ of $M_n^{(2)}$ and A is a vertex of that $M_l^{(2)}$, the segments described by A in transforming $M_n^{(2)}$ into the two polytopes $e_k e_l M_n^{(2)}$ and $e_l e_k M_n^{(2)}$ are the two pairs of sides, with the length $\sqrt{2(n-k)}$ and $\sqrt{2(n-l)}$, of a rectangle leading from A to the opposite vertex A'. So we find the coordinates of A' by adding to the coordinates of A the variations corresponding to the motions due to each of the operations e_k, e_l taken separately. So, in the case of three or more expansions we will have to use the extension of this rule to parallelopipeda and parallelotopes; to this geometrical composition of motions always corresponds the arithmetical addition of influences. So the general rule is proved.

By the way we still find the theorem:

THEOREM XXXV. "The operation e_k can still be applied to any expansion form deduced from $M_n^{(2)}$ in the symbol of coordinates of which the $n-k^{th}$ and the $n-k+1^{st}$ digit, i. e. the k^{th} and the $k+1^{st}$ digit counted from the end, are equal"

This theorem enables us to find immediately the expansion symbols of an expansion form deduced from $M_n^{(2)}$ with given coordinate symbol. We show this by the example **[5′ 4′ 4′ 3′ 3′ 2′ 2′ 2′ 1′ 1]** of art. 55.

In [5′ 4′ 4′ 3′ 3′ 2′ 2′ 2′ 1′ 1] five intervals occur, viz, if we represent the p^{th} digit from the end by d_p between (d_1, d_2), (d_2, d_3), (d_5, d_6), (d_7, d_8), (d_9, d_{10}). So we find $e_1 e_2 e_5 e_7 e_9 M_{10}$.

60. By means of the operations e_k we can deduce from $M_n^{(2)}$ all the possible polytopes the square bracketed symbol of coordinates of which winds up in a unit. If we wish to deduce from $M_n^{(2)}$ also all the forms with a square bracketed symbol ending in zero — which is a desideratum as to the treatment of the nets — we have to introduce the operation c of *contraction*. The subject of this contraction

is the group of limits $(l)_{n-1}$ of vertex import, sometimes denoted by g_0, the vertices of which form exactly all the vertices of the expansion form, each vertex taken once, and now the operation c consists in this: all these limits undergo a translational motion, of the same amount, towards the centre O of the expansion form, by which any of these limits gets a vertex or some vertices in common with some of the others. By this contraction the edges of the expansion form parallel to the axes of coordinates are annihilated.

We have now the general theorem:

THEOREM XXXVI. "By applying the contraction c to any expansion form all the digits of the symbol of coordinates of this form are diminished by a unit".

This theorem, which shows that the preceding one still holds for contraction forms deduced from $M_n^{(2)}$, is almost self evident. So, as the motion of the limit g_0 lying in that part of S_n where all the coordinates are positive takes place in the direction of the line making in that part of S_n equal angles whith the n axes, all the coordinates of the pattern vertex diminish by the same amount, and this process has to go on untill the smallest of the digits disappears. For then we once more obtain a polytope the symbol of coordinates of which satisfies the laws of the first part of theorem XXVIII (art 47).

Remark. By combining the theorems XXXV and XXXVI we can find the symbol in the operators c and e_k of *any* form deduced from $M_n^{(2)}$. But this process can be simplified by introducing the operation e_0 which transforms the centre O of $M_n^{(2)}$ considered as an infinitesimal measure polytope $M_n^{(0)}$ into $M_n^{(2)}$. Then the contraction symbol c can be shunted out by substituting $e_k\, e_l \ldots e_m\, M_n^{(0)}$ for $c\, e_k\, e_l \ldots e_m\, M_n^{(2)}$, but this implies that we replace $e_k\, e_l \ldots e_m\, M_n^{(2)}$ by $e_0\, e_k\, e_l \ldots e_m\, M_n^{(0)}$. This remark will be useful in part F of the next section (compare theorem LIII).

E. *Nets of potytopes.*

61. The theory of the nets derived from $M_n^{(2)}$ is based entirely on the consideration of the most simple of these nets, the net $N(M_n^{(2)})$ of the measure polytope itself. So we begin by the analytical representation of that net $N(M_n^{(2)})$.

By means of the symbol $[2a_1 + 1, 2a_2 + 1, \ldots, 2a_n + 1]$ the net of $M_n^{(2)}$ is decomposed into its measure polytopes, if $a_1, a_2, \ldots, a_n$ are arbitrary integers and the heavy square brackets mean that in order to obtain a definite $M_n^{(2)}$ of the net we have to permutate and to

take with either of the two signs the units printed in heavy type only. Of the M_n^2 brought to the fore by this symbol itself the centre is the point $2a_1, 2a_2, \ldots, 2a_n$. So $[2a_1, 2a_2, \ldots, 2a_n]$ may be called the "frame" of the net, and this symbol may be written quite as well with round or even without brackets, as the faculty of taking for the a_i all possible integer values includes permutation and changing of signs.

62. If we consider the net $N(M_n^{(2)})$ as a polytope[1]) of S_{n+1} with an infinite number of limits $(l)_n$ which instead of bending round in S_{n+1} fills S_n, we can apply to this polytope the expansions $e_1, e_2, \ldots, e_n$ and the contraction c, either separately or in possible combination; in this simple way the measure polytope nets $e_1 N(M_n)$, $e_2 N(M_n)$, etc. have been determined by Mrs. Stott. We introduce the corresponding analytical considerations by the following:

Theorem XXXVII. "Let any expansion or expansion and contraction form $(P)_n$ of $M_n^{(2)}$ be represented by the symbol of coordinates $[a_1, a_2, \ldots, a_{n-1}, a_n]$. Let $M_n^{(2a)}$ be the measure polytope with edge $2a$ concentric and coaxial to this $(P)_n$ and $N(M_n^{(2a)})$ the net of measure polytopes to which the $M_n^{(2a)}$ belongs. Let us suppose in each of the ∞^n measure polytopes of this net a concentric polytope equipollent to $(P)_n$. Then the vertices of all the ∞^n polytopes obtained in this manner cannot form together the vertices of a net, if a differs from a_1 and from $a_1 + 1$."

This theorem of a negative tendency can be proved thus. If we call two $(P)_n$ "adjacent" if the measure polytopes $M_n^{(2a)}$ concentric to them have this position with respect to each other, i. e. if these $M_n^{(2a)}$ are in $M_{n-1}^{(2a)}$ contact, and we consider the limits $(l)_{n-1}$ of the highest import of any two adjacent $(P)_n$ deduced from the common $M_{n-1}^{(2a)}$ of the two $M_n^{(2a)}$ concentric with these $(P)_n$, we see at once that these limits g_{n-1} coincide for $a = a_1$, whilst they are at edge distance from each other and form therefore the end polytopes of a prism for $a = a_1 + 1$. In all other cases two adjacent $(P)_n$ are either too near to each other or too far apart.

What we shall have to show farther is this that the vertices of the ∞^n polytopes $(P)_n$ do form together the vertices of a net in each of the cases $a = a_1$ and $a = a_1 + 1$. We prepare the general proof of this assertion by indicating by the special case of the threedimensional net of truncated cubes $[1 + \sqrt{2}, 1 + \sqrt{2}, 1]$ included in larger cubes $M_3^{(2a)}$, where $a = 2 + \sqrt{2}$, how the other constituents are to be found. This will give us occasion to introduce

[1]) Compare art. 39.

some new geometrical terms by the use of which the expression of general laws will be simplified.

In fig. 14 is represented in heavy lines one of the tC with centre O and an eighth part of the $M_3^{(2a)}$ surrounding it, viz. that part lying in the octant of the positive coordinates taken in the directions OV'_1, OV'_2, OV_3'. Now we make to correspond to the different limiting elements of the surrounding cube the limiting elements of the tC into which the first are transformed if the tC is deduced from the surrounding cube by truncation at vertices, edges and faces. So the triangle ABC of vertex import corresponds to the vertex V, the edge AA' (or the face of edge import which replaces it in an other case) corresponds to the edge VW_1, the octagonal face $B'BCC' \ldots$ corresponds to the face W_2VW_3. Then by reflecting the triangle ABC into the three faces of $M_3^{(2a)}$ through the corresponding vertex V as mirrors and by dealing in the same way with the edge AA' with respect to the two faces through the corresponding edge VW_1 and with the face $B'BCC' \ldots$ with respect to the corresponding face W_2VW_3 we get successively the eight triangular faces of an RCO with V, the four upright edges of a P_4 with V_1, the two end planes of a P_8 with V'_1 as centre. We simplify these expressions by saying that "multiplication" of the triangle ABC round V, of the edge AA' round VW_1, of the face $B'BCC' \ldots$ round W_2VW_3 generates the indicated polyhedra RCO, $P_4 = C$, P_8.

In fig. 14 have been represented in ordinary lines the RCO generated by the triangle ABC, the three cubes generated by the edges AA', BB', CC' and the three P_8 generated by the faces $B'BCC' \ldots$, $C'CAA' \ldots$, $A'ABB' \ldots$ From this diagram it is clear that the indicated RCO, C, P_8 fill up the interstitial space between the tC, i. e. that the net bearing in ANDREINI's memoir the number 22 exists; we facilitate the inspection of this diagram by adding a stereoscopic representation of it. [1])

The deduction of the coordinate symbols of the new constituents RCO, C, P_8 from those of the tC and its surrounding cube shows us, what we have to do in general in order to obtain the coordinate symbols of the new constituents.

We begin with RCO obtained by multiplying the triangle ABC round V. In order to get the representation of the triangle ABC with respect to the original axes we have to replace the square brackets of the symbol $[1 + \sqrt{2}, 1 + \sqrt{2}, 1]$ of tC by round ones. In order to represent that triangle with respect to new axes

[1]) The effect is enhanced if we place it so, as to have the small arrow at the left.

VV_1, VV_2, VV_3 we have to replace the digits of $(1+\sqrt{2}, 1+\sqrt{2}, 1)$ by their complements to $a = 2+\sqrt{2}$, giving $(1, 1, 1+\sqrt{2})$, i. e. $(1+\sqrt{2}, 1, 1)$. In order to multiply the last triangle round the new origin V we have to return to square brackets. So $[1+\sqrt{2}, 1, 1]$ is the symbol of the new constituent RCO. We repeat that the digits of this new symbol are the complements to $a = 2+\sqrt{2}$ of the digits of the "groundform" tC taken in inversed order.

In the case of the edge AA' and the cube derived from it we have to assume V_1, the centre of the cube, as new origin, and V_1V, $V_1V'_2$, $V_1V'_3$ as new axes. Thereby $x_1 = [1]$, $x_2 = 1+\sqrt{2}$, $x_3 = 1+\sqrt{2}$ is transformed into $x_1 = [1]$, $x'_2 = 1$, $x'_3 = 1$; so by multiplication we get $x_1 = [1]$, $x_2, x_3 = [1, 1]$ or shorter $[1]$, $[1, 1]$, which in this special case may be combined to $x_1, x'_2, x'_3 = [1, 1, 1]$ or shorter $[1, 1, 1]$, the cube.

Finally the face $A'ABB'\ldots$ represented by $x_1, x_2 = [1+\sqrt{2}, 1]$, $x_3 = 1+\sqrt{2}$ passes by multiplication into $x_1, x_2 = [1+\sqrt{2}, 1]$, $x_3 = [1]$ or shorter $[1+\sqrt{2}, 1][1]$.

So if we arrange the constituents in the order g_3, g_2, g_1, g_0 of decreasing import we get

$$\left.\begin{array}{llll} g_3 = [1+\sqrt{2}, & 1+\sqrt{2}, & 1] & \\ g_2 = [1+\sqrt{2}, & & 1] & [1] \\ g_1 = [\;\;1\;\;] & & [1 & , 1] \\ g_0 = [1+\sqrt{2}, & & 1 & , 1] \end{array}\right\},$$

the first and the last being semiregular polyhedra deduced from the cube, whilst the intermediate ones appear as prisms. We remark that the pairs of syllables of the symbols of g_2 and g_1 can be derived from the symbols of g_3 and g_0 by taking for g_2 the last two digits of g_3 and the last digit of g_0, for g_1 the last digit of g_3 and the last two digits of g_0.

Now it is obvious that in the general case of the polytope $(P)_n$ of S_n represented by $[a_1, a_2, \ldots, a_{n-1}, a_n]$ the introduced multiplication of the limits of different import, which multiplication can be performed for any value of the constant a, leads in general to $n+1$ constituents $g_n, g_{n-1}, \ldots, g_1, g_0$, represented by

$$\left.\begin{array}{ll} g_0 & = [a_1, a_2, a_3, \ldots\ldots, a_{n-2}, a_{n-1}, a_n] \\ g_{n-1} & = [a_2, a_3, a_4, \ldots\ldots, a_{n-1}, a_n]\,[a-a_1] \\ \ldots & \ldots\ldots\ldots\ldots \\ g_{n-k} & = [a_{k+1}, a_{k+2}, \ldots\ldots, a_n]\,[a-a_k, a-a_{k-1}, \ldots, a-a_1] \\ \ldots & \ldots\ldots\ldots\ldots \\ g_1 & = [a_n]\,[a-a_{n-1}, a-a_{n-2}, \ldots\ldots, a-a_2, a-a_1] \\ g_0 & = [a-a_n, a-a_{n-1}, a-a_{n-2}, \ldots\ldots, a-a_2, a-a_1] \end{array}\right\},$$

where g_n is given, g_0 is obtained by subtracting the digits of g_n from a and taking the differences in inverted order, while the two syllables of g_{n-k} are got by taking the last $n-k$ digits of g_n and the last k digits of g_0 for $k = 1, 2, \ldots, n-1$.

63. We now prove the following problem of positive tendency completing the preceding one.

THEOREM XXXVIII. "In either of the two cases $a = a_1$ and $a = a_1 + 1$ the vertices of the ∞^n polytopes $(P)_n$ of the preceding theorem do form together the vertices of a net. The constituents of this net are obtained by means of the algorithm developed at the end of the preceding article."

We march in the direction of the proof of this general theorem:

1°. by deducing from the symbol of coordinates of the given groundform $(P)_n$ the symbol representing all the repetitions of this polytope and therefore all the vertices of the system,

2°. by deriving from this new symbol the symbols of the polytopes different from the groundform the vertices of which belong to the system (which set of new constituents will prove to be equivalent to that obtained above by the geometrical multiplication introduced above),

3°. by showing that the system of polytopes obtained in this way fills space, i. e. that there is neither overlapping, nor hole.

Symbol of the total system of vertices. The symbol of a definite repetition of the groundform is

$$[2b_1 a + \boldsymbol{a}_1,\ 2b_2 a + \boldsymbol{a}_2, \ldots,\ 2b_{n-1} a + \boldsymbol{a}_{n-1},\ 2b_n a + \boldsymbol{a}_n], \ldots T)$$

where $b_1, b_2, \ldots, b_{n-1}, b_n$ is a definite set of arbitrarily chosen integers. So this symbol represents the total system of vertices, if the b_i denote all possible sets of integers.

From the symbol T we deduce the frame symbol

$$[2b_1 a,\ 2b_2 a, \ldots,\ 2b_{n-1} a,\ 2b_n a], \ldots\ldots\ldots\ldots F)$$

representing the system of vertices of a net of measure polytopes $M_n^{(2a)}$, one of which has the origin as vertex and the n spaces $x_i = 0$, $(i = 1, 2, \ldots, n)$ as limiting spaces.

Presumptive new constituents. The most general transformation by which the total system of vertices $T)$ passes into itself consists in a transport of $p_i a$ units from the permutable to the unmovable part of x_i, the n quantities p_i being integer. But this process is limited by the restriction that in the case of a new constituent sought the permutable parts placed within the same pair of square brackets have to satisfy the conditions of theorem XXVIII, from

which it ensues that the extent of the restriction depends on the number of syllables which the symbol of any constituent may contain. This number is evidently two at most. For the process can only afford besides the original minimum digit a_n one new minimum digit, viz. zero in the case $a = a_1$ and unity in the case $a = a_1 + 1$. So we have to hunt up only new constituents the symbols of which are either monosyllabic or composed of two syllables.

If we take *all* the p_i equal to one we find

$$[(2b_1 + 1)a + \boldsymbol{a}_1 - \boldsymbol{a}, (2b_2 + 1)a + \boldsymbol{a}_2 - \boldsymbol{a}, \ldots,$$
$$(2b_{n-1} + 1)a + \boldsymbol{a}_{n-1} - \boldsymbol{a}, (2b_n + 1)a + \boldsymbol{a}_n - \boldsymbol{a}],$$

or, if we replace negative permutable parts by the positive ones of the same absolute value, rearrange these positive parts according to decreasing order and substitute for brevity β' for $2b + 1$,

$$[\beta_1' a + \boldsymbol{a} - \boldsymbol{a}_n, \beta_2' a + \boldsymbol{a} - \boldsymbol{a}_{n-1}, \ldots, \beta'_{n-1} a + \boldsymbol{a} - \boldsymbol{a}_2, \beta'_n a + \boldsymbol{a} - \boldsymbol{a}_1], \ldots T')$$

winding up in zero for $a = a_1$ and in unity for $a = a_1 + 1$. So we find the repetitions of the new constituent g_0 of the last list of the preceding article. This form g_0 and the given form g_n we started from are the only constituents of measure polytope descent proper.

If we transform the first k digits of T' by the transport of a units from the permutable parts to the unmovable ones and put each of the two sets of digits, the set of the k transformed ones and the set of the $n—k$ untransformed ones, between square brackets, we get after rearranging, if β'_i still replaces $2b_i + 1$ and β_i is substituted for $2b_i$

$$[\beta'_1 a + \boldsymbol{a} - \boldsymbol{a}_k, \beta'_2 a + \boldsymbol{a} - \boldsymbol{a}_{k-1}, \ldots, \beta'_k a + \boldsymbol{a} - \boldsymbol{a}_1]$$
$$[\beta_{k+1} a + \boldsymbol{a}_{k+1}, \beta_{k+2} a + \boldsymbol{a}_{k+2}, \ldots, \beta_{n-1} a + \boldsymbol{a}_{n-1}, \beta_n a + \boldsymbol{a}_n], \ldots T'')$$

revealing the new constituent

$$[a - a_k, a - a_{k-1}, \ldots, a - a_1][a_{k+1}, a_{k+2}, \ldots, a_{n-1}, a_n],$$

a prismotope $(P_k; P_{n-k})$ with the constituents $(P)_k$ and $(P)_{n-k}$ represented by each of the two syllables of the symbol taken separately; if the digits of the second syllable correspond to the coordinates x_1, $x_2, \ldots, x_{n-k}$ and those of the first syllable to $x_{n-k+1}, x_{n-k+2}, \ldots, x_n$, this prismotope is the constituent g_{n-k} of the last list of the preceding article. In the latter case the different positions of $(P)_{n-k}$ are parallel to $O(X_1 X_2 \ldots X_{n-k})$, those of $(P)_k$ to $O(X_{n-k+1} X_{n-k+1} \ldots X_n)$. So we find again all the new constituents obtained formerly by geometrical multiplication.

No overlapping and no hole. By a translational motion in the direction of one of the axes over a distance $2a$ the system of vertices T) is transformed in itself; so, if the central measure polytope $[\overbrace{a, a, \ldots, a}^{n}]$ is filled exactly by the set of constituents found above, these constituents form a net By a reflection in one of the n spaces $x_i = 0$, $(i = 1, 2, \ldots, n)$, the system T) also is transformed in itself; so, if the part of the central measure polytope $M_n^{(2a)}$ containing the points with positive coordinates only is filled exactly, the constituents form a net. We indicate this part of the central measure polytope by the symbol $M_n^{(+a)}$.

We now prove the following lemma:

„Let $(P)_n^a$ be a constituent lying partially within $M_n^{(+a)}$ and $(P)_{n-1}^{a,\,b}$ any of its limits lying partially within $M_n^{(+a)}$. Then the set of polytopes obtained above always contains one and only one polytope $(P)_n^b$ having with $(P)_n^a$ the limit $(P)_{n-1}^{a,\,b}$ in common; this $(P)_n^b$ lies with respect to $(P)_n^a$ on the opposite side of $(P)_{n-1}^{a,\,b}$."

The condition that $(P)_n^a$ lies at least partially within $M_n^{(+a)}$ is fulfilled, if we consider that repetition of the chosen constituent the coordinates of the centre of which admit the values $+a$ and zero only. We find, if all the coordinates are zero the groundform contained in T), if all the coordinates are $+a$ a polytope contained in T'), if some coordinates are $+a$ and the other ones zero a polytope contained in T''). Now the first case, of the groundform, and the second case, of all coordinates $= +a$, are included in the third case, as we get them by putting $k = 0$ and $k = n$. So we can choose for $(P)_n^a$ the polytope

$$\underset{x_1,\, x_2,\, \ldots\ldots,\, x_k}{[a + \boldsymbol{a} - \boldsymbol{a}_k, a + \boldsymbol{a} - \boldsymbol{a}_{k-1}, \ldots, a + \boldsymbol{a} - \boldsymbol{a}_1]}\underset{x_{k+1},\, x_{k+2},\, \ldots\ldots,\, x_n}{[a_{k+1}, a_{k+2}, \ldots, a_{n-1}, a_n]}$$

where the x_i placed under the two syllables indicate the coordinates to which the two sets of digits refer, and occupy ourselves with the question how to get a limit $(l)_{n-1}$ of this prismotope. Now in general the limits $(l)_{n-1}$ of the prismotope $(P_k; P_{n-k})$ present themselves in two groups, viz. if $(P)_{k-1}$ is any limit $(l)_{k-1}$ of $(P)_k$ and $(P)_{n-k-1}$ any limit $(l)_{n-k-1}$ of $(P)_{n-k}$, in the two forms $(P_{k-1}; P_{n-k})$ and $(P_k; P_{n-k-1})$. So [1]), we have to consider the two different cases

[1]) For a limit $(P)_{n-1}^{a,\,b}$ lying at least partially within M_n^{+a} none of the coordinates may assume values equal to or surpassing $+a$ for *all* the vertices of that limit; therefore in the first case $(P_{k-1}; P_{n-k})$ we have to place between round brackets a certain number s_1 of the *largest* digits $[a + \boldsymbol{a} - \boldsymbol{a}_i]$ where $\boldsymbol{a} - \boldsymbol{a}_i$ is taken with the reversed sign, i. e. a_{k_1} $a_{k-1}, \ldots, a_{k-s_1+1}$ taken in inverted order.

$$\left.\begin{array}{l} \underset{x_1, x_2, \ldots\ldots, x_{k-s_1}}{[a+\boldsymbol{a}-\boldsymbol{a}_{k-s_1}, a+\boldsymbol{a}-\boldsymbol{a}_{k-s_1-1}, \ldots, a+\boldsymbol{a}-\boldsymbol{a}_1]} \underset{x_{k-s_1+1}, x_{k-s_1+2}, \ldots, x_k}{(a_{k-s_1+1}, a_{k-s_1+2}, \ldots, a_k)} \\ \hfill \underset{x_{k+1}, x_{k+2}, \ldots, x_n}{[a_{k+1}, a_{k+2}, \ldots, a_{n-1}, a_n]} \\ \underset{x_1, x_2, \ldots\ldots, x_k}{[a+\boldsymbol{a}-\boldsymbol{a}_k, a+\boldsymbol{a}-\boldsymbol{a}_{k-1}, \ldots, a+\boldsymbol{a}-\boldsymbol{a}_1]} \\ \underset{x_{k+1}, x_{k+2}, \ldots\ldots, x_{k+s_2}}{(a_{k+1}, a_{k+2}, \ldots\ldots, a_{k+s_2})} \underset{x_{k+s_2+1}, x_{k+s_2+2}, \ldots, x_n}{[a_{k+s_2+1}, a_{k+s_2+2}, \ldots, a_{n-1}, a_n]} \end{array}\right\},$$

which two limits $(l)_{n-1}$ admit as centres the points

$$\left.\begin{array}{l} \overset{k-s_1}{\overline{aa\ldots a}}\ \overset{s_1}{\overline{t_1 t_1 \ldots t_1}}\ \overset{n-k}{\overline{00\ldots 0}} \\ \overset{k}{\overline{aa\ldots a}}\ \overset{s_2}{\overline{t_2 t_2 \ldots t_2}}\ \overset{n-k-s_2}{\overline{00\ldots 0}} \end{array}\right\},$$

t_1 and t_2 being determined by the relations

$$s_1 t_1 = \sum_{i=k-s_1+1}^{k} a_i \quad , \quad s_2 t_2 = \sum_{i=k+1}^{k+s_2} a_i \quad ,$$

showing that we have $0 < t_i < a$ for $i = 1, 2$. So the centres of these two $(l)_{n-1}$ lie on the boundary of the measure polytope $M_n^{(+a)}$ and therefore the $(l)_{n-1}$ themselves lie partially within that measure polytope.

Now for each of the two cases there is only one constituent passing through the chosen limit $(l)_{n-1}$, viz.

$$\underset{x_1, x_2, \ldots, x_{k-s_1}}{[a+\boldsymbol{a}-\boldsymbol{a}_{k-s_1}, a+\boldsymbol{a}-\boldsymbol{a}_{k-s_1-1}, \ldots, a+\boldsymbol{a}-\boldsymbol{a}_1]} \underset{x_{k-s_1+1}, x_{k-s_1+2}, \ldots, x_n}{[a_{k-s_1+1}, a_{k-s_1+2}, \ldots, a_{n-1}, a_n]}$$

$$\underset{x_1, x_2, \ldots, x_{k+s_2}}{[a+\boldsymbol{a}-\boldsymbol{a}_{k+s_2}, a+\boldsymbol{a}-\boldsymbol{a}_{k+s_2-1}, \ldots, a+\boldsymbol{a}-\boldsymbol{a}_1]} \underset{x_{k+s_2+1}, x_{k+s_2+2}, \ldots, x_n}{[a_{k+s_2+1}, a_{k+s_2+2}, \ldots, a_{n-1}, a_n]}$$

So, all we have to do yet is to investigate the position of the centres. If we indicate these points by the letters G_a, G_{b_1}, G_{b_2}, G_{ab_1}, G_{ab_2} and we remark that for these five points we have

$$x_1 = x_2 = \ldots = x_{k-s_1},\ x_{k-s_1+1} = x_{k-s_1+2} = \ldots = x_k,$$
$$x_{k+1} = x_{k+2} = \ldots = x_{k+s_2},\ x_{k+s_2+1} = x_{k+s_2+2} = \ldots = x_n,$$

we find the following list of coordinates

	$x_1, \ldots$	$x_{k-s_1+1}, \ldots$	$x_{k+1}, \ldots$	$x_{k+s_2+1}, \ldots$
G_a	a	a	0	0
G_{b_1}	a	0	0	0
G_{b_2}	a	a	a	0
G_{ab_1}	a	t_1	0	0
G_{ab_2}	a	a	t_2	0

According to this list of the two triples (G_a, G_{b_1}, G_{ab_1}), (G_a, G_{b_2}, G_{ab_2}) of collinear points G_{ab_1} lies between G_a, G_{b_1}, and G_{ab_2} between

G_a, G_{b_2}. So the proof of the lemma is given. So neither of the two systems of constituents can admit holes.

In order to show that no two polytopes of any of the two systems can overlap we remark that by means of the symbols T), T'), T'') any polytope of the chosen system can be promoted to central polytope, which shows that not a single vertex can lie inside that polytope.

So we have proved completely now the theorem under consideration.

64. We now formulate the manner of deduction of all the measure polytope nets as follows:

Theorem XXXIX. "Let $G = [a_1, a_2, \ldots, a_{n-1}, a_n]$ be the symbol of coordinates of the "groundform" of the net. Deduce from it the symbol $O = [a - a_n, a - a_{n-1}, \ldots, a - a_2, a - a_1]$ of the "opposite form", where a is either a_1 or $a_1 + 1$. Derive from these two symbols G, O the mixed symbol I_k of the "intermediate forms" represented by

$$[a_{n-k+1}, a_{n-k+2}, \ldots, a_{n-1}, a_n] \qquad [a - a_{n-k}, a - a_{n-k-1}, \ldots, a - a_2, a - a_1],$$

of the two syllables of which the first contains the last k digits of G, the second the last $n - k$ digits of O. Then G, the forms I_k, $(k = n - 1, n - 2, \ldots, 2, 1)$, O are respectively the constituents $g_n, g_{n-1}, g_{n-2}, \ldots, g_2, g_1, g_0$ of the net."

"The frame of the constituent g_{n-k} is

$$[\beta_{n-k+1}a, \beta_{n-k+2}a, \ldots \beta_{n-1}a, \beta_n a, \beta'_{n-k}a, \beta'_{n-k-1}a, \ldots, \beta'_2 a, \beta'_1 a],$$

where we have $\beta_i = 2b_i$ and $\beta'_i = 2b_i + 1$, the b_i being integer and the digits of the first syllable being related to the odd, those of the second syllable being related to the even multiples of a".

"If (e, c), etc. indicates a net with an expansion groundform and a contraction opposite form, the theorem includes the four cases:

$$\begin{aligned}
a_n &= 1, a = a_1 \ldots\ldots\ldots\ldots\ldots (e, c),\\
a_n &= 1, a = a_1 + 1 \ldots\ldots\ldots\ldots (e, e),\\
a_n &= 0, a = a_1 \ldots\ldots\ldots\ldots\ldots (c, c),\\
a_n &= 0, a = a_1 + 1 \ldots\ldots\ldots\ldots (c, e)."
\end{aligned}$$

In this theorem the deduction of the intermediate constituents differs slightly from that given in the preceding article, the two methods passing into each other by interchanging k and $n - k$, and the two syllables. In the new form the succession of the different constituents is a more regular one, as the following examples prove.

Example I. The two nets with $\mathbf{[5'\,4'\,4'\,3'\,3'\,2'\,2'\,2'\,1'\,1]}$ as groundform admit the constituents:

$g_{10} \ldots [5'4'4'3'3'2'2'2'1'1]$	$g_{10} \ldots [5'4'4'3'3'2'2'2'1'1]$
	$g_9 \ldots [4'4'3'3'2'2'2'1'1]\,,\,[1]$
$g_8 \ldots [4'3'3'2'2'2'1'1]\,,[1\,0]\sqrt{2}$	$g_8 \ldots [4'3'3'2'2'2'1'1]\,,\,[1'1]$
$g_7 \ldots [3'3'2'2'2'1'1]\,,[1\,1\,0]\sqrt{2}$	$g_7 \ldots [3'3'2'2'2'1'1]\,,\,[1'1'1]$
$g_6 \ldots [3'2'2'2'1'1]\,,[2\,1\,1\,0]\sqrt{2}$	$g_6 \ldots [3'2'2'2'1'1]\,,\,[2'1'1'1]$
$g_5 \ldots [2'2'2'1'1]\,,[2\,2\,1\,1\,0]\sqrt{2}$	$g_5 \ldots [2'2'2'1'1]\,,\,[2'2'1'1'1]$
$g_4 \ldots [2'2'1'1]\,,[3\,2\,2\,1\,1\,0]\sqrt{2}$	$g_4 \ldots [2'2'1'1]\,,\,[3'2'2'1'1'1]$
$g_3 \ldots [2'1'1]\,,[3\,3\,2\,2\,1\,1\,0]\sqrt{2}$	$g_3 \ldots [2'1'1]\,,\,[3'3'2'2'1'1'1]$
$g_2 \ldots [1'1]\,,[3\,3\,3\,2\,2\,1\,1\,0]\sqrt{2}$	$g_2 \ldots [1'1]\,,\,[3'3'3'2'2'1'1'1]$
$g_1 \ldots [1]\,,[4\,3\,3\,3\,2\,2\,1\,1\,0]\sqrt{2}$	$g_1 \ldots [1]\,,\,[4'3'3'3'2'2'1'1'1]$
$g_0 \ldots [5\,4\,3\,3\,3\,2\,2\,1\,1\,0]\sqrt{2}$	$g_0 \ldots [5'4'3'3'3'2'2'1'1'1]$

Example II. The two nets with $\mathbf{[5\,4\,4\,3\,3\,2\,2\,2\,1\,0]}\sqrt{\mathbf{2}}$ as groundform admit the constituents:

$g_{10} \ldots [5443322210]\sqrt{2}$	$g_{10} \ldots [5\,4\,4\,3\,3\,2\,2\,2\,10]\sqrt{2}$
	$g_9 \ldots [1],[4\,4\,3\,3\,2\,2\,2\,10]\sqrt{2}$
$g_8 \ldots [43322210]\sqrt{2}\,,\,[10]\sqrt{2}$	$g_8 \ldots [1'1],[4\,3\,3\,2\,2\,2\,10]\sqrt{2}$
$g_7 \ldots [3322210]\sqrt{2}\,,\,[110]\sqrt{2}$	$g_7 \ldots [1'1'1],[3\,3\,2\,2\,2\,10]\sqrt{2}$
$g_6 \ldots [322210]\sqrt{2}\,,\,[2110]\sqrt{2}$	$g_6 \ldots [2'1'1'1],[3\,2\,2\,2\,10]\sqrt{2}$
$g_5 \ldots [22210]\sqrt{2}\,,\,[22110]\sqrt{2}$	$g_5 \ldots [2'2'1'1'1],[2\,2\,2\,10]\sqrt{2}$
$g_4 \ldots [2210]\sqrt{2}\,,\,[322110]\sqrt{2}$	$g_4 \ldots [3'2'2'1'1'1],[2\,2\,10]\sqrt{2}$
$g_3 \ldots [210]\sqrt{2}\,,\,[3322110]\sqrt{2}$	$g_3 \ldots [3'3'2'2'1'1'1],[2\,10]\sqrt{2}$
$g_2 \ldots [10]\sqrt{2}\,,\,[33322110]\sqrt{2}$	$g_2 \ldots [3'3'3'2'2'1'1'1],[10]\sqrt{2}$
$g_0 \ldots [5433322110]\sqrt{2}$	$g_0 \ldots [5'4'3'3'3'2'2'1'1'1]$

The nets of measure polytope extraction of the spaces S_3, S_4, S_5 are put on record in the Tables V and VI. The first column of these tables is concerned with the "name" of the net; it contains the system of operators e_k and c which are to precede the general symbol $N(M_n^2)$ in order to obtain the symbol of the net. This system of operators is in close connection with the consideration of the net of S_n as a simple polytope of S_{n+1}; for $a = a_1$ it is equal to the system of operators characterizing the groundform, for $a = a_1 + 1$ it consists of latter system completed by e_n. So of the three parts into which each of the three cases $n = 3$, $n = 4$, $n = 5$ has been subdivided, the first contains the nets (e, c), the second the nets (e, e), the third the nets (c, c). Therefore the question rises where the nets (c, e) are to be found.

The algorithm indicated in our last theorem immediately shows

that by interchanging the two extreme forms with one another the intermediate constituents return in inverted order of succession. This remark suggests an answer to the question raised just now. By taking the constituents $g_n, g_{n-1}, \ldots, g_1, g_0$ contained in the second, third, ..., $n+1^{st}$, $n+2^{nd}$ column of the same horizontal line corresponding to a certain net in reversed order of succession we get the constituents $g'_n, g'_{n-1}, \ldots, g'_1, g'_0$ of a net bearing in general an other name, the operators occurring in which are inscribed in the $n+3^{rd}$ column; this net with constituents with complementary import is essentially the same as the original one. So by inverting the order of succession of the imports the three groups (e, c), (e, e), (c, c) pass into (c, e), (e, e), (c, c), in other words the first group furnishes the group (c, e), whilst each of the other groups passes into itself. We have used this fact, to which we shall have to come back in part *F* of this section, in order to simplify the Tables V and VI. So on one hand the nets (c, e) have been omitted totally, whilst on the other the number of lines of the groups (e, e) and (c, c) have been diminished by writing down the nets in a transparent systematical order and omitting at any time the net appearing already in inverted order under the preceding ones.[1])

In the column under the heading *p.* some particularities of the nets have been inscribed. By *r.* we have indicated that the net is regular, by *s. p.* that it is "semiperiodic", i. e. that the two extreme forms are the same which implies the equality of any two constituents with complementary import.

The other columns will be explained later on.

A survey of the results contained in the tables suggests the following remarks:

a). There is a great difference in character between the constituents of a simplex net proper on one hand and those of a measure polytope net. All the constituents of a simplex net proper are expansion and contraction forms of the simplex, whilst we found just now that in a measure polytope net in general only two of the constituents, the groundform and the opposite form, are expansion and extraction forms of the measure polytope.[2])

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[1]) The cases $ce_2\,N(C_8)$, $ce_3\,N(C_8)$, etc. do not figure in the first third part of Table II contained in the memoir of Mrs. STOTT, as they appear already as expansion forms under either $N(C_{16})$ or $N(C_{24})$.

In order to spare room we have omitted in Table VI the column containing the name of the net taken in inversed order. For the upper and middle part it is always the symbol before M_5 under g_0 to which e_5 has been added, for the last part it is that symbol itself.

[2]) Compare for the prisms and prismotopes entering here my paper: "On the characteristic numbers of the prismotope", *Proceedings* of Amsterdam, vol. XIV, p. 424.

This difference in character implies a difference in the number of different positions a constituent of definite form may admit. In the case of a simplex net proper this number is *two* in general and only *one* if the form is central symmetric. In the case of a measure polytope net this number is *one* for the two extreme constituents, whilst the intermediate form I_k generally occurs in a number of different positions indicated by half the number of limits $M_k^{(2)}$ of $M_n^{(2)}$, i. e. in $2^{n-k-1}(n)_k$ different positions.

In the case of the simplex net we have considered as kind of constituent any polytope of the net with *equipollent* repetitions; when the partition cycle was a power cycle we have even been obliged to split up a *kind* of constituent into several *groups*, in order to keep the analytical treatment in contact with the geometrical facts. On account of the extreme transparency of the measure polytope nets we can allow ourselves to be less exacting and extend the notion of constituent here by admitting that the $2^{n-k-1}(n)_k$ different positions of the intermediate form I_k introduced above belong to the *same* constituent.

b). In order to be able to indicate the number of different constituents according to the new point of view we fall back on the different cases (e, c), (e, e), (c, c), (c, e) mentioned at the end of the last theorem. By generalizing the results of the two examples given above one finds immediately that the required number is in general $n - p + 1$, where p indicates the number of e's contained in the symbol. But this general number $n - p + 1$ is still to be considered as a maximum, i. e. under circumstances the number of constituents may become less. This decrease can be due to two different causes. If in the first place in one of the two groups (e, c), (c, c) of a net in S_n the expansion operator with the largest index is e_k, where $k < n - 1$, the constituents $g_k, g_{k+1}, \ldots, g_{n-2}$ are lacking together with g_{n-1}. If in the second place in one of the two groups (e, e), (c, c) a net is semiperiodic the equal constituents of complementary import may count for one constituent.

c). *Some* of the intermediate constituents may become measure polytopes, this being even the case with *all* the intermediate constituents of the net $e_n N(M_n)$. So by extending the notion of constituent still more the number of the different kinds of constituent is lessened in these cases, this number being unity for the net $e_n N(M_n)$.

d). By comparing the cases g_2 under $n = 4$ we remark that the prismotope $(4; 4)$ which is the measure polytope C_8 of S_4 is indicated by three different symbols; in the cases of the nets (e, c), of the nets (e, e), of the nets (c, c) we get successively:

$$[11]\,.\,[10]\sqrt{2},\quad [11]\,.\,[11],\quad [10]\sqrt{2}\,.\,[10]\sqrt{2}$$

corresponding (fig. 15) to the projections

$$\left.\begin{matrix} A\,B\,C\,D \\ E\,F\,G\,H \end{matrix}\right\},\qquad \left.\begin{matrix} A\,B\,C\,D \\ A\,B\,C\,D \end{matrix}\right\},\qquad \left.\begin{matrix} E\,F\,G\,H \\ E\,F\,G\,H \end{matrix}\right\}$$

on the planes $O\,X_1\,X_2$ and $O\,X_3\,X_4$, if in these symbols the successive digits refer to x_1, x_2, x_3, x_4. Of these the second, equal to $[11\;11]$ occurs in one position only, whilst the two others admit respectively six and three positions in accordance with the splitting up of x_1, x_2, x_3, x_4 in $(x_1, x_2), (x_3\, x_4)$, in $(x_1, x_3), (x_2, x_4)$, in $(x_1, x_4), (x_2, x_3)$.

F. *Polarity.*

65. If we polarize an expansion or a contraction form derived from the measure polytope $M_n^{(2)}$ of S_n with respect to a concentric spherical space (with ∞^{n-1} points) as polarisator we get a new polytope admitting one kind of limit $(l)_{n-1}$ and equal dispacial angles [1]), to which corresponds the inverted symbol of characteristic numbers of the original polytope. Moreover, if $[a_1, a_2, \ldots, a_{n-1}, a_n]$ is the coordinate symbol of the original polytope, this symbol represents also the limiting spaces S_{n-1} of the new polytope in space coordinates.

For the manner in which the process of truncation is transformed by inversion compare page 69 of Section I.

66. We now pass to:

Theorem XL. "Any polytope $(P)_n$ of measure polytope descent in S_n has the property that the vertices V_i adjacent to any arbitrary vertex V lie in the same space S_{n-1} normal to the line joining

---

[1]) Compare for this inversion page 68 of Section I.

By inversion of the measure polytope we find the cross polytope. Moreover we find in S_4, in the notation of the foot note of page 68, if $L\,e_1\,e_2\,e_3$ stands *now* for the "limiting bodies of the reciprocal polytope of $e_1\,e_2\,e_3\,C_8$",

$Le_1 = 64\ T(1_3, 3_{2+1})$,	$Lce_1 = 32\ P_3^2$,
$Le_2 = 96\ P^2{}_{2+1}$,	$Lce_2 = LC_{24} = 24\ O$,
$Le_3 = 64\ X$,	$Lce_3 = LC_{16} = 8\ C$,
$Le_1\,e_2 = 192\ T(1_{2+1}, 1_{2+1}, 2_{1+1+1})$,	$Lce_1\,e_2 = 96\ T(2_{2+1}, 2_{2+1})$,
$Le_1\,e_3 = 192$ symm. $P^1{}_{\text{deltoid}}$,	$Lce_1\,e_3 = 96\ P_3^2$,
$Le_2\,e_3 = 192$ symm. $P^1{}_{\text{deltoid}}$,	$Lce_2\,e_3 = 48\ P_4^1{}_{\text{(square)}}$,
$Le_1\,e_2\,e_3 = 384\ Y$,	$Lce_1\,e_2\,e_3 = 192\ T(1_3, 3_{2+1})$,

X representing a polyhedron limited by six faces, two groups of three equal deltoids connected in such a way as to give rise to an axis of period 3, and Y a tetrahedron limited by four unequal scalene triangles. For the shape of the tetrahedra Y compare problem 79 of vol. XI of the "Wiskundige Opgaven", where the projections of these tetrahedra on the four sets of axes of the polytope are given into the bargain.

this vertex V to the centre O of the polytope. The system of the spaces S_{n-1} corresponding in this way to the different vertices V of $(P)_n$ include an other polytope $(P)'_n$, the reciprocal polar of $(P)_n$ with respect to a certain concentric spherical space, unless $(P)_n$ be the cross polytope $ce_{n-1}\, M_n$ in which special case all the spaces S_{n-1} pass through the centre O."

After the first section of this memoir had been published we perceived that the analytical proof of the corresponding theorem XXII might have been replaced by a much simpler geometrical one [1]), applicable to any polytope $(P)_n$ deduced from a regular polytope, whether simplex or not, by the operations e_k and c.

This simple geometrical proof runs as follows:

All the vertices V_i adjacent to V lie on two spherical spaces (with ∞^{n-1} points), the circumscribed one with centre O and an other with centre V and radius VV_i equal to the edge. So they lie in the spherical space (with ∞^{n-2} points) common to these two spherical spaces and therefore in the space S_{n-1} normal to VO containing this intersection. If this S_{n-1} cuts VO in P we have $2\, VP.\, VO = \overline{VV_i}^2$ from which it ensues that the distance PO is the same for all the vertices V, i. e. that the spaces S_{n-1} are the polar spaces of the vertices V with respect to a definite spherical space (with ∞^{n-1} points) round O as centre.

Moreover the special case of the cross polytope, where P coincides with O, is self evident.

67. In the section concerned with the simplex we have explained by the laws of reciprocity why it may happen that two different groups of operations of expansion applied to the simplex produce under circumstances either two polytopes equal and concentric but of opposite orientation, or the same polytope. What corresponds to this here is that any polytope derived from M_n can also be derived from the cross polytope C_{2^n} of S_n which is the reciprocal polar of M_n. As we had already occasion to remark in art. 48 we shall have to come back to this assertion in the third section.

But the state of affairs with respect to equal measure polytope nets with different expansion symbols is a quite different one. In a joint paper of M^rs^. STOTT and myself published two years ago [2]) it is shown geometrically that we have in general the relations:

[1]) To some of the free copies at my disposal I added a post-scriptum, containing this remark, on page 69.

[2]) Compare the second foot note of art. 38 of Section I.

$$EN = cE'e_n N', \quad Ee_n N = E'e_n N', \quad cEN = cE'N',$$

where N and N' represent polarly related regular nets of S_n, whilst the sets of operations e_k, $(k = 1, 2, \ldots, n-1)$, contained in E and E' are complementary to each other, i.e. that E' contains the operations e_{n-k} complementary to the operations e_k of E and no other one. Now, in the case of the net of measure polytopes we have $N' = N$; so we get:

Theorem XLI. "We have the relations:

$$\begin{aligned}
e_a e_b e_c \ldots e_r e_s e_t \quad NM_n^{(2)} &= ce_{a'} e_{b'} e_{c'} \ldots e_{r'} e_{s'} e_{t'} e_n NM_n^{(2)}, \\
e_a e_b e_c \ldots e_r e_s e_t e_n NM_n^{(2)} &= e_{a'} e_{b'} e_{c'} \ldots e_{r'} e_{s'} e_{t'} e_n NM_n^{(2)}, \\
ce_a e_b e_c \ldots e_r e_s e_t \quad NM_n^{(2)} &= ce_{a'} e_{b'} e_{c'} \ldots e_{r'} e_{s'} e_{t'} \quad NM_n^{(2)},
\end{aligned}$$

under the conditions

$$a + t' = b + s' = c + r' = \ldots = r + c' = s + b' = t + a' = n;$$

then the constituents $g_0, g_1, g_2, \ldots, g_{n-2}, g_{n-1}, g_n$ of the one are equal to the constituents $g'_n, g'_{n-1}, g'_{n-2}, \ldots, g'_2, g'_1, g'_0$ of the other. So the nets $e_a e_b e_c \ldots e_r e_s e_t e_n NM_n^{(2)}$ and $ce_a e_b e_c \ldots e_r e_s e_t NM_n^{(2)}$ are semiperiodic under the conditions

$$a + t = b + s = c + r = \ldots = n.$$

In the latter cases there is an unpaired middle constituent for n even."

Proof. We prove each of the three relations by showing that the extreme constituents g_0, g_n of the net at the left of the sign of equality are equal to the constituents g'_n, g'_0 of the net at the right. But we suppose that it will do to enter into details for one of the three relations, say the second.

In the case of the net $e_a e_b e_c \ldots e_r e_s e_t e_n NM_n^{(2)}$, where as in art. 38 we suppose the indices of the $k+1$ factors $e_a, e_b, \ldots e_n$ to be arranged according to increasing values of the subscripts, the principal constituent g_n is, according to theorem XXXV:

$$[\overbrace{k', k', \ldots k'}^{n-t}, \overbrace{(k-1)', (k-1)', \ldots (k-1)'}^{t-s}, \overbrace{(k-2)', (k-2)', \ldots (k-2)'}^{s-r}, \ldots,$$
$$\overbrace{2', 2', \ldots 2'}^{c-b}, \overbrace{1', 1', \ldots 1'}^{b-a}, \overbrace{1, 1, \ldots 1}^{a}].$$

So we find according to theorem XXXIX for g_0 by subtraction from $k'+1$:

$$[\overbrace{k', k', \ldots k'}^{a}, \overbrace{(k-1)', (k-1)', \ldots (k-1)'}^{b-a}, \overbrace{(k-2)', (k-2)', \ldots (k-2)'}^{c-}, \ldots,$$
$$\overbrace{2', 2', \ldots 2'}^{s-r}, \overbrace{1', 1', \ldots 1'}^{t-s}, \overbrace{1, 1, \ldots 1}^{n-t}].$$

Likewise we get for the constituents g'_n and g'_0 of the second net represented by $e_{a'} e_{b'} e_{c'} \ldots e_{r'} e_{s'} e_{t'} e_n N M_n^{(2)}$ the same expressions in which the $a, b, c, \ldots r, s, t$ are dashed. From this it ensues that we shall have at the same time $g'_n = g'_0$ and $g'_0 = g_n$ under the conditions

$$a = n - t',\ b - a = t' - s',\ c - b = s' - r', \ldots$$
$$s - r = c' - b',\ t - s = b' - a',\ n - t = a',$$

giving immediately

$$a + t' = b + s' = c + r' = \ldots = r + c' = s + b' = t + a' = n.$$

These conditions pass into

$$a + t = b + s = c + r = \ldots = n,$$

if the two nets coincide in a semiperiodic one [1]).

Remark. If we count as one the two nets which pass into each other by interchanging the two extreme forms (and also the two nets N and $e_n N$ of measure polytopes only) the number of measure polytope nets is $8 + 2.5 = 18$ in S_4, $16 + 2.9 = 34$ in S_5, $32 + 2.19 = 70$ in S_6, $64 + 2.35 = 134$ in S_7, $128 + 2.71 = 270$ in S_8, etc.

68. The circumstances under which polarization of a measure polytope net leads to an other measure polytope net are easily indicated. For, though in the case of a net belonging to the family (e, e) the centres of all the constituents are the groups of centres of the different limits $(l)_0, (l)_1, (l)_2, \ldots, (l)_{n-1}, (l)_n$ of the net $N(M_n^{2m})$, m being the extension number, and these points form together the vertices of a net $N(M_n^m)$, it is only $N(M_n^2)$ itself which satisfies the condition that an $M_n^{(2)}$ the vertices of which are centres of the $M_n^{(2)}$ of the net includes only one vertex of this net. So, if we discard the case $ce_2 N(M_4) = N(C_{24})$, the net $N(M_n)$ and the one deduced from it by polarization form together the only pair of *two reciprocal nets* of measure polytope descent.

In general the system of vertices of a net obtained by polarizing a measure polytope net is the combination of several groups of centres of limits $M_k^{(2m)}$ of the measure polytopes of the net $N(M_n^{2m})$, m being the extension number. So we find in S_3:

[1]) In the case of the first relation, where we do not obtain the second member by dashing the subscripts $a, b, c, \ldots, r, s, t$ of the first, the proof is a bit more complicated. Here we find for g_n the expression given above, but for g_0 — as we have to subtract from k_1 instead of $k_1 + 1$ —

$$[\ \overbrace{k, k, \ldots k}^{a},\ \overbrace{k-1, k-1, \ldots k-1}^{b-a},\ \overbrace{k-2, k-2, \ldots k-2}^{c-b},\ \ldots,\ \overbrace{2, 2, \ldots 2}^{s-r},\ \overbrace{1, 1, \ldots 1}^{t-s},\ \overbrace{0, 0, \ldots 0}^{n-t}\ ]\sqrt{2},$$

etc.

in the case of	N	centres	of	limits	M_3,
,,	$e_1 N, c e_1 N, c e_1 e_2 N$	,,	,,	,,	M_3, M_0,
,,	$e_2 N, e_1 e_2 N$	,,	,,	,,	M_3, M_1, M_0,

whilst — as we remarked above — in the cases where e_3 occurs all the groups of centres contribute to the system of vertices. In the case of the groups M_3, M_0 a space filling double pyramid on a square base may be considered as the constituent of the reciprocal net, in the case of the three groups M_3, M_1, M_0 we are obliged to consider as constituent a polyhedron (5, 9, 6) which may be got by dividing the double pyramid mentioned into four equal parts by bisecting the pairs of parallel sides of the square base [1]).

G. *Symmetry, considerations of the theory of groups, regularity.*

69. We determine the spaces of symmetry Sy_{n-1} and consider successively the case of the measure polytope M_n of S_n and that of any polytope $(P)_m$ deduced from M_n by the operations of expansion and contraction.

Case of the measure polytope. Let us suppose Sy_{n-1} is a definite space of symmetry of M_n and let A_1 be a vertex of M_n not contained in Sy_{n-1}. Then the mirror image of A_1 with respect to Sy_{n-1} is an other vertex A_2 of M_n, which implies that $A_1 A_2$ is either an edge or a central diagonal of a certain limit M_k of M_n where k may be — the case of the edge included — one of the numbers $1, 2, \ldots, n-1$. Let S_k be the space containing that M_k. Then any edge $A_1 A'$ through A_1 of M_n not belonging to M_k is normal in A_1 to S_k and therefore to $A_1 A_2$; so these $n-k$ edges $A_1 A'$ are parallel to Sy_{n-1} and M_n can be generated by prismatizing M_k in these directions, i.e. Sy_{n-1} *is* a space of symmetry of M_n, if and only if its section S_{k-1} with S_k is a space of symmetry of M_k, which condition is fulfilled in the cases $k=1$, $k=2$ only. For in all the remaining cases $k=3, 4, \ldots, n-1$ (and also for $k=n$) the two simplexes $S(k)$ the vertices of which are the groups of vertices of M_k adjacent to A_1 and to A_2 are equal but of opposite orientation, which proves that the space S_{k-1} of S_k normally bisecting $A_1 A_2$ is no space of symmetry of M_k.

For $k=1$ the line $A_1 A_2$ is an edge, for $k=2$ it is a diagonal of a face. So the two groups of spaces Sy_{n-1} are the n spaces $x_i = 0$ and the $n(n-1)$ spaces $x_i \pm x_k = 0$; so the number of spaces Sy_{n-1} is n^2.

[1]) We defer further developments about reciprocal nets to an other paper also destined to complement art. 39; compare "Nieuw Archief voor Wiskunde", vol. X, p. 273.

Case of the polytope $(P)_m$ *deduced from the measure polytope.* The n^2 spaces Sy_{n-1} found above are spaces of symmetry for $(P)_m$; so here again the only question is if $(P)_m$ can admit a space of symmetry Sy_{n-1} which is no Sy_{n-1} for the M_n from which $(P)_m$ has been derived. We suppose that there is such an Sy_{n-1}, represent by M'_n the mirror image with respect to that Sy_{n-1} of the M_n from which $(P)_m$ has been derived by a set of e_k and c operations, and remark now that — as Sy_{n-1} is space of symmetry for the figure consisting of $(P)_m$ and the two measure polytopes M_n, M'_n — it must be possible to derive $(P)_m$ from M'_n by the same set of operations. This particularity presents itself in the case of the octagon $e_1(p_4)$ only, as the p_4 itself may be represented either as $[1, 1]$ or as $[1, 0]\sqrt{2}$. So we find:

THEOREM XLII. "The measure polytope $[\overbrace{1\ 1 \ldots 1}^{n}]$ of S_n and the polytopes deduced from it by expansion and contraction admit n^2 spaces Sy_{n-1} of symmetry, the n spaces $x_i = 0$ and the $n(n-1)$ spaces $x_i \pm x_k = 0$. Only in the case of the plane we have to add for $e_1(p_4)$ the four new axes of symmetry passing through pairs of opposite vertices of the octagon".

70. Moreover we find: [1])

THEOREM XLIII. "The order of the group of anallagmatic displacements of the measure polytope M_n of S_n and the polytopes deduced from it by expansion and contraction is $2^{n-1}.\ n!$"

"The order of the extended group of anallagmatic displacements of these polytopes, reflexions with respect to spaces Sy_{n-1} of symmetry included, is $2^n.\ n!$ In this extended group the first group of order $2^{n-1}.\ n!$ forms a perfect subgroup".

For $n=2$ these general results have to be completed in the known way for the octagon."

For the simple proof we refer to the article quoted.

71. Finally we have to apply to the polytopes and nets of measure descent the scale of regularity due to M^r^. ELTE. As to the theory we can only repeat here what has been remarked in the art^s^. 42 and 43, with omission of all that refers to the central symmetry of some of the polytopes of simplex extraction. So theorem XXV must take here the simpler form:

THEOREM XLIV. "Any two limiting elements $(l)_d$ belong to the same subgroup or to different subgroups, in the sense of the scale

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[1]) Compare "Report of the British Association", 1894, p. 563.

of regularity, according as their symbols of coordinates are equal or different."

As the application of ELTE's scale[1]) to polytopes and nets of measure descent is rather easy it may suffice to give some examples, both of polytopes and nets.

a). *Example* $[\mathbf{3'\,3'\,2'\,1'\,1}]$. Here we find four different groups of edges $(3', 2')$, $(2', 1')$, $(1', 1)\,(1, -1)$. So the contributions to the numerator are 1 from the vertices and $\frac{1}{2}$ from the edges and the fraction is $\dfrac{1+\frac{1}{2}}{5} = \dfrac{3}{10}$, the minimum value in S_5.

b). *Example* $[\mathbf{3\,3\,2\,1\,0}]\sqrt{\mathbf{2}}$. Here three groups of edges appear, viz. $(3, 2)\sqrt{2}$, $(2, 1)\sqrt{2}$, $(1, 0)\sqrt{2}$. So we find once more $\dfrac{3}{10}$.

c). *Example* $[\mathbf{1\,1\,0\,0\,0}]\sqrt{\mathbf{2}}$. Only one kind of edge, viz. $(1, 0)\sqrt{2}$. So we have to examine the faces. As it is clear that we find triangles $(\sqrt{2}, \sqrt{2}, 0)\;0\;0$ and squares $[\sqrt{2}, \sqrt{2}]\;0\;0\;0$, the degree of regularity is $\dfrac{4}{10} = \dfrac{2}{5}$.

d). *Example* $\boldsymbol{e}_4\,\boldsymbol{N}(\boldsymbol{M}_5{}^{(2)})$. The groundform $[1'\,1\,1\,1\,1]$ admits two kinds of edges $(1'\,1)\,1\,1\,1$ and $1'\,1\,1\,1\,[1]$ of a different character. So we find $\dfrac{1\frac{1}{2}}{6} = \dfrac{1}{4}$.

e). *Example* $\boldsymbol{ce}_1\,\boldsymbol{e}_2\,\boldsymbol{e}_3\,\boldsymbol{e}_4\,\boldsymbol{N}(\boldsymbol{M}_5{}^{(2)})$. Here we have to deal with four groups of constituents represented wich their frames in the table

$$\begin{array}{lllllll}
g_5 . [43210] \ldots\ldots & (2p_1 & , 2p_2 & , 2p_3 & , 2p_4 & , 2p_5 &)\,4 \\
g_3 \ldots [210][10] \ldots & (2p_1 & , 2p_2 & , 2p_3 & , 2p_4+1 & , 2p_5+1 &)\,4 \\
g_2 \ldots [10][210] \ldots & (2p_1 & , 2p_2 & , 2p_3+1 & , 2p_4+1 & , 2p_5+1 &)\,4 \\
g_0 \ldots\ldots [43210]. & (2p_1+1 & , 2p_2+1 & , 2p_3+1 & , 2p_4+1 & , 2p_5+1 &)\,4
\end{array}$$

So through the vertex 4, 3, 2, 1, 0 pass

$$\left.\begin{array}{llllll}
[\;\;\mathbf{4}, & \mathbf{3}, & \mathbf{2}, & \mathbf{1}, & \mathbf{0}] \ldots\ldots A_1 \\
[8+\mathbf{4}, & \mathbf{3}, & \mathbf{2}, & \mathbf{1}, & \mathbf{0}] \ldots\ldots A_2 \\
[4+\mathbf{0}, & 4+\mathbf{1}] & [\mathbf{2}, & \mathbf{1}, & \mathbf{0}] \ldots\ldots B \\
[4+\mathbf{0}, & 4+\mathbf{1}, & 4+\mathbf{2}] & [\mathbf{1}, & \mathbf{0}] \ldots\ldots C \\
[4+\mathbf{0}, & 4+\mathbf{1}, & 4+\mathbf{2}, & 4+\mathbf{3}, & 4+\mathbf{4}] \ldots\ldots D_1 \\
[4+\mathbf{0}, & 4+\mathbf{1}, & 4+\mathbf{2}, & 4+\mathbf{3}, & -4+\mathbf{4}] \ldots\ldots D_2
\end{array}\right\},$$

i. e. six polytopes and more in detail four cells $[4\,3\,2\,1\,0]$ and two prismotopes $[2\,1\,0][1\,0]$. Now the edges $(4\,3)\,2\,1\,0$ and $4\,3\,2\,(1\,0)$ belong to both the prismotopes, whilst each of the edges $4\,(3\,2)\,1\,0$

[1]) We stick here to the original scale (compare *Proceedings* of Amsterdam, vol. XV, p. 200).

and 4 3 (2 1) 0 belongs to only one. So there are two different kinds of edges and we find $\frac{2}{6} = \frac{1}{3}$.

Remark. In S_n the degree of regularity is a minimum, i. e. $\frac{3}{2n}$ for a polytope and $\frac{3}{2(n+1)}$ for a net,

1°. if the symbol of the polytope or that of the groundform of the net contains no zero,

2°. if the net admits a constituent g_{n-1}.

For in both cases there are at least two kinds of edges: in the first case the edges [1], in the second case the erect edges of the prisms g_{n-1} differ in character from the remaining ones.

The results about regularity have been indicated in the Tables IV, V, VI. In Table IV the regularity fraction is contained in column 5, whilst the subscripts in column 4 give the different groups of limits $(l)_n$. In Tables V and VI in the cases $n = 4$ and $n = 5$ the last column contains the regularity fraction, the last but one [1]) the different groups of limits $(l)_k$, whilst the part $n = 3$ of Table V contains two columns more, one indicating the number of the ANDREINI diagram of the net, the other indicating the particularities of the edges passing through a vertex (see ANDREINI's list, page 30—32 of the memoir quoted in art. 22).

Section III: POLYTOPES AND NETS DERIVED FROM THE CROSS POLYTOPE.

A. *The symbol of coordinates.*

72. In this section which is so closely related to the immediately preceding one that it may be considered as a mere supplement of the latter we have to start from the cross polytope $C_{2n}^{(2)}$ of S_n represented by the symbol $[1\overbrace{00\ldots0}^{n-2}]\sqrt{2}$ and to remember that we are to prove by and by that there is no difference whatever between the offspring of this cross polytope and that of the measure polytope $[\overbrace{11\ldots1}^{n}]$ of S_n.

For $n = 3, 4, 5$ we have successively in the symbols of M^rs^. STOTT: [2])

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[1]) The numbers of the different groups of limits $(l)_k$ for $k > 1$ have been found in the manner indicated for the simplex in Table III, but we have judged it of no importance to insert an analogous table for the measure polytope.

[2]) For the deduction of the e and c symbols from the symbols of coordinates compare part D of this section.

In Table IV second column are inscribed the e and c symbols of the polytopes deduced from the cross polytope corresponding to the symbols of coordinates of the third column.

$n = 3.$

$[100]\sqrt{2} = O$	$[1'1\,1] = e_2\,O = RCO$		$[1\,1\,1] = ce_2\,O =$
$[210]\sqrt{2} = e_1\,O = tO$	$[2'1'1] = e_1 e_2\,O = tCO$	$[110]\sqrt{2} = ce_1\,O = CO$	$[1'1'1] = ce_1 e_2\,O =$

$n = 4.$

$[1\,000]\sqrt{2} = C_{16}$	$[3\,2\,1\,0]\sqrt{2} = e_1 e_2 C_{16}$		$[2\,2\,1\,0]\sqrt{2} = ce_1 e_2 C_1$
$[2\,100]\sqrt{2} = e_1 C_{16}$	$[2'1'1\,1] = e_1 e_3 C_{16}$	$[1100]\sqrt{2} = ce_1 C_{16} = C_{24}$	$[1'1'1\,1] = ce_1 e_3 C_1$
$[2\,110]\sqrt{2} = e_2 C_{16}$	$[2'1'1'1] = e_2 e_3 C_{16}$	$[1110]\sqrt{2} = ce_2 C_{16}$	$[1'1'1'1] = ce_2 e_3 C_1$
$[1'111] = e_3 C_{16}$	$[3'2'1'1] = e_1 e_2 e_3 C_{16}$	$[1111] = ce_3 C_{16} = C_8$	$[2'2'1'1] = ce_1 e_2 e_3 C_1$

$n = 5.$

$[1\,0\,000]\sqrt{2} = C_{32}$	$[3\,2\,2\,1\,0]\sqrt{2} = e_2 e_3 C_{32}$		$[2\,2\,2\,1\,0]\sqrt{2} = ce_2 e_3 C_3$
$[2\,1\,000]\sqrt{2} = e_1 C_{32}$	$[2'1'1'1\,1] = e_2 e_4 C_{32}$	$[1\,1\,000]\sqrt{2} = ce_1 C_{32}$	$[1'1'1'1\,1] = ce_2 e_4 C_3$
$[2\,1\,100]\sqrt{2} = e_2 C_{32}$	$[2'1'1'1'1] = e_3 e_4 C_{32}$	$[1\,1\,100]\sqrt{2} = ce_2 C_{32}$	$[1'1'1'1'1] = ce_3 e_4 C_3$
$[2\,1\,110]\sqrt{2} = e_3 C_{32}$	$[4\,3\,2\,1\,0]\sqrt{2} = e_1 e_2 e_3 C_{32}$	$[1\,1\,110]\sqrt{2} = ce_3 C_{32}$	$[3\,3\,2\,1\,0]\sqrt{2} = ce_1 e_2 e_3 C_3$
$[1'1\,111] = e_4 C_{32}$	$[3'2'1'1\,1] = e_1 e_2 e_4 C_{32}$	$[1\,1\,111] = ce_4 C_{32}$	$[2'2'1'1\,1] = ce_1 e_2 e_4 C_3$
$[3\,2\,100]\sqrt{2} = e_1 e_2 C_{32}$	$[3'2'1'1'1] = e_1 e_3 e_4 C_{32}$	$[2\,2\,100]\sqrt{2} = ce_1 e_2 C_{32}$	$[2'2'1'1'1] = ce_1 e_3 e_4 C_3$
$[3\,2\,110]\sqrt{2} = e_1 e_3 C_{32}$	$[3'2'2'1'1] = e_2 e_3 e_4 C_{32}$	$[2\,2\,110]\sqrt{2} = ce_1 e_3 C_{32}$	$[2'2'2'1'1] = ce_2 e_3 e_4 C_3$
$[2'1'111] = e_1 e_4 C_{32}$	$[4'3'2'1'1] = e_1 e_2 e_3 e_4 C_{32}$	$[1'1'111] = ce_1 e_4 C_{32}$	$[3'3'2'1'1] = ce_1 e_2 e_3 e_4 C_3$

B. *The characteristic numbers.*

73. From the preceding section concerned with the measure polytope can be gathered the symbols with the characteristic numbers of the polytopes deduced from the cross polytope, the symbols of coordinates of which wind up in a unit, as these polytopes also belong to the offspring proper of the measure polytope. So we have only to add a couple of examples about polytopes, the symbols of coordinates of which end in zero.

Example **[2 1 1 0]**, method working from two sides [1]).

The number of vertices is $2^3 . 4!$ divided by $2!$, i. e. $8 . 24 : 2 = 96$.

The number of the edges passing through the pattern vertex is six, for this vertex is united by edges to the vertices:

$$\begin{array}{lll} 1\,2\,1\,0, & 2\,0\,1\,1, & 2\,0\,1-1, \\ 1\,1\,2\,0, & 2\,1\,0\,1, & 2\,1\,0-1. \end{array}$$

So the number of edges is $\dfrac{96 . 6}{2} = 288$.

In order to find spaces containing limiting bodies we consider successively the equations:

$$\pm x_1 = 2, \quad \pm x_1 \pm x_2 = 3, \quad \pm x_1 \pm x_2 \pm x_3 \pm x_4 = 4.$$

The equations $\pm x_i = 2$ give 8 forms $[1\,1\,0]$, i. e. 8 $C\,O$ of vertex import.

[1]) In the two examples we omit the common factor $\sqrt{2}$.

The equations $\pm x_i \pm x_j = 3$ give 24 forms (2 1) [1 0], i. e. 24 P_4 of edge import.

The equations $\Sigma \pm x_i = 4$ give 16 forms (2 1 1 0), i. e. 16 CO of body import.

So, we find 24 CO and 24 C, i. e. 48 polyhedra, and therefore $\frac{1}{2}(24 \times 14 + 24 \times 6) = 240$ faces.

So the result is (96, 288, 240, 48) in accordance with the law of Euler.

Example **[3 2 1 1 0]**, direct method.

The number of vertices is $2^4 . 5!$ divided by $2!$, i. e. $1920 : 2 = 960$.

The edges split up into three groups (32), (21), (10). Through the pattern vertex pass: *one* edge (32), *two* edges (21) — on account of the two digits 1 — and *four* edges (10) — on account of the two digits 1 and of the faculty to make the last digit to correspond either to $+x_5$ or to $-x_5$.

So there are in toto

480 edges (32), 960 edges (21), 1920 edges (10),

i. e. 3360 edges.

The faces split up into six groups, viz. the triangles (211) and (110), the squares (32) (10), (21) (10) and [10] and the hexagon (321).

In the pattern vertex concur:

one triangle (211),
two triangles (110), on account of $\pm x_5$,
four squares (32) (10), on account of the two digits 1 and of $\pm x_5$,
" " (21) (10), " " " " " " " " " " " ",
two " [10], " " " " " " ",
two hexagons (321), " " " " " " "[1].

So we find:

$$960 \left(\frac{3 \text{ triangles}}{3} + \frac{10 \text{ squares}}{4} + \frac{2 \text{ hexagons}}{6} \right)$$
$$= 960 \text{ triangles} + 2400 \text{ squares} + 320 \text{ hexagons},$$

i. e. 3680 faces.

The limiting bodies split up into the seven groups:

$$(3211) = tT, (321)(10) = P_6, (32)(110) = P_3, (2110) = CO,$$
$$(32)[10] = (21)[10] = P_4, [110] = CO.$$

[1]) In the case [10] the difference between $+x_5$ and $-x_5$ has no effect, on account of the square brackets.

Of these seven polyhedra concur, on account of the reasons given above, in the pattern vertex in the indicated order:

$$1\ tT, \quad 4\ P_6, \quad 2\ P_3, \quad 2\ CO,$$
$$2\ P_4, \quad 2\ P_4, \quad 1\ CO.$$

So we find:

$$960\left(\frac{tT}{12}+\frac{3\ CO}{12}+\frac{4\ P_6}{12}+\frac{4\ P_4}{8}+\frac{2\ P_3}{6}\right)$$
$$= 80\ tT + 240\ CO + 320\ P_6 + 480\ P_4 + 320\ P_3,$$

i. e. 1440 limiting polyhedra.

Finally the limiting polytopes split up into four groups:

(32110), (321)[10], (32)[110], [2110]

and so we find:

$$32\ e_1 e_3 S(5),\ 80\ (6\,;4),\ 40\ P_{CO},\ 10\ c\, e_1 e_3 C_8,$$

i. e. 162 limiting polyhedra.

So the result is (960, 3360, 3680, 1440, 162) in accordance with the law of Euler.

With respect to the import we have still to add that we pass to the complementary import, if a polytope of the measure polytope family is regarded as a polytope of cross polytope descent. So in the first of the two examples where the cross polytope import has been indicated the result is complementary to that registered in Table IV read from left to right.

C. *Extension number and truncation integers and fractions.*

74. THEOREM XLV. "The new polytopes, all with edges of length unity, can be found by means of a regular extension of the cross polytope followed by a regular truncation, either at the vertices alone, or at the vertices and the edges, or at the vertices, edges and faces, etc."

For the proof we refer to the arts. 15 and 56.

Here the limit $(l)_{n-1}$ of the highest import, i. e. g_{n-1}, corresponds to the equation $x_1 + x_2 + \ldots + x_n =$ constant. So the extension number is the sum of the digits of the new polytope divided by the sum of the digits of the cross polytope, i. e. by $\sqrt{2}$. So the extension number of [3′ 3′ 2′ 1′ 1] is $5 + 9\sqrt{2}$ divided by $\sqrt{2}$, i. e. $9 + \frac{5}{2}\sqrt{2}$.

We can stick here to the method of measuring the amount of the different truncations on the edges. But we must point out a

difficulty underlying this method. So, in the case of truncation of an octahedron (fig. 16) at the edge BC, it makes a difference whether we choose BA or BC' as the edge on which we determine the amount of truncation. For if we move the truncating plane (through BC normal to OM, where M is the midpoint of BC) parallel to itself untill it passes through O it contains the other extremity A of the edge BA, while it bisects the edge BC'. This difficulty can be overcome by stipulating that the edge to be chosen may not contain a vertex opposite to one of the vertices of the limit at which the truncation takes place. But this implies always that we measure quite as well on the line MO joining the centre of that limit to the centre of the polytope. So if the truncating space cuts MO in P the amount of truncation is $\frac{MP}{MO}$. Now the complement $\frac{PO}{MO}$ of this quantity can be deduced immediately from the symbol of coordinates $[a_1, a_2, \ldots, a_n]$ of the cross polytope form considered. If we suppose that the truncation takes place at the limit $(l)_{k-1}$ of the corresponding extended cross polytope $[1, \overline{0, \ldots, 0}^{\,n-1}]\sum\limits_1^n a_i$ lying in the space represented by $x_1 + x_2 + \ldots + x_k =$ constant it is immediately evident that $\frac{PO}{MO}$ is equal to the quotient of the sum $\sum\limits_1^k a_i$ of the first k digits of the symbol of the truncated polytope by the corresponding sum of the extended cross polytope, i. e. by $\sum\limits_1^n a_i$. So from $\frac{PO}{MO} = \frac{\sum\limits_1^k a_i}{\sum\limits_1^n a_i}$ we deduce:

$$\text{amount of truncation} = \frac{MP}{MO} = \frac{\sum\limits_{k+1}^n a_i}{\sum\limits_1^n a_i}.$$

We illustrate this theory by the example **[3′3′2′1′1]** for which we have determined above the extension number. Here we find moreover

$$\sum_2^5 a_i = 4 + 6\sqrt{2},\ \sum_3^5 a_i = 3 + 3\sqrt{2},\ \sum_4^5 a_i = 2 + \sqrt{2},\ a_5 = 1$$

and therefore

$$\frac{4 + 6\sqrt{2}}{5 + 9\sqrt{2}},\quad \frac{3 + 3\sqrt{2}}{5 + 9\sqrt{2}},\quad \frac{2 + \sqrt{2}}{5 + 9\sqrt{2}},\quad \frac{1}{5 + 9\sqrt{2}}$$

as the amount of truncation at $(l)_0$, $(l)_1$, $(l)_2$, $(l)_3$. As these numbers

$$\frac{2}{137}(44+3\sqrt{2}), \frac{3}{137}(13+4\sqrt{2}), \frac{1}{137}(8+13\sqrt{2}), \frac{1}{137}(9\sqrt{2}-5)$$

are rather impractical, we only put on record in Table IV the results relating to the cross polytope forms proper, where the denominator and the numerator of the fraction $\frac{MP}{MO}$ are both integer multiples of $\sqrt{2}$. Here the result $9 \mid 6, 3, 1$ corresponding to $[3\,3\,2\,1\,0]\sqrt{2}$ expresses that the amount of truncation at $(l)_0$, $(l)_1$, $(l)_2$ is respectively $\frac{2}{3}, \frac{1}{3}, \frac{1}{9}$.

D. *Expansion and contraction symbols.*

75. What we have to prove here is:

THEOREM XLVI „The expansion e_k, $(k=1, 2, 3, .., n-2)$, applied to the cross polytope $C_{2^n}^{(2)}$ of S_n changes the symbol of coordinates $[1\overline{00\ldots0}^{\,n-1}]\sqrt{2}$ of that regular polytope by addition of $\sqrt{2}$ to the first $k+1$ digits into $[2\overline{11..1}^{\,k}\,\overline{00..0}^{\,n-k-1}]\sqrt{2}$, whilst in the case of e_{n-1} where application of this rule would give a symbol without zero we have to add unity instead of $\sqrt{2}$ to all the digits, giving $[1'\overline{11\ldots1}^{\,n-1}]$".

Proof. We treat the cases $k<n-1$ and $k=n-1$ separately.

Case $k<n-1$. The operation e_k acts upon the limits $(l)_k = S(k+1)$ of the cross polytope. Now the centre M of the limit $(l)_k$ represented by

$$x_1, x_2, \ldots, x_{k+1} = (1\overline{00..0}^{\,k})\sqrt{2}, \qquad x_{k+2} = x_{k+3} = \ldots = x_n = 0$$

has the coordinates

$$x_1 = x_2 = \ldots = x_{k+1} = \frac{\sqrt{2}}{k+1}, \qquad x_{k+2} = x_{k+3} = \ldots = x_n = 0.$$

If we move this limit $(l)_k$ parallel to itself in the direction OM to a position $(l)'_k$ for which the centre M' satisfies the relation $OM' = \lambda . OM$, where λ is to be determined, we find for the coordinates of M'

$$x_1 = x_2 = \ldots = x_{k+1} = \frac{\lambda\sqrt{2}}{k+1}, \qquad x_{k+2} = x_{k+3} = \ldots = x_n = 0.$$

So by this motion the coordinates $x_1, x_2, \ldots, x_{k+1}$ of any vertex A of $(l)_k$ increase by $\dfrac{(\lambda-1)\sqrt{2}}{k+1}$, whilst the coordinates $x_{k+2}, x_{k+3}, \ldots x_n$ of this point remain zero. So $(l)'_k$ is represented by

$$x_1, x_2, \ldots, x_{k+1} = \Big(1 + \overbrace{\frac{\lambda-1}{k-1}, \frac{\lambda-1}{k-1}, \ldots, \frac{\lambda-1}{k-1}}^{k}\Big)\sqrt{2},$$
$$x_{k+2} = x_{k+3} = \ldots = x_n = 0,$$

from which it ensues that the symbol of coordinates of the new polytope becomes

$$\Big[1 + \overbrace{\frac{\lambda-1}{k-1}, \frac{\lambda-1}{k-1}, \ldots, \frac{\lambda-1}{k-1}}^{k}, \overbrace{0, \ldots, 0}^{n-k-1}\Big]\sqrt{2}.$$

So the new polytope satisfies the law of the equality of all the edges expressed in theorem XXVIII if, and only if, we have either $\lambda = 1$ or $\lambda = k$. As $\lambda = 1$ corresponds to the cross polytope itself, we have to take $\lambda = k$ in which case we find $[2\ \overbrace{11\ldots1}^{k}\ \overbrace{00\ldots0}^{n-k-1}]$ as the theorem requires it.

Case $k = n - 1$. We consider the limit $(l)_{n-1} = S(n)$ represented by

$$x_1, x_2, \ldots, x_n = (1\ \overbrace{00\ldots0}^{n-1})\sqrt{2}$$

with the centre M, the coordinates of which are

$$x_1 = x_2 = \ldots = x_n = \frac{\sqrt{2}}{n},$$

and move this $(l)_{n-1}$ parallel to itself in the direction OM to a position the centre M' of which is determined by the relation $OM' = \lambda . OM$. Then we find in the way indicated above for the symbol of coordinates of the new polytope

$$\Big[1 + \frac{\lambda-1}{n}, \overbrace{\frac{\lambda-1}{n}, \ldots, \frac{\lambda-1}{n}}^{n-1}\Big]\sqrt{2}.$$

So, if we discard immediately the supposition $\lambda = 1$ leading back to the original cross polytope, the new polytope the symbol of which contains no zero satisfies the law of theorem XXVIII, if — and only if — we have

$$\left(1 + \frac{\lambda-1}{n}\right) : \frac{\lambda-1}{n} = (1+\sqrt{2}) : 1$$

giving $\lambda - 1 = \frac{1}{2} n \sqrt{2}$. So we find the polytope with the symbol

$$[1 + \tfrac{1}{2}\sqrt{2}, \overbrace{\tfrac{1}{2}\sqrt{2}, \ldots, \tfrac{1}{2}\sqrt{2}}^{n-1}]\sqrt{2} = [1 + \sqrt{2}, \overbrace{1, 1, \ldots, 1}^{n-1}]$$

in accordance with the statement of the theorem.

By the way we find:

Theorem XLVII. "In the expansion e_k the limits $S(k+1)$ of $C_{2^n}^{(2)}$ are moved away from the centre to a distance equal to k times the original distance for $k < n-1$ and to a distance equal to $1 + \frac{1}{2}n\sqrt{2}$ times the original distance for $k = n-1$".

This comes true, for this extension corresponds in both cases to that deduced from the sum of the digits of the symbol of coordinates of the new polytope.

As the distance OM was $\sqrt{\dfrac{2}{k+1}}$, it becomes $k\sqrt{\dfrac{2}{k+1}}$ for $k < n-1$ and $\dfrac{n+\sqrt{2}}{\sqrt{n}}$ for $k = n-1$.

76. Theorem XLVIII. "The influence of any number of expansions $e_k, e_l, e_m, \ldots$ of $C_2^{n(2)}$ on its symbol $[\overbrace{100\ldots0}^{n-1}]\sqrt{2}$ is found by adding the influences of each of the expansions taken separately".

Proof. Here likewise, in the succession of two expansions the subject of the second is to be what its original subject has become under the influence of the first. So in the case of $e_2 e_1 O$ of the octahedron (fig. 17[a]) the original subject of e_2 (the triangle) is transformed by e_1 into a hexagon (fig. 17[b]) and now the hexagon is moved out, in the case $e_1 e_2 O$ the linear subject of e_1 (the edge) is transformed by e_2 into a square (fig. 17[c]) and now this square is moved out; in both cases the result (fig. 17[d]) is the same, a tCO. In general, for $k > l$, in the case $e_k e_l C_{2n}^{(2)}$ the subject $S(k+1)$ of e_k is transformed by e_l into the form $e_l\ S(k+1)$ of the same number of dimensions, while in the case $e_l e_k C_{2n}^{(2)}$ the subject $S(l+1)$ of e_l is transformed by e_k into an $n-1$-dimensional limit g_l of import l. Here also the geometrical condition: "that the two new positions of any vertex shall be separated by the length of an edge" leads to the ordinary composition of the motions of the centre according to the rule of the parallelogram in the case of two expansions, etc.

By the way we find:

Theorem XLIX. "The operation e_k can still be applied to any polytope deduced from $C_{2^n}^{(2)}$ in the symbol of coordinates of which the $k+1^{st}$ and the $k+2^{nd}$ digit are equal."

We indicate by means of this theorem the expansion symbol of

the example $[5'4'4'3'3'2'2'2'1'1]$ of art. 55, considered as a descendent of $[\overline{1\,0\,0\,..\,0}^{\,9}]$. Of the five intervals $\sqrt{2}$, indicated by (d_1, d_2), (d_3, d_4), (d_5, d_6), (d_8, d_9), (d_9, d_{10}) the first corresponds to the original interval of the symbol of coordinates of $C_{2^{10}}^{(2)}$ whilst according to the theorem the others result from the four operations e_2, e_4, e_7, e_8. But as the symbol winds up in a unit instead of a zero we have to add e_9. So we find $e_2\, e_4\, e_7\, e_8\, e_9\; C_{2^{10}}^{(2)}$.

77. By means of the operations e_k we can deduce from $C_{2n}^{(2)}$ all the possible polytopes the square bracketed symbols of coordinates of which are characterized by the fact that there is an interval $\sqrt{2}$ between the first and the second digits. If we wish to deduce from $C_{2n}^{(2)}$ also polytopes with square bracketed symbols the two digits d_1, d_2 of which are equal we have to follow Mrs. Stott by introducing the operation c of contraction, the subject of which is the group of limits $(l)_{n-1}$ of vertex import. With respect to this operation we can prove the theorem:

Theorem L. "By applying the contraction c to any expansion form deduced from $C_{2n}^{(2)}$ the largest digit of the symbol of coordinates of this form is diminished by $\sqrt{2}$."

Proof. Here we have to consider the two cases of the symbol of coordinates, winding up either in 1 or in 0.

Case $[1+(a+1)\sqrt{2},\ 1+a\sqrt{2},\ 1+b\sqrt{2},\ldots,1]$. — If we replace $1+(a+1)\sqrt{2}$ by $1+a\sqrt{2}$ the limit g_0 represented by

$$x_1 = 1+(a+1)\sqrt{2},\qquad x_2, x_3,\ldots,x_n = (1+a\sqrt{2}, 1+b\sqrt{2},\ldots,1)$$

passes into

$$x_1 = 1+a\sqrt{2},\qquad x_2, x_3,\ldots,x_n = (1+a\sqrt{2}, 1+b\sqrt{2},\ldots,1),$$

i.e. that limit $(l)_{n-1}$ moves parallel to the axis OX_1 towards the centre O over a distance $\sqrt{2}$. Evidently application of this process to *all* the limits g_0 corresponds to a substitution of $1+a\sqrt{2}$ for the digit $1+(a+1)\sqrt{2}$ within the square brackets. Evidently any two adjacent limits represented originally by

$$x_1 = 1+(a+1)\sqrt{2},\qquad x_2, x_3,\ldots x_n = (1+a\sqrt{2}, 1+b\sqrt{2},\ldots,1),$$
$$x_2 = 1+(a+1)\sqrt{2},\qquad x_1, x_3,\ldots x_n = (1+a\sqrt{2}, 1+b\sqrt{2},\ldots,1),$$

which were separated by the right prism

$$x_1, x_2 = (1+(a+1)\sqrt{2}, 1+a\sqrt{2}),\qquad x_3,\ldots,x_n = (1+b\sqrt{2},\ldots,1),$$

pass into the two limits

$$x_1 = 1 + a\sqrt{2}, \qquad x_2, x_3, \ldots, x_n = (1 + a\sqrt{2}, 1 + b\sqrt{2}, \ldots, 1),$$
$$x_2 = 1 + a\sqrt{2}, \qquad x_1, x_3, \ldots, x_n = (1 + a\sqrt{2}, 1 + b\sqrt{2}, \ldots, 1),$$

which are in contact with each other by the $n - 2$-dimensional polytope

$$x_1 = 1 + a\sqrt{2},\ x_2 = 1 + a\sqrt{2},\ x_3,\ x_4, \ldots, x_n = (1 + b\sqrt{2}, \ldots, 1).$$

Case $[a + 1, a, b, \ldots, 0]\sqrt{2}$. — Here we have to consider the influence of the replacing of $a + 1$ by a. The proof runs exactly in the same lines.

Remark. By combining the theorems XLVIII and XLIX we can find the symbols in c and e_k of *any* form deduced from $C_{2n}^{(2)}$. But this process can be simplified by introducing the operation e_0 which transforms the centre O of $C_{2n}^{(2)}$ considered as an infinitesimal cross polytope $C_{2n}^{(0)}$ into $C_{2n}^{(2)}$. Then the contraction symbol c can be shunted out by substituting $e_k\, e_l \ldots e_m\, C_{2n}^{(0)}$ for $ce_k\, e_l \ldots e_m\, C_n^{\prime(2)}$, but this implies that we replace $e_k\, e_l \ldots e_m\, C_{2n}^{(2)}$ by $e_o\, e_k\, e_l \ldots e_m\, C_{2n}^{(0)}$. This remark — corresponding literally to that of art. 60 — will also be useful in part F of this section.

Meanwhile we *have* shown now that any coordinate symbol between square brackets satisfying the laws of the first part of theorem XXVIII (art. 47) can be interpreted both ways, either as a form deduced from the measure polytope or as a descendent from the cross polytope. So we have proved the following theorem already stated implicitly in art. 48:

Theorem LI. "The families of polytopes deduced from the two patriarchs, measure polytope and cross polytope, are identical."

E. *Nets of polytopes.*

78. In accordance with the last theorem the net of measure polytopes $N(M_n^{(2)})$ can also be considered as a net $N(ce_{n-1}\, C_{2n}^{(2)})$ of polytopes $ce_{n-1}\, C_{2n}^{(2)}$. So the nets put on record for $n = 3, 4, 5$ can be transcribed as nets of cross polytope descent.

But instead of doing this we point out a particularity of the case $n = 4$. For $n = 4$ both the half measure polytopes $\pm \frac{1}{2}[1, 1, 1, 1]$ are cells C_{16} and in relation with this fact we find a new fourdimensional net of regular polytopes, i. e. S_4 possesses besides the measure polytope net exceptionally a cross polytope net too. If we suppose that the net $N(M_4^{(2)})$ be composed of alternate white and black polytopes, so that two $M_4^{(2)}$ with a common $M_3^{(2)}$ differ in colour, and that each white $M_4^{(2)}$ is truncated at one set of eight vertices, so as to retain a $+\frac{1}{2}[1, 1, 1, 1]$, whilst

each black $M_4^{(2)}$ is truncated in the same way so as to retain a $-\frac{1}{2}[1, 1, 1, 1]$, the interstitial spaces between these two sets of *inclined* $C_{16}(2\sqrt{2})$ can be filled up by a third set of *erect* $C_{16}(2\sqrt{2})$, and we obtain a fourdimensional net formed by three equally numerous groups of cells $C_{16}(2\sqrt{2})$ with the property that all the polytopes of the same group are equipollent. Moreover we can transform the net $N(M_4^{(2)})$ of alternate white and black polytopes into a net of regular cells $C_{24}^{(2)}$ by decomposing each white $M_4^{(2)}$ into eight mutually congruent pyramids with the centre of the polytope as common vertex and the eight limiting cubes of the polytope as bases, and uniting each of these white pyramids to the black measure polytope with which it is in contact by its base [1]). Now what concerns us here is that by treating the new regular net $N(C_{16})$ in the same way in which the net $N(M_4)$ has been treated we find several new fourdimensional nets; for these nets the reader may compare Table II of the memoir of M^rs^. STOTT quoted several times [2]).

Remark. In art. 64 we have seen that with respect to measure polytope nets any net (c, e) is also a net (e, c). This particularity does not present itself for the nets deduced from $N(C_{16})$. So here we will have to distinguish four cases [3]), i. e. (e, c), (e, e), (c, c) and moreover (c, e).

79. We have seen that the vertices of the net $N(M_4^{(2)})$ can be represented by the symbol $[2a_1 + 1, 2a_2 + 1, 2a_3 + 1, 2a_4 + 1]$ where the a_i are arbitrary integers. By considering the point $x_i = 1$, $(i = 1, 2, 3, 4)$, as the new origin of parallel axes and omitting the square brackets we get for the coordinates of these vertices

$$2a_1,\ 2a_2,\ 2a_3,\ 2a_4.$$

From this we deduce that the vertices of the net $N(C_{16}(2\sqrt{2}))$ can be represented by the same coordinate values under addition of the condition that $\sum_1^4 a_i$ has a defined character of parity. If we choose the condition "$\sum_1^4 a_i$ is even" we get for the three sets of $C_{16}(2\sqrt{2})$ the coordinate symbols:

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[1]) Compare p. 242 of vol. II of my textbook "Mehrdimensionale Geometrie" or *Proceedings* of the Academy of Amsterdam, vol. X, p. 536, 537.

[2]) In the part of that Table concerned with the nets deduced from $N(C_{16})$ the P_T of the line with the number 28 ought to find a place in the same column in the line with the number 27. Moreover we can add in the last column of the line 29 that this net is equal to that of line 47.

The fact that several nets of this part are equal to nets deduced from cell C_{24} will be explained in part F of this section.

[3]) In (e, c), etc. the first letter is related to C_{16}, the second to C_{24}.

$$\begin{array}{lll}
\text{I} \ldots\ldots & [2a_1 + \mathbf{2}, 2a_2 + \mathbf{0}, 2a_3 + \mathbf{0}, 2a_4 + \mathbf{0}], & \Sigma a_i \text{ even}, \\
\text{II} \ldots\ldots & \tfrac{1}{2}[2a_1 + 1 + \mathbf{1}, 2a_2 + 1 + \mathbf{1}, 2a_3 + 1 + \mathbf{1}, 2a_4 + 1 + \mathbf{1}], & \text{,, odd}, \\
\text{III} \ldots\ldots & -\tfrac{1}{2}[2a_1 + 1 + \mathbf{1}, 2a_2 + 1 + \mathbf{1}, 2a_3 + 1 + \mathbf{1}, 2a_4 + 1 + \mathbf{1}], & \text{,, even}.
\end{array}$$

Of these three sets I represents the erect group, while II and III form the two inclined groups.

If we wish to represent analytically the fourdimensional nets derived from $N(C_{16})$ we have to start from the three symbols I, II, III, and to study the influence of the operations e_k, c. As to the representation of all the vertices of these new nets by coordinate symbols these influences can be split up into two inadequate parts; of these the first deals with the variation in form of any C_{16} of each of the three groups, whilst the second is concerned with the variation of the distance of any two C_{16}. We treat each of these two parts for itself.

a) *Variation in shape.* We know the influence of the operations e_k, c on the coordinate symbol $[2000]$ of the central $C_{16}(2\sqrt{2})$ of the erect group and from this we can deduce the corresponding influences on the $C_{16}(2\sqrt{2})$ of each of the inclined groups by means of the transformations of coordinates by which $[2\,0\,0\,0]$ passes into $\frac{1}{2}[1\,1\,1\,1]$ and $-\frac{1}{2}[1\,1\,1\,1]$.

The formulae corresponding to the first transformation are

$$\left.\begin{aligned}
2y_1 &= x_1 + x_2 + x_3 + x_4 \\
2y_2 &= x_1 + x_2 - x_3 - x_4 \\
2y_3 &= x_1 - x_2 + x_3 - x_4 \\
2y_4 &= x_1 - x_2 - x_3 + x_4
\end{aligned}\right\} ;$$

by changing the sign of y_1 we get formulae corresponding to the second transformation. In the following small table we put on record the result of the first transformation:

$1\,[2\,0\,0\,0]$	$\frac{1}{2}[1\,1\,1\,1]$
$e_1\,[4\,2\,0\,0]$	$\frac{1}{2}[3\,3\,1\,1]$
$e_2\,[4\,2\,2\,0]$	$[4\,2\,2\,0]$
$e_3\,[1'1\,1\,1]\sqrt{2}$	$\frac{1}{2}[2'1\,1\,1]$ and $-\frac{1}{2}[1'1'1', \sqrt{2}-1]$
$e_1 e_2\,[6\,4\,2\,0]$	$[6\,4\,2\,0]$
$e_1 e_3\,[2'1'1\,1]\sqrt{2}$	$\frac{1}{2}[3+2\sqrt{2}, 3\,1\,1]$ and $-\frac{1}{2}[3+\sqrt{2}, 3+\sqrt{2}, 1', \sqrt{2}-1]$
$e_2 e_3\,[2'1'1'1]\sqrt{2}$	$[4+2\sqrt{2}, 2\,2\,0]$ and $-\frac{1}{2}[4+\sqrt{2}, 2+\sqrt{2}, 2+\sqrt{2}, \sqrt{2}]$
$e_1 e_2 e_3\,[3'2'1'1]\sqrt{2}$	$[6+2\sqrt{2}, 4\,2\,0]$ and $-\frac{1}{2}[6+\sqrt{2}, 4+\sqrt{2}, 2+\sqrt{2}\ \sqrt{2}]$
$ce_1\,[2\,2\,0\,0] = C_{24}^{(2\sqrt{2})}$ ...	$[2\,2\,0\,0]$
$ce_2\,[2\,2\,2\,0]$	$-\frac{1}{2}[3\,1\,1\,1]$
$ce_3\,[1\,1\,1\,1]\sqrt{2} = C_8^{(2\sqrt{2})}$...	$[2\,0\,0\,0]\sqrt{2}$ and $-\frac{1}{2}[1\,1\,1\,1]\sqrt{2}$
$ce_1 e_2\,[4\,4\,2\,0]$	$-\frac{1}{2}[5\,3\,1\,1]$
$ce_1 e_3\,[1'1'1\,1]\sqrt{2}$	$[2+2\sqrt{2}, 2\,0\,0]$ and $-\frac{1}{2}[1'1'1\,1]\sqrt{2}$
$ce_2 e_3\,[1'1'1'1]\sqrt{2}$	$-\frac{1}{2}[3+2\sqrt{2}, 1\,1\,1]$ and $-\frac{1}{2}[3+\sqrt{2}, 1'1'1']$
$ce_1 e_2 e_3\,[2'2'1'1]\sqrt{2}$	$-\frac{1}{2}[5+2\sqrt{2}, 3\,1\,1]$ and $-\frac{1}{2}[5+\sqrt{2}, 3+\sqrt{2}, 1+\sqrt{2}, 1+\sqrt{2}]$

b) *Variation in distance.* We account for the variation of the distance of any two sixteencells due to the extension of these cells by multiplying the immovable parts of the digits of the three symbols of coordinates given above for the three groups of sixteencells by a certain constant. This constant is the extension number itself when the operation e_4 is lacking, i. e. in the two general cases (e, c) and (c, c) of nets deduced from $N(C_{16})$; in the remaining general cases (e, e) and (c, e) we have to add $\sqrt{2}$ to that multiplier in order to create room for the intermediate prisms with $2\sqrt{2}$ as height.

As we start from $[2\,0\,0\,0]$ the extension number is half the sum of the digits. So we find for the multiplier the values given in the following table

(e, c)	(e, e)	(c, c)	(c, e)
	$e_4 \ldots 1+\sqrt{2}$		$ce_4 \ldots \sqrt{2}$
$e_1 \ldots 3$	$e_1e_4 \ldots 3+\sqrt{2}$	$ce_1 \ldots 2$	$ce_1e_4 \ldots 2+\sqrt{2}$
$e_2 \ldots 4$	$e_2e_4 \ldots 4+\sqrt{2}$	$ce_2 \ldots 3$	$ce_2e_4 \ldots 3+\sqrt{2}$
$e_3 \ldots 1+2\sqrt{2}$	$e_3e_4 \ldots 1+3\sqrt{2}$	$ce_3 \ldots 2\sqrt{2}$	$ce_3e_4 \ldots 3\sqrt{2}$
$e_1e_2 \ldots 6$	$e_1e_2e_4 \ldots 6+\sqrt{2}$	$ce_1e_2 \ldots 5$	$ce_1e_2e_4 \ldots 5+\sqrt{2}$
$e_1e_3 \ldots 3+2\sqrt{2}$	$e_1e_3e_4 \ldots 3+3\sqrt{2}$	$ce_1e_3 \ldots 2+2\sqrt{2}$	$ce_1e_3e_4 \ldots 2+3\sqrt{2}$
$e_2e_3 \ldots 4+2\sqrt{2}$	$e_2e_3e_4 \ldots 4+3\sqrt{2}$	$ce_2e_3 \ldots 3+2\sqrt{2}$	$ce_2e_3e_4 \ldots 3+3\sqrt{2}$
$e_1e_2e_3 \ldots 6+2\sqrt{2}$	$e_1e_2e_3e_4 \ldots 6+3\sqrt{2}$	$ce_1e_2e_3 \ldots 5+2\sqrt{2}$	$ce_1e_2e_3e_4 \ldots 5+3\sqrt{2}$

80. By means of the preceding developments we can find the three net symbols for all the different nets deduced from $N(C_{16})$. But this work can be reduced by the remark that it will do to use only the net symbol of the erect group in the cases of the seven nets $1, e_1, e_2, e_1e_2, ce_1, ce_2, ce_1e_2$, while we want these of two groups only for the eight nets $e_3, e_1e_3, e_2e_3, e_1e_2e_3, ce_3, ce_1e_3, ce_2e_3, ce_1e_2e_3$, and all the three symbols in the remaining cases where e_4 occurs. The proof of this assertion is based on the following theorem, where we distinguish the three sets of cases just indicated as the set without e_3 and e_4, the set with e_3 and without e_4, and the set with e_4:

Theorem LII. "Any of the three net symbols represents all the vertices of the net in the set without e_3 and e_4, two thirds of all the vertices in the set with e_3 and without e_4, one third of all the vertices in the set with e_4".

This theorem is an immediate consequence of the following lemma: "Any limiting tetrahedron of the net $N(C_{16})$ is common to two C_{16} belonging to different groups, any limiting triangle is common to three C_{16} no two of which belong to the same group".

The first part of this lemma is evident by itself. As to the second part related to a face we state that the angle formed by the two spaces of adjacent tetrahedra $ABCD$, $ABCD'$ of C_{16} at the common face ABC is 120° (see my paper: "On the angles of the regular polytopes, etc.", *Amer. Journ. of Math.*, vol. XXXI, p. 307), from which it ensues that any face is common to three C_{16}; as any two of these three C_{16} have a limiting tetrahedron in common they belong to different groups, etc.

The lemma just proved immediately shows the truth of the theorem. If, after having driven asunder the cells $C_{16}(2\sqrt{2})$ of the net $N(C_{16})$ so as to create room for the extension recorded above, the extended C_{16} receive the shape exacted by the character of the net under consideration by means of a regular truncation, the contact of the cells — belonging to different groups — by faces will remain uninfluenced if the operations e_3, e_4 do not yet present themselves, the truncations being then restricted either to the vertices alone or to vertices and edges; so, as any vertex of the net belongs at least to one face and each face belongs to three polytopes of the set without e_3, e_4, one of each group, each vertex of the net must be contained in each of the three net symbols of any case of that set.

So in this case the net itself can be represented by any of the three symbols, which includes that the constituents furnished by one symbol are identical with those furnished by each of the two others, though constituents of polytope and body import of one symbol may become under certain circumstances constituents respectively of vertex and edge import of an other.

Now the state of affairs changes as soon as e_3 makes its appearance. This operation still *preserves* the contact by limiting bodies of body import between cells belonging to different groups, but it *annihilates* at the same time face contact between limiting bodies of body import of the same cell. So here the limiting bodies of body import of any constituent have been split up into two sets P and Q dividing the vertices equally between them, in such a way that any two of these limits which were in face contact before belong to different sets. So here the arrangement of the three groups A, B, C of constituents is such that any constituent of group A is in body contact by its set of limits P with constituents of group B, by its set of limits Q with constituents of group C. So each of the three net symbols contains all the vertices of one group and only half the number of vertices of each of the two other groups, i.e. $\frac{2}{3}$ of the total amount.

Finally, in the set with e_4, two cells — belonging to different groups — cannot have a vertex in common; so here each net symbol represents only $\frac{1}{3}$ of the system of vertices.

We now indicate schematically how we can determine all the constituents of the different nets of C_{16}. To that end we have

1°. to deduce from the preceding developments the net symbols necessary in every case,

2°. to calculate the coordinates of the centres of the different constituents, by multiplying the coordinates of a vertex, of the midpoint of an edge, of the centre of a face and of the centre of a limiting body of $[2, 0, 0, 0]$ by the extension number,

3°. to determine the vertices contained in the net symbols, lying at the same minimum distance from these centres.

As we shall have to consider the "extended" vertex, midpoint of edge, centre of face, or centre of limiting body mentioned sub 2° as new origin of parallel axes of coordinates in order to be able to obtain the simplest representation of the sets of vertices mentioned sub 3° we will denote this extended point henceforth by O'.

Of each of the three sets we will treat some examples, of the first $e_1\, e_2\, N(C_{16})$ and $ce_1\, e_2\, N(C_{16})$, of the second $e_2\, e_3\, N(C_{16})$ and $ce_1\, e_3\, N(C_{16})$, of the third $e_1\, e_4\, N(C_{16})$, $e_1\, e_2\, e_3\, e_4\, N(C_{16})$ and $ce_1\, e_2\, e_3\, e_4\, N C_{16}$. Afterwards we will put on record the coordinate symbols of all the constituents in Table VII.

81. *Case* $\boldsymbol{e_1\, e_2\, N(C_{16})}$. Net symbol

$$[12\, a_1 + \mathbf{6},\ 12\, a_2 + \mathbf{4},\ 12\, a_3 + \mathbf{2},\ 12\, a_4 + \mathbf{0}],\ \sum_1^4 a_i \text{ even}.$$

Here the constituent of polytope import is $[6, 4, 2, 0] = e_1\, e_2\, C_{16}$. There are no constituents of body and face import as the operations e_4 and e_3 do not present themselves. So we have only to determine the polytopes of edge and vertex import.

Edge gap prism. By extension the centre $1, 1, 0, 0$ of the edge $(2, 0)\, 0\, 0$ of $[2, 0, 0, 0]$ becomes $6, 6, 0, 0$. By putting in the net symbol $a_i = 0$, $(i = 1, 2, 3, 4)$, we find among others the vertices $(6, 4)\, [2, 0]$ and by putting $a_1 = a_2 = 1$, $a_3 = a_4 = 0$, and taking the movable digits $6, 4$ with the negative sign we find also the vertices $(6, 8)\, [2, 0]$; with respect to the new axes with the point $6, 6, 0, 0$ as new origin O' these two groups of vertices can be represented together by the symbol $[2, 0]\, [2, 0]$. So we find a measure polytope C_8 which is to be interpreted here as a prism on a cube, P_C.

Vertex gap polytope. By extension of the vertex $2, 0, 0, 0$ of $[2, 0, 0, 0]$ we get $12, 0, 0, 0$ as new origin O'. By substituting $a_i = 0, (i = 1, 2, 3, 4)$, in the first place and $a_1 = 2$, $a_i = 0, (i = 2, 3, 4)$, in the second (with the movable digit 6 taken negatively) we put in evidence the two sets of vertices $6[4, 2, 0]$ and $18[4, 2, 0]$, i.e. with respect to O' the vertices $[6][4, 2, 0]$ contained in the net symbol. But this symbol still contains other vertices lying at the same minimum distance $2\sqrt{14}$ from O', i.e. all the vertices represented with respect to that point by $[6, 4, 2, 0]$ and no other. So we find e.g. the point $4, 6, 2, 0$, with the coordinates $16, 6, 2, 0$ with respect to the original axes, by considering the vertices $12a_1 + 4$, $12a_2 - 6$, $12a_3 + 2$, $12a_4$ and putting $a_1 = a_2 = 1$ and $a_3 = a_4 = 0$, etc. So the result is that the constituent of vertex import is a $[6, 4, 2, 0] = e_1 e_2 C_{16}$ and therefore identical with the constituents of polytope import.

Case $\boldsymbol{ce_1 e_2 N(C_{16})}$. Net symbol

$$[10a_1 + \mathbf{4}, 10a_2 + \mathbf{4}, 10a_3 + \mathbf{2}, 10a_4 + \mathbf{0}], \sum_1^4 a_i \text{ even.}$$

Here the constituent of polytope import is $[4, 4, 2, 0] = ce_1 e_2 C_{16}$. As in the preceding case of $e_1 e_2 N(C_{16})$ the constituents of body and of face import are lacking. Moreover by the contraction the original edge and therefore also the constituent P_C of edge import is annihilated, i.e. P_C is reduced to its base C. We verify this analytically as follows. By extension of the midpoint $1, 1, 0, 0$ of the edge $(2, 0)$ $0, 0$ of $[2, 0, 0, 0]$ we get $5, 5, 0, 0$ as new origin O'. Now the vertices at minimum distance from O' contained in the net symbol are found by putting $a_i = 0, (i = 1, 2, 3, 4)$, giving $4, 4[2, 0]$, and $a_1 = a_2 = 1$, $a_3 = a_4 = 0$ (with the two digits 4 taken negatively) giving $6, 6[2, 0]$, i.e. with respect to O' the two squares $1, 1[2, 0]$ and $-1, -1[2, 0]$ forming two opposite faces of a cube with O' as centre.

Finally we remark that the contraction c does not affect the constituent of vertex import. This is easily verified by determining the vertices at minimum distance from the point O' with the coordinates $10, 0, 0, 0$ presenting itself here.

82. *Case* $\boldsymbol{e_2 e_3 N(C_{16})}$. As the operation e_3 presents itself here we have to find besides the constituent $[2'1'1'1]\sqrt{2} = e_2 e_3 C_{16}$ of polytope import those of face, of edge and of vertex import, and in order to be able to gather all the vertices of these constituents we have to use two of the three net symbols. But we prefer to

investigate how far we can proceed in this way by using the first net symbol only. This much more complicated symbol is
$[4(2+\sqrt{2})a_1+4+\sqrt{2},\ 4(2+\sqrt{2})a_2+2+\sqrt{2},\ 4(2+\sqrt{2})a_3+2+\sqrt{2},\ 4(2+\sqrt{2})a_4+\sqrt{2}]$,
Σa_i being even. We abridge it into the following form, clear by itself:

$$[4+\sqrt{2},\ 2+\sqrt{2},\ 2+\sqrt{2},\ \sqrt{2}],\ (8+4\sqrt{2})\ \overline{a_1, a_2, a_3, a_4},\ \sum_1^4 a_i \text{ even},$$

where $\overline{a_1, a_2, a_3, a_4}$ preceded by the common factor $8+4\sqrt{2}$ represents the immovable part.

Face gap prismotope. By extension the centre $\frac{2}{3}, \frac{2}{3}, \frac{2}{3}, 0$ of the face $(2, 0, 0)\ 0$ of $[2, 0, 0, 0]$ passes into the new origin O' with the coordinates $\frac{4}{3}(2+\sqrt{2}),\ \frac{4}{3}(2+\sqrt{2}),\ \frac{4}{3}(2+\sqrt{2}),\ 0$. By supposing the four a_i of the net symbol to disappear we get inter alia the set of vertices $(4+\sqrt{2},\ 2+\sqrt{2},\ 2+\sqrt{2})\ [\sqrt{2}]$, i. e. a P_3. These are the only vertices contained in the net symbol above mentioned lying at minimum distance $\frac{4}{3}\sqrt{3}$ from O', but as we shall see immediately the two other net symbols contain other vertices partaking of this property. However, in order to sharpen our analytic tools, we leave these other net symbols alone for a moment and try to deduce these lacking vertices from the simple properties of the prismotope with two regular generating polygons in planes perfectly normal to each other. By means of the P_3 just found we know that one of these polygons is a triangle, and the character of the other polygon can be deduced from its circumradius. For the relation $\rho_1^2+\rho_2^2=\rho^2$ between the circumradii ρ_1, ρ_2, ρ of the two generating polygons and the prismotope itself gives, as we have $\rho=\frac{4}{3}\sqrt{3}$ and $\rho_1=\frac{2}{3}\sqrt{6}$, $\rho_2=\frac{2}{3}\sqrt{6}$, i. e. the second polygon is also a triangle and the prismotope a $(3;3)$. We have therefore only to find a third position of the first triangle, the two end planes of P_3 containing already two positions, and this third position can be found by remarking that the centres of these three equipollent triangles are the vertices of an equilateral triangle with O' as centre. So, if p, q, r, s are the coordinates of the centre of this third position we have that the triangle with the three vertices

$$\begin{array}{cccc} \frac{8}{3}+\sqrt{2}, & \frac{8}{3}+\sqrt{2}, & \frac{8}{3}+\sqrt{2}, & \sqrt{2} \\ \frac{8}{3}+\sqrt{2}, & \frac{8}{3}+\sqrt{2}, & \frac{8}{3}+\sqrt{2}, & -\sqrt{2} \\ p & , \quad q & , \quad r & , \quad s \end{array}$$

must admit

$$\frac{4}{3}(2+\sqrt{2}),\ \frac{4}{3}(2+\sqrt{2}),\ \frac{4}{3}(2+\sqrt{2}),\qquad 0$$

as centre. From this it ensues that we have

$$p=q=r=\frac{8}{3}+2\sqrt{2},\ s=0,$$

furnishing $(4+2\sqrt{2},\ 2+2\sqrt{2},\ 2+2\sqrt{2}),\ 0$ for the third position of the first triangle [1]). Indeed the part of the second net symbol corresponding to $[4+2\sqrt{2}, 2, 2, 0]$, i. e.

$$[4+2\sqrt{2}, 2, 2, 0],\ (4+2\sqrt{2})\ \overline{2a_1+1,\ 2a_2+1,\ 2a_3+1,\ 2a_4+1},\ \sum_4^1 a_i \text{ odd},$$

gives for $a_1 = a_2 = a_3 = 0$ and $a_4 = -1$ the set of vertices represented by

$$(4+2\sqrt{2}-\mathbf{0},\ 4+2\sqrt{2}-\mathbf{2},\ 4+2\sqrt{2}-\mathbf{2})\ 4+2\sqrt{2}-\mathbf{4}-\mathbf{2\sqrt{2}},$$

i. e. $(4+2\sqrt{2},\ 2+2\sqrt{2},\ 2+2\sqrt{2})\ 0$.

Edge gap prism. By extension the centre $1, 1, 0, 0$ of the edge $(2, 0)\ 0, 0$ of $[2, 0, 0, 0]$ gives $2(2+\sqrt{2}),\ 2(2+\sqrt{2}),\ 0,\ 0$ for the coordinates of O'. By reducing the first net symbol to this point as new origin we get

$$[4+\sqrt{2},\ 2+\sqrt{2},\ 2+\sqrt{2},\ \sqrt{2}],\ (4+2\sqrt{2})\ \overline{2a_1-1,\ 2a_2-1,\ 2a_3,\ 2a_4},\ \sum_1^4 a_i \text{ even}.$$

By putting $a_i = 0$, $(i = 1, 2, 3, 4)$, and taking the permutable digits in the indicated order and with the positive sign we find the vertex $-\sqrt{2},\ -(2+\sqrt{2}),\ 2+\sqrt{2},\ \sqrt{2}$ lying at minimum distance $2\sqrt{(4+2\sqrt{2})}$ from O'. As this distance is smaller than $4+\sqrt{2}$ we are obliged, in order to find all the vertices contained in that symbol lying at that distance from O', to put $a_3 = a_4 = 0$ and to take either $a_1 = a_2 = 0$ or $a_1 = a_2 = 1$. So we find the 32 vertices $\frac{1}{2}[2+\sqrt{2}, \sqrt{2}][2+\sqrt{2}, \sqrt{2}]$, where the $\frac{1}{2}$ refers to the first syllable corresponding to the coordinates x_1, x_2 only. Now we have furthermore to examine the other two net symbols. For O' as origin the second net symbol is

$$\left.\begin{matrix} [4+2\sqrt{2}, & 2 & , & 2 & , & 0\ ] \\ -\frac{1}{2}[4+\sqrt{2}, & 2+\sqrt{2}, & & 2+\sqrt{2}, & & \sqrt{2}] \end{matrix}\right\},\ (4+2\sqrt{2})\ \overline{2a_1,\ 2a_2,\ 2a_3+1,\ 2a_4+1},\ \sum_1^4 a_i \text{ odd},$$

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[1]) Until now we have only used implicitly the condition that the planes of the generating polygons are perfectly normal to each other, in the equation $\rho_1^2+\rho_2^2=\rho^2$. As the plane $x_1+x_2+x_3=0$, $x_4=0$ is parallel to those of the first triangle, the plane $x_1=x_2=x_3$ perfectly normal to it must be parallel to those of the second. We verify this by the following table of the nine vertices of the prismotope

$$\left|\begin{matrix} 4+\sqrt{2}, 2+\sqrt{2}, 2+\sqrt{2}, & \sqrt{2} & 2+\sqrt{2}, 4+\sqrt{2}, 2+\sqrt{2}, & \sqrt{2} & 2+\sqrt{2}, 2+\sqrt{2}, 4+\sqrt{2}, & \sqrt{2} \\ 4+\sqrt{2}, 2+\sqrt{2}, 2+\sqrt{2}, & -\sqrt{2} & 2+\sqrt{2}, 4+\sqrt{2}, 2+\sqrt{2}, & -\sqrt{2} & 2+\sqrt{2}, 2+\sqrt{2}, 4+\sqrt{2}, & -\sqrt{2} \\ 4+2\sqrt{2}, 2+2\sqrt{2}, 2+2\sqrt{2}, & 0 & 2+2\sqrt{2}, 4+2\sqrt{2}, 2+2\sqrt{2}, & 0 & 2+2\sqrt{2}, 2+2\sqrt{2}, 4+2\sqrt{2}, & 0 \end{matrix}\right|,$$

the three rows forming the positions of the first triangle and the three columns (of sets of coordinates) those of the second. So for the triangle of the first column we have $x_1-x_2=2$, $x_2=x_3$, etc.

By continuing this research it can be verified, that each of the three net symbols contains the six vertices of a P_3 with two positions of the first triangle, i. e. two rows of the table of the nine vertices, as end planes.

the two sets of permutable digits having to be combined with the same set of immovable ones. Here we find only vertices lying at a greater distance from O', unless we take $a_1 = a_2 = 0$. So we get for $a_3, a_4 = (0, -1)$ by means of the upper half of the symbol the 16 new vertices $[2, 0][2(1+\sqrt{2}), 0]$, by means of the lower the 16 vertices $x_1, x_2 = \frac{1}{2}[2+\sqrt{2}, \sqrt{2}], x_3, x_4 = -\frac{1}{2}[2+\sqrt{2}, \sqrt{2}]$ already contained in the set $\frac{1}{2}[2+\sqrt{2}, \sqrt{2}][2+\sqrt{2}, \sqrt{2}]$ deduced from the first symbol. From this may be deduced that the two halves of the third symbol will furnish the two sets $[2, 0][2(1+\sqrt{2}), 0]$ and $x_1, x_2 = \frac{1}{2}[2+\sqrt{2}, \sqrt{2}]$, $x_3, x_4 = \frac{1}{2}[2+\sqrt{2}, \sqrt{2}]$.

So the result is a polytope with 48 vertices represented by the combination of the two symbols $\frac{1}{2}[2+\sqrt{2}, \sqrt{2}][2+\sqrt{2}, \sqrt{2}]$ and $[2, 0][2(1+\sqrt{2}), 0]$. It proves to be a P_{tC}. For, by applying on the tC represented by the symbol $[\sqrt{2}][2+\sqrt{2}, 2+\sqrt{2}, \sqrt{2}]$ the transformation

$$\left.\begin{aligned} x_1 + x_2 &= y_1\sqrt{2} \\ x_1 - x_2 &= y_2\sqrt{2} \end{aligned}\right\}, \quad \left.\begin{aligned} x_3 + x_4 &= y_3\sqrt{2} \\ x_3 - x_4 &= y_4\sqrt{2} \end{aligned}\right\}$$

we get $\frac{1}{2}[2+\sqrt{2}, \sqrt{2}][2+\sqrt{2}, \sqrt{2}]$ for the 32 vertices $[\sqrt{2}][2+\sqrt{2}][2+\sqrt{2}, \sqrt{2}]$ and $[2, 0][2(1+\sqrt{2}), 0]$ for the remaing 16 vertices $[\sqrt{2}][\sqrt{2}][2+\sqrt{2}, 2+\sqrt{2}]$.

Vertex gap polytope. By extension the vertex $2, 0, 0, 0$ of $[2, 0, 0, 0]$ gives $4(2+\sqrt{2}), 0, 0, 0$ for O'. With respect to this origin the first net symbol is

$$[4+\sqrt{2}, 2+\sqrt{2}, 2+\sqrt{2}, \sqrt{2}], (8+4\sqrt{2})\,\overline{a_1 - 1, a_2, a_3, a_4, \sum_1^4 a_i \text{even}},$$

which can be reduced to

$$[4+\sqrt{2}, 2+\sqrt{2}, 2+\sqrt{2}, \sqrt{2}], (8+4\sqrt{2})\ \overline{a_1, a_2, a_3, a_4, \sum_1^4 a_i \text{odd}}.$$

By taking in this last symbol $a_1, a_2, a_3, a_4 = [1, 0, 0, 0]$ and putting the digit $4+\sqrt{2}$ always where the 1 stands with the opposite sign of it, we get the 192 vertices $[4+3\sqrt{2}, 2+\sqrt{2}, 2+\sqrt{2}, \sqrt{2}]$ lying at the minimum distance $4(1+\sqrt{2})$ from O'.

With respect to the same origin O' the second net symbol is

$$\left.\begin{matrix} [4+2\sqrt{2}, & 2 & , & 2 & , & 0\ ] \\ -\frac{1}{2}[4+\sqrt{2}, & 2+\sqrt{2}, & & 2+\sqrt{2}, & & \sqrt{2}] \end{matrix}\right\}, (4+2\sqrt{2})\ \overline{2a_1 - 1, 2a_2 + 1, 2a_3 + 1, 2a_4 + 1, \sum_1^4 a_i \text{ odd}},$$

the immovable part of which can be reduced to

$$(4+2\sqrt{2})\,\overline{2a_1 + 1, 2a_2 + 1, 2a_3 + 1, 2a_4 + 1, \sum_1^4 a_i \text{even}}.$$

By considering the three groups of cases

$$a_i = 0, (i = 1, 2, 3, 4) —, a_1, a_2, a_3, a_4 = (-1, -1, 0, 0,) —,$$
$$a_i = -1, (i = 1, 2, 3, 4),$$

and adding to the immovable parts the permutable ones taken in any order, generally affected by the sign which tends to *decrease* the absolute value of the coordinate but — in connection with the negative sign before the lower half of the symbol which exacts an odd number of negative permutable digits — with exception of the smallest of these digits $\sqrt{2}$ the sign of which is to be chosen inversely so as to *increase* the absolute value of the coordinate, we get by the upper half the 96 new vertices $[4 + 2\sqrt{2}, 2 + 2\sqrt{2}, 2 + 2\sqrt{2}, 0]$ and by the lower the 96 vertices $\frac{1}{2}[4+3\sqrt{2}, 2+\sqrt{2}, 2+\sqrt{2}, \sqrt{2}]$, obtained above. So the result is a polytope with 288 vertices represented by the combination of the symbols

$$[4+3\sqrt{2}, 2+\sqrt{2}, 2+\sqrt{2}, \sqrt{2}], [4+2\sqrt{2}, 2+2\sqrt{2}, 2+2\sqrt{2}, 0].$$

As we will prove in section V this polytope with the characteristic numbers (288, 576, 336, 48) limited by $48\,tC$ is $ce_1 e_2 C_{24}$.

Case $\boldsymbol{ce_1 e_3 N(C_{16})}$. Besides $[1'1'1\,1]\sqrt{2} = ce_1 e_3 C_{16}$ we have to look out for the face gap filling and the polytope of vertex import, the edge gap filling being reduced by contraction to the base polyhedron of the prism occurring in the case of $e_1 e_3 N(C_{16})$.

Face gap prismotope. Here we get for the new origin O' the coordinates $\frac{2}{3}(2 + 2\sqrt{2}), \frac{2}{3}(2 + 2\sqrt{2}), \frac{2}{3}(2 + 2\sqrt{2}), 0$, as $2 + 2\sqrt{2}$ is the extension number.

So the first and the second net symbol are

$$[2 + \sqrt{2}, 2 + \sqrt{2}, \sqrt{2}, \sqrt{2}], \; (4 + 4\sqrt{2}) \; \overline{a_1 - \tfrac{1}{3}, \; a_2 - \tfrac{1}{3}, \; a_3 - \tfrac{1}{3}, \; a_4}, \; \sum_1^4 a_i \text{ even},$$

$$\left.\begin{matrix} [2 + 2\sqrt{2}, & 2 & , & 0 & , & 0\,] \\ -\frac{1}{2}[2 + \sqrt{2}, & 2 + \sqrt{2}, & & \sqrt{2}, & & \sqrt{2}] \end{matrix}\right\}, \; (2 + 2\sqrt{2}) \; \overline{2a_1 + \tfrac{1}{3}, \; 2a_2 + \tfrac{1}{3}, \; 2a_3 + \tfrac{1}{3}, \; 2a_4 + 1}, \; \sum_1^4 a_i \text{ odd}.$$

By taking in the first symbol $a_i = 0, (i = 1, 2, 3, 4)$, we find the vertices $\left(\frac{2 - \sqrt{2}}{3}, \frac{2 - \sqrt{2}}{3}, \frac{-4 - \sqrt{2}}{3}\right)[2]$ lying at minimum distance $\frac{4}{3}\sqrt{3}$ from O', i. e. a P_3; by substituting in the upper half of the second symbol $a_i = 0, (i = 1, 2, 3), a_4 = -1$ we get moreover $\left(\frac{2 + 2\sqrt{2}}{3}, \frac{2 + 2\sqrt{2}}{3}, \frac{-4 + 2\sqrt{2}}{3}\right) 0$, the third triangle of the prismotope $[3; 3]$ to be found.

Vertex gap polytope. The new origin is $2(2+2\sqrt{2}), 0, 0, 0$ and the first and second net symbol become, in the shortest form possible,

$$[2+\sqrt{2},\ 2+\sqrt{2},\ \sqrt{2},\ \sqrt{2}],\ (4+4\sqrt{2})\ \overline{a_1\quad,\ a_2\quad,\ a_3\quad,\ a_4}\quad,\ \sum_1^4 a_i \text{ odd},$$

$$\left.\begin{array}{l}[2+2\sqrt{2},\quad 2\quad,\ 0\ ,\ 0\ ]\\ -\tfrac{1}{2}[2+\sqrt{2},\ 2+\sqrt{2},\ \sqrt{2},\ \sqrt{2}]\end{array}\right\},\ (2+2\sqrt{2})\ \overline{2a_1+1,\ 2a_2+1,\ 2a_3+1,\ 2a_4+1},\ \sum_1^4 a_i \text{ even}.$$

Putting into the first symbol $a_1, a_2, a_3, a_4 = [1, 0, 0, 0]$ and combining with the a_i differing from zero one of the two digits $2+\sqrt{2}$ taken with the sign tending to decrease the absolute value of the coordinate we get the 192 vertices $[2+3\sqrt{2}, 2+\sqrt{2}, \sqrt{2}, \sqrt{2}]$. Putting into the upper half of the second symbol $a_i = 0, (i=1,2,3,4)$, we find moreover the 96 vertices $[2+2\sqrt{2}, 2+2\sqrt{2}, 2\sqrt{2}, 0]$. So the result is a polytope with 288 vertices which will prove later on to admit the characteristic numbers (288, 864, 720, 144) and to be $e_2\, C_{24}$.

83. *Case* $\boldsymbol{e_1 e_4 N(C_{16})}$. Here the extension number is $3+\sqrt{2}$. So we have to reduce the three net symbols

$$[4,2,0,0],(6+2\sqrt{2})\ \overline{a_1\quad,\ a_2\quad,\ a_3\quad,\ a_4}\quad,\sum_1^4 a_i \text{ even},$$

$$\tfrac{1}{2}[3,3,1,1],(3+\sqrt{2})\overline{2a_1+1,2a_2+1,2a_3+1,2a_4+1},\sum_1^4 a_i \text{ odd},$$

$$-\tfrac{1}{2}[3,3,1,1],(3+\sqrt{2})\overline{2a_1+1,2a_2+1,2a_3+1,2a_4+1},\sum_1^4 a_i \text{ even}$$

for the constituents of body, face, edge, vertex import to the new origins $(3+\sqrt{2})\,\overline{\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}}$, $(3+\sqrt{2})\,\overline{\tfrac{2}{3},\tfrac{2}{3},\tfrac{2}{3},0}$, $(3+\sqrt{2})\,\overline{1,1,0,0}$, $(3+\sqrt{2})\,\overline{2,0,0,0}$ respectively, the constituent of polytope import being $[4,2,0,0] = e_1\, C_{16}$.

Body gap prism. The three net symbols become

$$[4,2,0,0],(3+\sqrt{2})\overline{2a_1-\tfrac{1}{2},2a_2-\tfrac{1}{2},2a_3-\tfrac{1}{2},2a_4-\tfrac{1}{2}},\sum_1^4 a_i \text{ even},$$

$$\tfrac{1}{2}[3,3,1,1],(3+\sqrt{2})\overline{2a_1+\tfrac{1}{2},2a_2+\tfrac{1}{2},2a_3+\tfrac{1}{2},2a_4+\tfrac{1}{2}},\sum_1^4 a_i \text{ odd},$$

$$-\tfrac{1}{2}[3,3,1,1],(3+\sqrt{2})\overline{2a_1+\tfrac{1}{2},2a_2+\tfrac{1}{2},2a_3+\tfrac{1}{2},2a_4+\tfrac{1}{2}},\sum_1^4 a_i \text{ even}.$$

By making the a_i to disappear the first and the third[1]) symbol give the sets of vertices $\left(\frac{5-\sqrt{2}}{2}, \frac{1-\sqrt{2}}{2}, \frac{-3-\sqrt{2}}{2}, \frac{-3-\sqrt{2}}{2}\right)$, $\left(\frac{5+\sqrt{2}}{2}, \frac{1+\sqrt{2}}{3}, \frac{-3+\sqrt{2}}{2}, \frac{-3+\sqrt{2}}{2}\right)$ each of which corresponds to a (2100), i. e. to a tT. So the result is a P_{tT}, all the vertices of the second symbol lying at larger distance from O' than the circumradius $\sqrt{13}$ of this P_{tT}.

Face gap prismotope. Here the three net symbols are

$$[4,2,0,0], (3+\sqrt{2})\, \overline{2a_1-\tfrac{2}{3}, 2a_2-\tfrac{2}{3}, 2a_3-\tfrac{2}{3}, 2a_4}, \ \sum_1^4 a_i \text{ even},$$

$$\tfrac{1}{2}[3,3,1,1], (3+\sqrt{2})\, \overline{2a_1+\tfrac{1}{3}, 2a_2+\tfrac{1}{3}, 2a_3+\tfrac{1}{3}, 2a_4+1}, \ \sum_1^4 a_i \text{ odd},$$

$$-\tfrac{1}{2}[3,3,1,1], (3+\sqrt{2})\, \overline{2a_1+\tfrac{1}{3}, 2a_2+\tfrac{1}{3}, 2a_3+\tfrac{1}{3}, 2a_4+1}, \ \sum_1^4 a_i \text{ even}.$$

By taking in the first symbol $a_i = 0$, $(i = 1, 2, 3, 4)$, in the second $a_i = 0$, $(i = 1, 2, 3)$, $a_4 = -1$, in the third $a_i = 0$, $(i = 1, 2, 3, 4)$, we get the three hexagons

$$\left.\begin{array}{llll} (2-\frac{2}{3}\sqrt{2}, & -\frac{2}{3}\sqrt{2}, & -2-\frac{2}{3}\sqrt{2}) & 0 \\ (2+\frac{1}{3}\sqrt{2}, & \frac{1}{3}\sqrt{2}, & -2+\frac{1}{3}\sqrt{2}) & -\sqrt{2} \\ (2+\frac{1}{3}\sqrt{2}, & \frac{1}{3}\sqrt{2}, & -2+\frac{1}{3}\sqrt{2}) & \sqrt{2} \end{array}\right\}.$$

So the result is a $[6;\ 3]$.

Edge gap prism. Now the three net symbols become

$$[4,2,0,0], (3+\sqrt{2})\, \overline{2a_1-1, 2a_2-1, 2a_3, 2a_4}, \ \sum_1^4 a_i \text{ even},$$

$$\tfrac{1}{2}[3,3,1,1], (3+\sqrt{2})\, \overline{2a_1, 2a_2, 2a_3+1, 2a_4+1}, \ \sum_1^4 a_i \text{ odd},$$

$$-\tfrac{1}{2}[3,3,1,1], (3+\sqrt{2})\, \overline{2a_1, 2a_2, 2a_3+1, 2a_4+1}, \ \sum_1^4 a_i \text{ even}.$$

By taking in the first symbol $a_3 = a_4 = 0$ and either $a_1 = a_2 = 0$ or $a_1 = a_2 = 1$, in the second $a_1 = a_2 = 0$ and $a_3, a_4 = (-1, 0)$, in the third $a_1 = a_2 = 0$ and either $a_3 = a_4 = 0$ or $a_3 = a_4 = -1$ and by combining with the not disappearing immovable digits the greater permutable ones, generally affected by the sign tending to decrease the absolute value of the coordinate but — on account of the sign before $\frac{1}{2}[3, 3, 1, 1]$ of the second and the third symbol —

[1]) That one of the three symbols must remain inactive in the generation of the body gap prism is an immediate consequence of the lemma of art. 80.

with exception of one of the permutable units, we get successively the three quadruples of vertices

$$\tfrac{1}{2}[1+\sqrt{2}, -1+\sqrt{2}]\,0, 0 -, (1, -1)\,\tfrac{1}{2}[\sqrt{2}, \sqrt{2}] -, (1, -1)(\sqrt{2}, -\sqrt{2})$$

lying at minimum distance $\sqrt{6}$ from O'. These 12 points form the vertices of a prism P_O with octahedral base; each of the three quadruples just found lies in a plane passing through the axis of the prism and consists of a pair of opposite vertices of each of the two limiting octahedra. The equations of the three planes are

$$x_3 = 0, x_4 = 0 -, x_1 + x_2 = 0, x_3 = x_4 -, x + x_2 = 0, x_3 + x_4 = 0.$$

So the axis of the prism is represented by $x_3 = 0$, $x_4 = 0$, $x_1 + x_2 = 0$.

Moreover it is easily verified that the three quadrangles are rectangles with sides $2\sqrt{2}$ and 4. As we can unite the second and third symbols the P_O can be represented by the two symbols $\frac{1}{2}[1+\sqrt{2}, -1+\sqrt{2}]\,0, 0$ and $(1, -1)[\sqrt{2}, \sqrt{2}]$.

Vertex gap polytope. Finally the three net symbols are, in the simplest form,

$$[4,2,0,0], (6+2\sqrt{2})\ \overline{a_1 \quad , a_2 \quad , a_3 \quad , a_4 \quad}, \sum_1^4 a_i \text{ odd},$$

$$\tfrac{1}{2}[3,3,1,1], (3+\sqrt{2})\ \overline{2a_1+1, 2a_2+1, 2a_3+1, 2a_4+1}, \sum_1^4 a_i \text{ even},$$

$$-\tfrac{1}{2}[3,3,1,1], (3+\sqrt{2})\ \overline{2a_1+1, 2a_2+1, 2a_3+1, 2a_4+1}, \sum_1^4 a_i \text{ odd}.$$

By taking for a_1, a_2, a_3, a_4 in the first symbol $[1, 0, 0, 0]$, in the second either $0, 0, 0, 0$ or $(-1, -1, 0, 0)$ or $-1, -1, -1, -1$, in the third either $(-1, 0, 0, 0)$ or $(-1, -1, -1, 0)$, and by assigning to the permutable digits the sign which decreases the absolute value of the coordinate, we find the three sets of 48 points represented by the symbols

$$[2+\sqrt{2}, 2, 0, 0], \tfrac{1}{2}[2+\sqrt{2}, 2+\sqrt{2}, \sqrt{2}, \sqrt{2}], \ -\tfrac{1}{2}[2+\sqrt{2}, 2+\sqrt{2}, \sqrt{2}, \sqrt{2}],$$

which can be reduced to

$$[2+2\sqrt{2}, 2, 0, 0], [2+\sqrt{2}, 2+\sqrt{2}, \sqrt{2}, \sqrt{2}].$$

These 144 points prove to be the vertices of the polytope $e_3\,C_{24}$ with the characteristic numbers (144, 576, 672, 240).

$\boldsymbol{e_1 e_2 e_3 e_4 N(C_{16})}$. Extension number $6+3\sqrt{2}$, three net symbols

$$[6+\sqrt{2}, 4+\sqrt{2}, 2+\sqrt{2}, \sqrt{2}]\,,\ (12+6\sqrt{2})\ \overline{a_1 \quad, a_2 \quad, a_3 \quad, a_4}\ ,\ \sum_1^4 a_i \text{ even},$$

$$\left.\begin{array}{l}[6+2\sqrt{2},\ 4\ ,\ 2\ ,\ 0] \\ -\tfrac{1}{2}[6+\sqrt{2}, 4+\sqrt{2}, 2+\sqrt{2}, \sqrt{2}]\end{array}\right\},\ (6+3\sqrt{2})\ \overline{2a_1+1, 2a_2+1, 2a_3+1, 2a_4+1}, \sum_1^4 a_i \text{ odd},$$

$$\left.\begin{array}{l}[6+2\sqrt{2},\ 4\ ,\ 2\ ,\ 0] \\ \tfrac{1}{2}[6+\sqrt{2}, 4+\sqrt{2}, 2+\sqrt{2}, \sqrt{2}]\end{array}\right\},\ (6+3\sqrt{2})\ \overline{2a_1+1, 2a_2+1, 2a_3+1, 2a_4+1}, \sum_1^4 a_i \text{ even},$$

which are to be reduced to the new origins, indicated in the preceding example $e_1 e_4 N(C_{16})$. But in the case of the body gap we will mention only the first net symbol and the lower part of the third, which lead to the desired result.

Body gap prism. We find

$$[6+\sqrt{2}, 4+\sqrt{2}, 2+\sqrt{2}, \sqrt{2}], (6+3\sqrt{2})\ \overline{2a_1-\tfrac{1}{2}, 2a_2-\tfrac{1}{2}, 2a_3-\tfrac{1}{2}, 2a_4-\tfrac{1}{2}}, \sum_1^4 a_i \text{ even},$$

$$\tfrac{1}{2}[6+\sqrt{2}, 4+\sqrt{2}, 2+\sqrt{2}, \sqrt{2}], (6+3\sqrt{2})\, 2a_1+\tfrac{1}{2}, 2a_2+\tfrac{1}{2}. 2a_3+\tfrac{1}{2}, 2a_4+\tfrac{1}{2}, \sum_1^4 a_i \text{ even},$$

giving by means of the suppositions of the preceding example the prism P_{tO}, the two bases of which are

$$(3-\tfrac{1}{2}\sqrt{2}, 1-\tfrac{1}{2}\sqrt{2}, -1-\tfrac{1}{2}\sqrt{2}, -3-\tfrac{1}{2}\sqrt{2}),$$
$$(3+\tfrac{1}{2}\sqrt{2}, 1+\tfrac{1}{2}\sqrt{2}, -1+\tfrac{1}{2}\sqrt{2}, -3+\tfrac{1}{2}\sqrt{2}).$$

Face gap prismotope. Here we have

$$[6+\sqrt{2}, 4+\sqrt{2}, 2+\sqrt{2}, \sqrt{2}]\,,\ (6+3\sqrt{2})\ \overline{2a_1-\tfrac{2}{3}, 2a_2-\tfrac{2}{3}, 2a_3-\tfrac{2}{3}, 2a_4}\ ,\ \sum_1^4 a_i \text{ even},$$

$$\left.\begin{array}{l}[6+2\sqrt{2},\ 4\ ,\ 2\ ,\ 0] \\ -\tfrac{1}{2}[6+\sqrt{2}, 4+\sqrt{2}, 2+\sqrt{2}, \sqrt{2}]\end{array}\right\},\ (6+3\sqrt{2})\ \overline{2a_1+\tfrac{1}{3}, 2a_2+\tfrac{1}{3}, 2a_3+\tfrac{1}{3}, 2a_4+1}, \sum_1^4 a_i \text{ odd},$$

$$\left.\begin{array}{l}[6+2\sqrt{2},\ 4\ ,\ 2\ ,\ 0] \\ \tfrac{1}{2}[6+\sqrt{2}, 4+\sqrt{2}, 2+\sqrt{2}, \sqrt{2}]\end{array}\right\},\ (6+3\sqrt{2})\ \overline{2a_1+\tfrac{1}{3}, 2a_2+\tfrac{1}{3}, 2a_3+\tfrac{1}{3}, 2a_4+1}, \sum_1^4 a_i \text{ even},$$

giving by means of the suitable substitutions easily found successively

$$\begin{array}{llll}(2-\sqrt{2}, & -\sqrt{2}, & -2-\sqrt{2}) & [\sqrt{2}], \\ (2+\sqrt{2}, & \sqrt{2}, & -2+\sqrt{2}) & -\sqrt{2}, \\ (\ 2\ , & 0\ , & 2\ ) & -2\sqrt{2}, \\ (2+\sqrt{2}, & \sqrt{2}, & -2+\sqrt{2}) & \sqrt{2}, \\ (\ 2\ , & 0\ , & 2\ ) & 2\sqrt{2},\end{array}$$

which can be combined to

$$(2-\sqrt{2}, -\sqrt{2}, -2-\sqrt{2})[\sqrt{2}]-, (2+\sqrt{2}, \sqrt{2}, -2+\sqrt{2})[\sqrt{2}]-, (2, 0, -2)[2\sqrt{2}],$$

representing together a prismotope $[6; 6]$.

Edge gap prism. We get

$$[6+\sqrt{2}, 4+\sqrt{2}, 2+\sqrt{2}, \sqrt{2}], \ (6+3\sqrt{2})\ \overline{2a_1-1, 2a_2-1, 2a_3 \quad , 2a_4 \quad}, \ \sum_1^4 a_i \text{ even},$$

$$\left.\begin{matrix}[6+2\sqrt{2}, & 4 & , & 2 & , & 0] \\ -\tfrac{1}{2}[6+\sqrt{2}, & 4+\sqrt{2}, & & 2+\sqrt{2}, & & \sqrt{2}]\end{matrix}\right\}, \ (6+3\sqrt{2})\ \overline{2a_1 \quad , 2a_2 \quad , 2a_3+1, 2a_4+1}, \ \sum_1^4 a_i \text{ odd},$$

$$\left.\begin{matrix}[6+2\sqrt{2}, & 4 & , & 2 & , & 0] \\ \tfrac{1}{2}[6+\sqrt{2}, & 4+\sqrt{2}, & & 2+\sqrt{2}, & & \sqrt{2}]\end{matrix}\right\}, \ (6+3\sqrt{2})\ \overline{2a_1 \quad , 2a_2 \quad , 2a_3+1, 2a_4+1}, \ \sum_1^4 a_i \text{ even},$$

giving by means of the suitable substitutions

$$\begin{array}{ll}
\tfrac{1}{2}[2+2\sqrt{2}, 2\sqrt{2}] & [2+\sqrt{2}, \sqrt{2}], \\
[\ 2 \ , \ 0] & -\tfrac{1}{2}[2+3\sqrt{2}, \sqrt{2}], \\
\tfrac{1}{2}[2+\sqrt{2}, \sqrt{2}] & -\tfrac{1}{2}[2+2\sqrt{2}, 2\sqrt{2}], \\
[\ 2 \ , \ 0] & \tfrac{1}{2}[2+3\sqrt{2}, \sqrt{2}], \\
\tfrac{1}{2}[2+\sqrt{2}, \sqrt{2}] & \tfrac{1}{2}[2+2\sqrt{2}, 2\sqrt{2}],
\end{array}$$

which can be combined into

$$\tfrac{1}{2}[2+2\sqrt{2}, 2\sqrt{2}]\,[2+\sqrt{2}, \sqrt{2}]-, \ [2, 0]\,[2+3\sqrt{2}, \sqrt{2}]-, \ \tfrac{1}{2}[2+\sqrt{2}, \sqrt{2},]\,[2+2\sqrt{2}, 2\sqrt{2}],$$

representing together the 96 vertices of a P_{tCO}. For the transformation

$$\left.\begin{matrix} x_1+x_2=y_1\sqrt{2} \\ x_1-x_2=y_2\sqrt{2}\end{matrix}\right\}, \qquad \left.\begin{matrix} x_3+x_4=y_3\sqrt{2} \\ x_3-x_4=y_4\sqrt{2}\end{matrix}\right\}$$

gives immediately

$$y_2=[\sqrt{2}] \quad , \quad y_1, y_3, y_4=[4+\sqrt{2}, 2+\sqrt{2}, \sqrt{2}].$$

Vertex gap polytope. Finally we have to deal with

$$[6+\sqrt{2}, 4+\sqrt{2}, 2+\sqrt{2}, \sqrt{2}], \ (12+6\sqrt{2})\ \overline{a_1 \quad , a_2 \quad , a_3 \quad , a_4 \quad}, \ \sum_1^4 a_i \text{ odd},$$

$$\left.\begin{matrix}[6+2\sqrt{2}, & 4 & , & 2 & , & 0] \\ -\tfrac{1}{2}[6+\sqrt{2}, & 4+\sqrt{2}, & & 2+\sqrt{2}, & & \sqrt{2}]\end{matrix}\right\}, \ (6+3\sqrt{2})\ \overline{2a_1+1, 2a_2+1, 2a_3+1, 2a_4+1}, \ \sum_1^4 a_i \text{ even},$$

$$\left.\begin{matrix}[6+2\sqrt{2}, & 4 & , & 2 & , & 0] \\ \tfrac{1}{2}[6+\sqrt{2}, & 4+\sqrt{2}, & & 2+\sqrt{2}, & & \sqrt{2}]\end{matrix}\right\}, \ (6+3\sqrt{2})\ \overline{2a_1+1, 2a_2+1, 2a_3+1, 2a_4+1}, \ \sum_1^4 a_i \text{ odd},$$

giving by adequate substitutions

$$\begin{array}{r}
[6+5\sqrt{2}, 4+\sqrt{2}, 2+\sqrt{2}, \sqrt{2}], \\
\tfrac{1}{2}[6+3\sqrt{2}, 4+3\sqrt{2}, 2+3\sqrt{2}, \sqrt{2}], \\
\tfrac{1}{2}[6+4\sqrt{2}, 4+2\sqrt{2}, 2+2\sqrt{2}, 2\sqrt{2}], \\
-\tfrac{1}{2}[6+3\sqrt{2}, 4+3\sqrt{2}, 2+3\sqrt{2}, \sqrt{2}], \\
-\tfrac{1}{2}[6+4\sqrt{2}, 4+2\sqrt{2}, 2+2\sqrt{2}; 2\sqrt{2}],
\end{array}$$

i. e.

$$[6+5\sqrt{2}, 4+\sqrt{2}, 2+\sqrt{2}, \sqrt{2}]-, [6+3\sqrt{2}, 4+3\sqrt{2}, 2+3\sqrt{2}, \sqrt{2}]-, [6+4\sqrt{2}, 4+2\sqrt{2}, 2+2\sqrt{2}, 2\sqrt{2}],$$

representing together the 1152 vertices of the polytope $e_1\, e_2\, e_3\, C_{24}$.

Case $\boldsymbol{ce_1 e_2 e_3 e_4 N}(\boldsymbol{C}_{16})$. Extension number $5+3\sqrt{2}$, three net symbols

$$[4+\sqrt{2}, 4+\sqrt{2}, 2+\sqrt{2}, \sqrt{2}], (10+6\sqrt{2})\ \overline{a_1, a_2, a_3, a_4}, \sum_1^4 a_i \text{ even},$$

$$\left.\begin{matrix} -\frac{1}{2}[5+2\sqrt{2}, 3, 1, 1] \\ -\frac{1}{2}[5+\sqrt{2}, 3+\sqrt{2}, 1+\sqrt{2}, 1+\sqrt{2}] \end{matrix}\right\}, (5+3\sqrt{2})\ \overline{2a_1+1, 2a_2+1, 2a_3+1, 2a_4+1}, \sum_1^4 a_i \text{ odd},$$

$$\left.\begin{matrix} \frac{1}{2}[5+2\sqrt{2}, 3, 1, 1] \\ \frac{1}{2}[5+\sqrt{2}, 3+\sqrt{2}, 1+\sqrt{2}, 1+\sqrt{2}] \end{matrix}\right\}, (5+3\sqrt{2})\ \overline{2a_1+1, 2a_2+1, 2a_3+1, 2a_4+1}, \sum_1^4 a_i \text{ even},$$

which are to be reduced to the new origins, to be formed according to the indications of the preceding example. Here the polytope of edge import is lacking. In the case of the body gap we mention only the first net symbol and the lower part of the third, which lead to the desired result.

Body gap prism. We find

$$[4+\sqrt{2}, 4+\sqrt{2}, 2+\sqrt{2}, \sqrt{2}], (5+3\sqrt{2})\ \overline{2a_1-\tfrac{1}{2}, 2a_2-\tfrac{1}{2}, 2a_3-\tfrac{1}{2}, 2a_4-\tfrac{1}{2}}, \sum_1^4 a_i \text{ even},$$

$$\tfrac{1}{2}[5+\sqrt{2}, 3+\sqrt{2}, 1+\sqrt{2}, 1+\sqrt{2}], (5+3\sqrt{2})\ \overline{2a_1+\tfrac{1}{2}, 2a_2+\tfrac{1}{2}, 2a_3+\tfrac{1}{2}, 2a_4+\tfrac{1}{2}}, \sum_1^4 a_i \text{ even},$$

giving by means of the substitutions $a_i = 0, (i = 1, 2, 3, 4)$, the prism P_{tT}, the two bases of which are

$$\left(\frac{3-\sqrt{2}}{2}, \frac{3-\sqrt{2}}{2}, \frac{-1-\sqrt{2}}{2}, \frac{-5-\sqrt{2}}{2}\right),$$
$$\left(\frac{3+\sqrt{2}}{2}, \frac{3+\sqrt{2}}{2}, \frac{-1+\sqrt{2}}{2}, \frac{-5+\sqrt{2}}{2}\right),$$

Face gap prismotope. Here we have

$$[4+\sqrt{2}, 4+\sqrt{2}, 2+\sqrt{2}, \sqrt{2}], (5+3\sqrt{2})\ \overline{2a_1-\tfrac{2}{3}, 2a_2-\tfrac{2}{3}, 2a_3-\tfrac{2}{3}, 2a_4}, \sum_1^4 a_i \text{ even},$$

$$\left.\begin{matrix} -\frac{1}{2}[5+2\sqrt{2}, 3, 1, 1] \\ -\frac{1}{2}[5+\sqrt{2}, 3+\sqrt{2}, 1+\sqrt{2}, 1+\sqrt{2}] \end{matrix}\right\}, (5+3\sqrt{2})\ \overline{2a_1+\tfrac{1}{3}, 2a_2+\tfrac{1}{3}, 2a_3+\tfrac{1}{3}, 2a_4+1}, \sum_1^4 a_i \text{ odd},$$

$$\left.\begin{matrix} \frac{1}{2}[5+2\sqrt{2}, 3, 1, 1] \\ \frac{1}{2}[5+\sqrt{2}, 3+\sqrt{2}, 1+\sqrt{2}. 1+\sqrt{2}] \end{matrix}\right\}, (5+3\sqrt{2})\ \overline{2a_1+\tfrac{1}{3}, 2a_2+\tfrac{1}{3}, 2a_3+\tfrac{1}{3}, 2a_4+1}, \sum_1^4 a_i \text{ even},$$

giving by means of the suitable substitutions easily found

$$\begin{array}{ll}
(\tfrac{2}{3}-\sqrt{2}, \tfrac{2}{3}-\sqrt{2}, -\tfrac{4}{3}-\sqrt{2}) & [\sqrt{2}], \\
(\tfrac{2}{3}+\sqrt{2}, \tfrac{2}{3}+\sqrt{2}, -\tfrac{4}{3}+\sqrt{2}) & -\sqrt{2}, \\
(\tfrac{2}{3}, \tfrac{2}{3}, -\tfrac{4}{3}) & -2\sqrt{2}, \\
(\tfrac{2}{3}+\sqrt{2}, \tfrac{2}{3}+\sqrt{2}, -\tfrac{4}{3}+\sqrt{2}) & \sqrt{2}, \\
(\tfrac{2}{3}, \tfrac{2}{3}, -\tfrac{4}{3}) & 2\sqrt{2},
\end{array}$$

which can be telescoped into

$$\left(\tfrac{2}{3}-\sqrt{2}, \tfrac{2}{3}-\sqrt{2}, -\tfrac{4}{3}-\sqrt{2}\right)[\sqrt{2}]-,$$
$$\left(\tfrac{2}{3}+\sqrt{2}, \tfrac{2}{3}+\sqrt{2}, -\tfrac{4}{3}+\sqrt{2}\right)[\sqrt{2}]-, \left(\tfrac{2}{3}, \tfrac{2}{3}, -\tfrac{4}{3}\right)[2\sqrt{2}],$$

representing together the vertices of a prismotope $[6; 3]$.

Vertex gap polytope. Here we find finally

$$[4+\sqrt{2}, 4+\sqrt{2}, 2+\sqrt{2}, \sqrt{2}], \ (10+6\sqrt{2})\,\overline{a_1, a_2, a_3, a_4}, \ \sum_1^4 a_i \text{ odd},$$

$$\left.\begin{array}{l} -\tfrac{1}{2}[5+2\sqrt{2}, 3, 1, 1] \\ -\tfrac{1}{2}[5+\sqrt{2}, 3+\sqrt{2}, 1+\sqrt{2}, 1+\sqrt{2}] \end{array}\right\}, \ (5+3\sqrt{2})\,\overline{2a_1+1, 2a_2+1, 2a_3+1, 2a_4+1}, \ \sum_1^4 a_i \text{ even},$$

$$\left.\begin{array}{l} \tfrac{1}{2}[5+2\sqrt{2}, 3, 1, 1] \\ \tfrac{1}{2}[5+\sqrt{2}, 3+\sqrt{2}, 1+\sqrt{2}, 1+\sqrt{2}] \end{array}\right\}, \ (5+3\sqrt{2})\,\overline{2a_1+1, 2a_2+1, 2a_3+1, 2a_4+1}, \ \sum_1^4 a_i \text{ odd},$$

giving — as it ought to do — quite the same result as in the preceding example.

84. Probably after all the indications contained in the treatment of several special cases Table VII would be quite clear by itself but for the first column of the part corresponding to the second extreme polytope and the last column but one; so we have to add a few words about these two columns. [1])

In the two special cases treated in art. 81 the vertex polytopes proved to be polytopes all the vertices of which can be represented by one symbol, i. e. polytopes of measure polytope-extraction, viz. $ce_1\, e_2\, e_3\, C_8 = e_1\, e_2\, C_{16}$. But in the five cases studied in the arts. 82, 83 we had to deal with vertex polytopes the vertices of which cannot be represented by one symbol only, i. e. with forms which do not belong to the measure polytope family. These forms were said to be derivable from the cell C_{24} by applying respectively the sets of operations $ce_1\, e_2$, e_2, e_3, $e_1\, e_1\, e_3$. Now in part F of this section will be shown, not only that *all* the forms appearing here as vertex polytopes — whether their vertices are represented by one, two or three symbols — can be deduced from cell C_{24} by applying the operations e_k and c, but also by which set of operations any required result is to be obtained. This set of operations is indicated for all possible cases in the first column of the part of Table VII corresponding to g_0. So in the second case of art. 82 we found $e_2\, C_{24}$; but as the general theory (compare Theorem LV) demands $ce_1\, e_3\, C_{24}$ which is equal to $e_2\, C_{24}$, we have inscribed $ce_1\, e_3\, C_{24}$. [2])

The remark of the second foot note of art. 78 — that several nets deduced from $N(C_{16})$ are equal to nets deduced from $N(C_{24})$ —

[1]) The very last column will be explained later on.

[2]) The deduction of the symbols contained in the Table by applying the operations e_k and c to the cell C_{24}, i. e. to $[1, 1, 0, 0]\sqrt{2}$, will be given in the last section of this memoir.

must now be generalized to this: "Every net deduced from $N(C_{16})$ can at the same time be deduced from $N(C_{24})$." Now the last column but one indicates the name of the corresponding C_{24}-net. So we have $e_3\, e_4\, N(C_{16}) = e_1\, e_4\, N(C_{24})$, etc.

We must remember that the symbols given in Table VII have to be completed by applying the transformations indicated in art. 79. Moreover we fix our attention on the particular form in which the symbols of each constituent appear. Every prismotope g_2 is decomposed as to its vertices into two or three fourdimensional prisms, one of which degenerates in some cases into a regular polygon; of the fourdimensional prisms g_3, g_1 the first is determined by its two bases, whilst the latter appears as prismotope $(4;4)$ or as a combination of prismotopes, etc.

F. *Polarity.*

85. In art. 67 we remarked that in S_n any polytope derived by means of the operations e_k with or without c from the measure polytope M_n can also be derived from the cross polytope C_{2^n}. In art. 77 we stated this result in the form of theorem LI after having demonstrated it by showing that the *total* set of symbols of coordinates of the group derived from C_{2^n} is equal to that of the group derived from M_n. We have to come back to this result once more here, in order to indicate how it depends on the laws of reciprocity and what is the general relation between the two symbols of expansion operations of the *same* polytope deduced from M_n on one hand and from C_{2^n} on the other, which couples of symbols have been given for $n = 3, 4, 5$ (compare the foot note in art. 72) in the first and the second column of Table IV.

It goes without saying that the dependence between theorem LI and the laws of reciprocity merely consists in this that the polar reciprocal polytope of a regular polytope A of S_n with respect to a concentric spherical space is an other regular polytope A' and that in this polarity the vertices, edges, faces, etc. of the one correspond to the limits $(l)_{n-1}, (l)_{n-2}, (l)_{n-3}$, etc. of the other. So we have still only to deduce the relation between the two symbols of the same polytope. This task can be performed by comparing the first two columns of Table IV with each other and by generalizing for an arbitrary n the outcome of this comparison. So for $a < b < \ldots < s < t < n-1$ we immediately deduce from Table IV the following general laws:

$$\left.\begin{array}{rl} e_a\, e_b \ldots e_s\, e_t\, e_{n-1}\, M_n = & e_{n-t-1}\, e_{n-s-1} \ldots e_{n-b-1}\, e_{n-a-1}\, e_{n-1}\, C_{2^n} \\ c\, e_a\, e_b \ldots e_s\, e_t\, e_{n-1}\, M_n = & e_{n-t-1}\, e_{n-s-1} \ldots e_{n-b-1}\, e_{n-a-1}\, C_{2^n} \\ e_a\, e_b \ldots e_s\, e_t\, M_n = & c\, e_{n-t-1}\, e_{n-s-1} \ldots e_{n-b-1}\, e_{n-a-1}\, e_{n-1}\, C_{2^n} \\ c\, e_a\, e_b \ldots e_s\, e_t\, M_n = & c\, e_{n-t-1}\, e_{n-s-1} \ldots e_{n-b-1}\, e_{n-a-1}\, C_{2^n} \end{array}\right\}.$$

The proof of these general laws can be based on the remark that each pair of polytopes forming the two members of any of these four equations admits the same symbol of coordinates; if k is the number of the symbols $e_a, e_b, \ldots, e_s, e_t$ these symbols of coordinates are successively:

$$\left.\begin{array}{r} [1+(k+1)\sqrt{2}, \overbrace{1+k\sqrt{2}}^{n-t-1}, \overbrace{1+(k-1)\sqrt{2}}^{t-s}, \ldots, \overbrace{1+\sqrt{2}}^{b-a}, \overbrace{1}^{a}\,] \\ [k+1, \overbrace{k, \ldots, k}^{n-t-1}, \overbrace{k-1, \ldots, k-1}^{t-s}, \ldots, \overbrace{1, \ldots, 1}^{b-a}, \overbrace{0, \ldots, 0}^{a}] \\ [\overbrace{1+k\sqrt{2}}^{n-t}, \overbrace{1+(k-1)\sqrt{2}}^{t-s}, \ldots, \overbrace{1+\sqrt{2}}^{b-a}, \overbrace{1}^{a}\,] \\ [k\;, \overbrace{k-1, \ldots, k-1}^{t-s}, \ldots, \overbrace{1, \ldots, 1}^{b-a}, \overbrace{0, \ldots, 0}^{a}] \end{array}\right\}.$$

By introducing the operation e_0 corresponding to the generation of the regular polytopes starting from a point and representing this point for M_n by P_0, for C_{2^n} by P_0' we can unite these four general laws in:

Theorem LIII. "The two polytopes

$$e_a\, e_b\, e_c \ldots e_r\, e_s\, e_t\, P_0 \;,\; e_{a'}\, e_{b'}\, e_{c'} \ldots e_{r'}\, e_{s'}\, e_{t'}\, P_0'$$

are equal [1]) if and only if we have generally

$$a+t' = b+s' = c+r' = \ldots = r+c' = s+b' = t+a' = n-1."$$

86. The influence of theorem LIII on the results laid down in art.^{s} 65 and 66 is evident.

By polarizing an expansion or contraction form derived from the cross polytope C_{2^n} of S_n with respect to a concentric spherical space (with ∞^{n-1} points) as polarisator we get a new polytope admitting one kind of limit $(l)_{n-1}$ and equal dispacial angles[2]), to which corresponds the inverted symbol of characteristic numbers of the original polytope, etc.

---

[1]) This theorem gives for M_n and C_{2^n} what theorem XXIII contains about the two differently orientated positions of the simplex; it holds not only for M_n and C_{2^n}, n being general, but also for the polytopes C_{120} and C_{600} of S_4 and in the same way there exists a theorem analogous to theorem XXIII for the cell C_{24} of S_4 in its two different positions with respect to the system of coordinates. We shall have to come back to this point in the fifth section of this memoir.

[2]) Compare for S_4 the foot note of art. 65.

THEOREM LIV. "Any polytope $(P)_n$ of cross polytope descent in S_n has the property that the vertices V_i adjacent to any arbitrary vertex V lie in the same space S_{n-1} normal to the line joining this vertex V to the centre O of the polytope. The system of the spaces S_{n-1} corresponding in this way to the different vertices V of $(P)_n$ include an other polytope $(P)'_n$, the reciprocal polar of $(P)_n$ with respect to a certain concentric spherical space. But in the case of the cross polytope itself these spaces pass through the centre."

This theorem is a mere transcription of theorem XL.

87. If we apply the general relations of polarity, which have led us in art. 67 to theorem XLI, to the particular case of the polarly related nets $N(C_{16})$ and $N(C_{24})$ of S_4 we get:

THEOREM LV. "If the sets of operations E and E' are complementary to each other, i. e. if E' contains the operations e_{4-k} complementary to the operations e_k of E and no other, we have

$$EN(C_{16}) = cE'e_4 N(C_{24}),\ Ee_4 N(C_{16}) = E'e_4 N(C_{24}),\ cEN(C_{16}) = cE'N(C_{24}),$$
$$cEe_4 N(C_{16}) = E'N(C_{24})\text{''}$$

An analytical proof of this theorem would require a more ample knowledge of the net symbols of the nets deduced from $N(C_{24})$ than we have at our disposal, after having nearly finished the third part of this memoir. We will therefore invert the order of ideas, i. e. we will content ourselves here by giving the analytical form of the geometric facts and use theorem LV and the last column but one of Table VII based on it in the last section of this memoir dealing with the extra regular polytopes, to facilitate and control the deduction of the polytopes and nets, deduced from C_{24}. There we shall have occasion to apply the same principle to the polarly related polytopes C_{600} and C_{120}.[1])

88. The connection between C_8, C_{16}, C_{24} according to which the $C_{24}^{(2)}$ can be split up with respect to its vertices into a $C_8^{(2)}$ and a $C_{16}^{(2\sqrt{2})}$ and with repect to its limiting spaces into a $C_8^{(2\sqrt{2})}$ and a $C_{16}^{(4)}$ leads to connections between the polytopes and the nets which cannot be deduced from polarity only. So we find:

$$C_{24} = ce_2\, C_8\ (= ce_1\, C_{16}),\ e_1\, C_{24} = e_1\, e_2\, C_{16}$$

and

$$N(C_{24}) = ce_1\, N(C_{16}) = ce_4\, N(C_{16}) = ce_3\, N(C_{24}).$$

But there is still a more striking coincidence to be indicated, viz. that the nets $e_2\, N(C_{16})$ and $e_1\, e_2\, N(C_{16})$ are respectively equal to the nets

[1]) We defer the investigation of the reciprocal nets of those given in Table VII to the paper announced in the foot note of art. 68.

$ce_1 e_3 N(C_8)$ and $ce_1 e_2 e_3 N(C_8)$, the constituent C_8 forming at the same time the g_1 of the former couple and the g_2 of the latter. We shall have occasion to profit by this coincidence in the next article.

G. *Symmetry, considerations of the theory of groups, regularity.*

89. On account of the fact that the offspring of the cross polytope is identical with that of the measure polytope, the theorems XLII and XLIII may be applied to any form of cross polytope descent.

So we have only to add a few lines with respect to the regularity and for the same reason this task has to be performed with respect to the nets deduced from $N(C_{16})$ only.

If we individualize the 31 nets of Table VII by an N bearing the rank number as subscript we can say that the nets N_1, N_{17}, N_{24} are regular and that the degree of regularity of the nets N_3, N_5 with two equal extreme constituents is known, as these nets are at the same time measure polytope nets. As moreover each of the 26 remaining nets admits faces of at least two different shapes, the degree of regularity of each of these nets is either $\frac{4}{10}$ or $\frac{3}{10}$, according to its having only one kind or more than one kind of edge. But now it is immediately clear that any net admitting a constituent g_3 has at least two different kinds of edges, as the erect edges of the fourdimensional g_3, characterized by the property that the same coordinates of the two end points differ by unity, cannot be at the same time edges of the groundform in any of its three orientations. So we have still to treat the twelve cases N_2, N_4, N_6, N_7, N_8, N_{18}, N_{19}, N_{20}, N_{21}, N_{22}, N_{23}, N_{27} forming two different groups, one of the nets N_{18}, N_{19}, N_{27} with groundforms admitting only one kind of edge and one containing the other nine not characterized by this property. Now we can decide the question with respect to any of the nets of these two groups with the least amount of trouble by means of the following general problem, where G is the groundform given in Table VII, P the pattern vertex obtained by omitting the square brackets of G, whilst Q_i and Q'_i represent the vertices of the net adjacent to P, of which Q_i are and Q'_i are not vertices of G:

"Determine the repetitions r of G (in its three orientations) with P as vertex and examine whether or not all the vertices Q_i and Q'_i are vertices of the same number of these repetitions (G included)".

The first case must present itself for the three nets N_{18}, N_{19}, N_{27}. For the groundform of each of these nets admits one kind of edge and its repetitions containing P are grouped regularly round P;

so these repetitions must be arranged in the same manner round every edge. But this decides that the arrangement of *all* the constituents round every edge is the same, as there is only one other constituent, viz g_0. So the degree of regularity of N_{18}, N_{19}, N_{27} is $\frac{2}{5}$.

In all the nine cases of the second group there are two or more different kinds of edge and the degree of regularity is $\frac{3}{10}$. From these cases we treat a couple of examples.

Example $e_1\,\mathbf{N}(C_{16})$. All the repetitions of the groundform are represented by the system of the three symbols

$$\left.\begin{array}{l} [6a_1 \quad + \mathbf{4},\ 6a_2 \quad + \mathbf{2},\ 6a_3 \quad + \mathbf{0},\ 6a_4 \quad + \mathbf{0}],\ \Sigma a_i \text{ even} \\ \frac{1}{2}[6a_1 + 3 + \mathbf{3},\ 6a_2 + 3 + \mathbf{3},\ 6a_3 + 3 + \mathbf{1},\ 6a_4 + 3 + \mathbf{1}],\ \Sigma a_i \text{ odd} \\ -\frac{1}{2}[6a_1 + 3 + \mathbf{3},\ 6a_2 + 3 + \mathbf{3},\ 6a_3 + 3 + \mathbf{1},\ 6a_4 + 3 + \mathbf{1}],\ \Sigma a_i \text{ even} \end{array}\right\}.$$

So the repetitions r with 4, 2, 0, 0 as vertex are:

$$\left.\begin{array}{rllll} [& \mathbf{4}, & \mathbf{2}, & \mathbf{0}, & \mathbf{0}] \ldots r_1 \\ [6 & -\mathbf{2},\ 6 & -\mathbf{4}, & \mathbf{0}, & \mathbf{0}] \ldots r_2 \\ \frac{1}{2}[& 3 + \mathbf{1}, & 3 - \mathbf{1}, & -6 + 3 + \mathbf{3}, & 3 - \mathbf{3}] \ldots s_1 \\ \frac{1}{2}[& 3 + \mathbf{1}, & 3 - \mathbf{1}, & 3 - \mathbf{3}, & -6 + 3 + \mathbf{3}] \ldots s_2 \\ -\frac{1}{2}[& 3 + \mathbf{1}, & 3 - \mathbf{1}, & 3 - \mathbf{3}, & 3 - \mathbf{3}] \ldots t_1 \\ -\frac{1}{2}[& 3 + \mathbf{1}, & 3 - \mathbf{1}, & -6 + 3 + \mathbf{3}, & -6 + 3 + \mathbf{3}] \ldots t_2 \end{array}\right\}$$

which may be indicated by the symbols $r_1, r_2, \ldots, t_2$. Now the adjacent vertex 2, 4, 0, 0 is vertex of the six repetitions and 4, 0, 2, 0 of r_1, s_2, t_1 only. So there are two kinds of edges and the degree of regularity is $\frac{3}{10}$.

Example $e_3\,\mathbf{N}(C_{16})$. If we telescope $[pp_1 + Q_1, pp_2 + Q_2, pp_3 + Q_3, pp_4 + Q_4]$ into $[q_1, q_2, q_3, q_4,]\ (p)\ \overline{p_1, p_2, p_3, p_4}$ the repetitions of the groundform $[2 + \sqrt{2}, \sqrt{2}, \sqrt{2}, \sqrt{2}]$ can be represented by

$$[2 + \sqrt{2},\ \sqrt{2},\ \sqrt{2},\ \sqrt{2}],\ (1 + 2\sqrt{2})\ \overline{2a_1,\ 2a_2,\ 2a_3,\ 2a_4},\ \sum_1^4 a_i \text{ even},$$

$$\left.\begin{array}{l} \frac{1}{2}[1 + 2\sqrt{2},\ 1,\ 1,\ 1] \\ \frac{1}{2}[1 + \sqrt{2},\ 1 + \sqrt{2},\ 1 + \sqrt{2},\ 1 - \sqrt{2}] \end{array}\right\},\ (1 + 2\sqrt{2})\ \overline{2a_1 + 1,\ 2a_2 + 1,\ 2a_3 + 1,\ 2a_4 + 1},\ \sum_1^4 a_i \text{ odd},$$

$$\left.\begin{array}{l} -\frac{1}{2}[1 + 2\sqrt{2},\ 1,\ 1,\ 1] \\ -\frac{1}{2}[1 + \sqrt{2},\ 1 + \sqrt{2},\ 1 + \sqrt{2},\ 1 - \sqrt{2}] \end{array}\right\},\ (1 + 2\sqrt{2})\ \overline{2a_1 + 1,\ 2a_2 + 1,\ 2a_3 + 1,\ 2a_4 + 1},\ \sum_1^4 a_i \text{ even}.$$

So the repetitions r with $2 + \sqrt{2}, \sqrt{2}, \sqrt{2}, \sqrt{2}$ as vertex are only

$$\begin{array}{l} [\mathbf{2} + \sqrt{\mathbf{2}},\ \sqrt{\mathbf{2}},\ \sqrt{\mathbf{2}},\ \sqrt{\mathbf{2}}], \\ [1 + 2\sqrt{2} + \mathbf{1} - \sqrt{\mathbf{2}},\ 1 + 2\sqrt{2} - \mathbf{1} + \sqrt{\mathbf{2}},\ 1 + 2\sqrt{2} - (\mathbf{1} + \sqrt{\mathbf{2}}),\ 1 + 2\sqrt{2} - (\mathbf{1} + \sqrt{\mathbf{2}})]. \end{array}$$

Now $\sqrt{2}, 2 + \sqrt{2}, \sqrt{2}, \sqrt{2}$ is vertex of both, whilst on the other hand $2 + \sqrt{2}, \sqrt{2}, \sqrt{2}, -\sqrt{2}$ is vertex of the first only. So two kinds of edges, degree of regularity $\frac{3}{10}$.

The very last column of Table VII contains the results.

Section IV:

Polytopes and nets derived from the half measure polytope.

A. *The symbol of coordinates.*

90. Several times we have commemorated the fact that the eight vertices of a cube can be split up into two groups of four points, the vertices of two regular tetrahedra, and that with respect to the cube the vertices of each group may be said to be non adjacent, i. e. not connected by an edge of the cube — see e. g. the introduction of section I and the foot note of art. 4. [1]) Also that the sixteen vertices of an eightcell can be split up into two groups of eight non adjacent points, the vertices of two regular sixteencells (compare e. g. art. 78). So in general the 2^n vertices of the measure polytope M_n of space S_n can be split up into two groups of 2^{n-1} non adjacent points, but the polytopes of which these groups of points are the vertices are not regular for $n > 4$. So in the case $n = 5$ there are two different kinds of limits $(l)_4$, viz. cells C_{16} forming what remains of the limiting eightcells of M_5 and simplexes $S(5)$ replacing the vanished vertices of M_5. In relation with their generation we call these new polytopes *half measure polytopes* and we investigate in this section these polytopes and the nets which can be derived from them.

In the cases $[111]$ and $[1111]$ of cube and eightcell we have represented the two half measure polytopes by the symbols $\pm\frac{1}{2}[111]$ and $\pm\frac{1}{2}[1111]$ respectively. Likewise we indicate by $\pm\frac{1}{2}[\overline{11\ldots1}^{\,n}]$ the two half measure polytopes into which M_n can be decomposed according to the vertices, where $+\frac{1}{2}[\overline{11\ldots1}^{\,n}]$ includes all the vertices of which an *even* number of coordinates is negative and $-\frac{1}{2}[\overline{11\ldots1}^{\,n}]$ all the vertices of which an *odd* number of coordinates is negative. These symbols immediately reveal a difference in character between the half measure polytopes of S_{2n} and S_{2n+1} which we will represent for short by HM_{2n} and HM_{2n+1}. In the case of HM_{2n} the polytope admits a centre of symmetry, as the reversion of the signs of all the coordinates of any vertex furnishes an other vertex of the same group; on the contrary in the case of HM_{2n+1} every vertex is

[1]) The result mentioned contains a numerical error; it ought to be replaced by

$$(\tfrac{1}{4}(7+\sqrt{2}),\ \tfrac{1}{4}(3+\sqrt{2}),\ \tfrac{1}{4}(\sqrt{2}-1),\ \tfrac{1}{4}(5+3\sqrt{2})),$$
$$(\tfrac{1}{4}(7+3\sqrt{2}),\ \tfrac{1}{4}(3-\sqrt{2}),\ \tfrac{1}{4}(1+\sqrt{2}),\ \tfrac{1}{4}(5+\sqrt{2})),$$

see „Wiskundige Opgaven", vol. XI, problem 96.

opposite to the simplex replacing the opposite vertex of the measure polytope. So in this respect HM_{2n} presents analogy to measure polytope and cross polytope, whilst HM_{2n+1} imitates the simplex.

We still remark that the case $n = 2$ is exceptional in this sense that the corresponding HM_2 is a line, i. e. a diagonal of the square, instead of a form of two dimensions; as we shall see this remark is essential in the theory of the nets derived from the half measure polytopes.

91. It is easy to prove that the half measure polytopes partake of the two properties characterizing the semiregular polytopes considered in the preceding sections, i. e. that all the vertices are of the same kind and all the edges of the same length, here $2\sqrt{2}$. Indeed we already solved incidentally in art. 47 the more general question:

"Under what circumstances will the symbols

$$\pm \tfrac{1}{2}[a_1, a_2, \ldots, a_n]$$

represent the vertices of polytopes in S_n, all the edges of which have the same length, say $2\sqrt{2}$"?

For the length $2\sqrt{2}$ of the edges the solution takes the form of

THEOREM LVI. "The symbol $\pm \tfrac{1}{2}[a_1, a_2, \ldots, a_n]$ for which

$$a_1 \geqq a_2 \geqq \ldots\ldots \geqq a_n$$

represents a polytope admitting the required properties under the conditions: $a_{n-1} = a_n = 1$ and the difference between any two unequal adjacent digits equal to 2".

So we find

in S_3 the two forms $\tfrac{1}{2}[111]$, $\tfrac{1}{2}[311]$,
„ S_4 „ four „ $\tfrac{1}{2}[1111]$, $\tfrac{1}{2}[3311]$, $\tfrac{1}{2}[3111]$, $\tfrac{1}{2}[5311]$,
„ S_5 „ eight „ $\tfrac{1}{2}[11111]$, $\tfrac{1}{2}[33311]$, $\tfrac{1}{2}[33111]$, $\tfrac{1}{2}[31111]$,
$\tfrac{1}{2}[55311]$, $\tfrac{1}{2}[53311]$, $\tfrac{1}{2}[53111]$, $\tfrac{1}{2}[75311]$,

etc., which are represented in the following table by other symbols referring to T, C_{16} and HM; these symbols will be explained later on. [1])

$$n = 3$$

$$\tfrac{1}{2}[1\,1\,1] = T = HM_3, \qquad \tfrac{1}{2}[3\,1\,1] = tT = e_2\, HM_3.$$

$$n = 4$$

$$\left.\begin{aligned}\tfrac{1}{2}[1111] &= \quad C_{16} = \quad HM_4\\ \tfrac{1}{2}[3311] &= e_1\, C_{16} = e_2\, HM_4\end{aligned}\right\}, \quad \left.\begin{aligned}\tfrac{1}{2}[3111] &= \quad ce_2\, C_{16} = \quad e_3\, HM_4\\ \tfrac{1}{2}[5311] &= ce_1\, e_2\, C_{16} = e_2\, e_3\, HM_4\end{aligned}\right\}.$$

[1]) We remark here that the symbols e before HM_n are related to the limits of M_n.

$$n = 5$$

$$\left.\begin{array}{l} \tfrac{1}{2}[1\,1\,1\,1\,1] = HM_5 \\ \tfrac{1}{2}[3\,3\,3\,1\,1] = e_2\,HM_5 \\ \tfrac{1}{2}[3\,3\,1\,1\,1] = e_3\,HM_5 \\ \tfrac{1}{2}[3\,1\,1\,1\,1] = e_4\,HM_5 \end{array}\right\}, \qquad \left.\begin{array}{l} \tfrac{1}{2}[5\,5\,3\,1\,1] = e_2\,e_3\,HM_5 \\ \tfrac{1}{2}[5\,3\,3\,1\,1] = e_2\,e_4\,HM_5 \\ \tfrac{1}{2}[5\,3\,1\,1\,1] = e_3\,e_4\,HM_5 \\ \tfrac{1}{2}[7\,5\,3\,1\,1] = e_2\,e_3\,e_4\,HM_5 \end{array}\right\}.$$

We introduce for these forms and for the corresponding ones in spaces of a higher number of dimensions the collective "half measure polytope descendent", which we abreviate to *hmpd.*

B. *The characteristic numbers.*

92. It is not difficult to determine the characteristic numbers of HM_n for a general n. For, if a_p and a'_p denote the numbers of limits $(l)_p$ of M_n and HM_n respectively, we have the relations

$$\left.\begin{array}{lcll} a'_0 & = \tfrac{1}{2}a_0 & & \\ a'_1 & = & & a_2 \\ a'_2 & = & & 4\,a_3 \\ a'_3 & = & a_3 & +\tfrac{1}{2}(n)_4\;a_0 \\ a'_4 & = & a_4 & +\tfrac{1}{2}(n)_5\;a_0 \\ \cdots & & \cdots & \cdots \\ a'_p & = & a_p & +\tfrac{1}{2}(n)_{p+1}a_0 \\ \cdots & & \cdots & \cdots \\ a'_{n-1} & = & a_{n-1} & +\tfrac{1}{2}(n)_n\;a_0 \end{array}\right\},$$

where at the right the numbers are arranged in two columns of which the first contains old, the second new limits. Indeed the process transforming M_n into HM_n — which may be called an alternate truncation — destroys half the number of vertices, all the edges, all the faces, and maintains all the other limits $(l)_3$, $(l)_4, \ldots, (l)_{n-1}$ of M_n but in an altered shape, bringing new sets of edges, faces, limiting bodies, etc. into existence. Now each face of M_n produces an edge of HM_n, each limiting body of M_n — becoming a T — produces four triangular faces of HM_n and finally in general any set of $p+1$ vertices of M_n adjacent to a vertex destroyed produces a regular simplex $S(p+1)$ forming a limit $(l)_p$ of HM_n, for $p = 4, 5, \ldots, n-1$. This accounts for all the relations given above. Now, as the characteristic numbers of M_n are given by the equation

$$a_p = (n)_p\,2^{n-p}\ ,\ (p = 0, 1, 2, \ldots, n-1),$$

we find for HM_n:

$$a'_0 = 2^{n-1}, \quad a'_1 = (n)_2\, 2^{n-2}, \quad a'_2 = (n)_3\, 2^{n-1},$$
$$a'_p = (n)_p\, 2^{n-p} + (n)_{p+1}\, 2^{n-1}, \quad p = 3, 4, \ldots, n-1.$$

Neither is it difficult to prove that the characteristic numbers a'_p satisfy the law of Euler. To that end we go back to the relations given above and transform the Eulerian expression $a'_0 - a'_1 + a'_2 - \ldots + (-1)^{n-1} a'_{n-1}$ into

$$\left[\tfrac{1}{2} a_0 - a_3 + a_4 - \ldots + (-1)^{n-1} a_n\right]$$
$$- \left[a_2 - 4a_3 + \tfrac{1}{2} a_0 \left\{(n)_4 - (n)_5 + \ldots + (-1)^n (n)_n\right\}\right],$$

of which two sums between square brackets the first contains the contributions of the first column (old elements) and the second of the second (new elements). Now we add to each of these two sums between square brackets $\frac{1}{2} a_0 - a_1 + a_2$. So we get

$$\left[a_0 - a_1 + a_2 - \ldots + (-1)^{n-1} a_n\right]$$
$$- \left[\tfrac{1}{2} a_0 - a_1 + 2a_2 - 4a_3 + \tfrac{1}{2} a_0 \left\{(n)_4 - (n)_5 + \ldots + (-1)^n (n)_n\right\}\right].$$

But as we have

$$- a_1 + 2a_2 - 4a_3 = \tfrac{1}{2} a_0 \left\{- (n)_1 + (n)_2 - (n)_3\right\}$$

the second sum disappears, as it is equal to

$$\tfrac{1}{2} a_0 \left\{1 - (n)_1 + (n)_2 - (n)_3 + \ldots + (-1)^n (n)_n\right\} = \tfrac{1}{2} a_0 (1-1)^n.$$

So we find that the Eulerian expression of HM_n is equal to that of M_n and has therefore the value 2 for n odd and the value 0 for n even, etc.

We give here the results up to $n = 8$. They are

$n = 5 \ldots$ (16, 80, 160, 120, 26),
$n = 6 \ldots$ (32, 240, 640, 640, 252, 44),
$n = 7 \ldots$ (64, 672, 2240, 2800, 1624, 532, 78),
$n = 8 \ldots$ (128, 1792, 7168, 10752, 8288, 4032, 1136, 144).

In the outset we remarked that HM_5 admits two kinds of limits $(l)_4$, viz. cells C_{16} and simplexes $S(5)$. Here we remember that in general for $n > 4$ the HM_n is bounded by two kinds of limits $(l)_{n-1}$, viz. limits HM_{n-1} forming what remains of the limits M_{n-1} of M_n and limits $S(n)$ replacing the vanished vertices of M_n. It will be useful to call the HM_{n-1} the "original", the $S(n)$ the "truncation" limits.

93. In the cases of the offspring of simplex, measure polytope, and cross polytope we have used two different methods for the determination of the characteristic numbers, one fulfilling the exigencies for $n < 6$ as far as these numbers only are concerned, an

other giving for $n > 5$ not only the characteristic numbers but also the numbers of any limit of any kind; here we will do likewise.

So in the case of the polytopes connected with HM_5 in the manner indicated in theorem LVI we have to determine:

1°. the number of vertices according to general principles,

2°. the number of edges concurring in any vertex and thereby the total number of edges,

3°. the limiting polytopes $(l)_4$, which limits reveal at the same time the limiting bodies $(l)_3$,

4°. the number of faces (by means of Euler's rule).

But before applying this method to a definite example we give some further explanation with respect to the equations of the four-dimensional spaces containing the limits $(l)_{n-1}$ of the *hmpd.* deduced from HM_n in S_n, as this will save us trouble in the exposition of the direct method.

If $\frac{1}{2}[a_1 a_2 \ldots a_n]$ is the symbol of coordinates, where the digits have been arranged in diminishing order, we consider the vertices represented by

$$\begin{array}{cc} (a_1 \; a_2 \; \ldots \; a_p) & \frac{1}{2}[a_{p+1} \; a_{p+2} \; \ldots \; a_n] \\ x_1, x_2, \ldots, x_p & x_{p+1}, x_{p+2}, \ldots, x_n \end{array}$$

lying in the space S_{n-1} represented by the equation

$$x_1 + x_2 + \ldots + x_p = a_1 + a_2 + \ldots + a_p.$$

Evidently these vertices will determine a limit $(l)_{n-1}$ of the polytope, if $(a_1 a_2 \ldots a_p)$ and $\frac{1}{2}[a_{p+1} a_{p+2} \ldots a_n]$ represent polytopes $(P)_{p-1}$ and $(P)_{n-p}$ respectively, this $(l)_{n-1}$ being then a prismotope which may be denoted by $(P_{p-1}; P_{n-p})$. Now $(a_1 a_2 \ldots a_p)$ always represents a $(P)_{p-1}$, unless all the digits $a_1 a_1, \ldots, a_p$ are equal, in which case $(a_1 a_2 \ldots a_p)$ is a petrified syllable. On the other hand $\frac{1}{2}[a_{p+1} a_{p+2} \ldots a_n]$ always represents a $(P)_{n-p}$, unless we have either $p = n - 2$, or $p = n - 1$; for, as we remarked already $p = n - 2$ gives the syllable $\frac{1}{2}[11]$, i. e. a line segment instead of a square, and $p = n - 1$ gives a vertex only instead of an edge.

To this we have to add a few words only about the extreme cases $p = 1$ and $p = n$. For $p = 1$ we find the polytope with the coordinate symbol $\frac{1}{2}[a_2 a_3 \ldots a_n]$ lying in a space S_{n-1} represented by $\pm x_i = a_1$; it can be deduced from HM_{n-1}. For $p = n$ the result is different for n even and n odd, the polytope having as HM_n itself a centre of symmetry in the first case and two different limits, either a vertex and an $(l)_{n-1}$ or two differently shaped $(l)_{n-1}$, opposite to each other in the second. Or otherwise,

as follows. For n even the diagonals of M_n split up into two groups of non adjacent ones, of those bearing vertices belonging also to HM_n and of those bearing vertices cut off by the alternate truncation leading from M_n to HM_n; the 2^{n-2} diagonals of the first group are normal to two limits of vertex import [1]) in the considered polytope, whilst the 2^{n-2} diagonals of the second are normal to two limits which may be called of truncation import as they are derived from truncation limits of HM_n in passing to the polytope under consideration. For n odd there is only one group of diagonals of M_n, each of which bears only one vertex of HM_n; so each of these diagonals is normal to two differently shaped limits of the polytope, to one limit of vertex and to one limit of truncation import. But in the two cases, of n even and n odd, we have to deal with the two equations $\Sigma \pm x_i = \Sigma a_i$ and $\Sigma \pm x_i = \Sigma a_i - 2$, the last digit $a_n = 1$ having to be taken with the positive sign for limits of vertex import, with the negative sign for limits of truncation import.

After this introduction we treat a definite example.

94. *Case* $\frac{1}{2}$ **[5 3 3 1 1]**.

The number of vertices is 2^4 times 5! divided by 2^2, i. e. 480. The vertices adjacent to the pattern vertex 5 3 3 1 1 are

	5 1 3 3 1	
3 3 5 1 1	5 3 1 3 1	5 3 3 —1 —1
3 5 3 1 1	5 1 3 1 3	
	5 3 1 1 3	

which may be indicated by the brackets and the negative sign after the two units in the symbol

5 3 3 1 1 (—)

So seven edges concur in any vertex, i. e. the total number of edges is half the product of 480 and 7, i. e. 1680.

Now we have to pass to the limiting polytopes.

The spaces S_4 represented by $\pm x_i = 5$ give $2.(5)_1 = 10$ limits $\frac{1}{2}[3\,3\,1\,1]$ of polytope import.

[1]) Also the import of the different limits $(l)_{n-1}$ of HM_n will be considered in relation with the limits $(l)_{n-1}$ of M_n. So the equations $\pm x_i = a_1$ will give limits of $(l)_{n-1}$ import, the equations $\pm x_i \pm x_j = a_1 + a_2$ will give limits of $(l)_{n-2}$ import, etc., this series ending in general in limits of body and limits of vertex import, as no edge or face of M_n partakes in the limitation of HM_n.

The spaces S_4 represented by $\pm x_i \pm x_j = 5 + 3$ give $2^2.(5)_2 =$ 40 limits (53) $\frac{1}{2}[3\ 1\ 1]$ of body import.

The spaces S_4 represented by $\Sigma \pm x_i = 13$ give 2^4 limits (5 3 3 1 1) of vertex import, where $(5\ 3\ 3\ 1\ 1) = (4\ 2\ 2\ 0\ 0)$. [1])

The spaces S_4 represented by $\Sigma \pm x_i = 11$ give 2^4 limits $(5\ 3\ 3\ 1 - 1)$ introduced by the alternate truncation.

So the limiting polytopes are

$$10\, e_1\, C_{16} + 40\, P_{tT} + 16\, e_2\, S(5) + 16\, e_1\, e_3\, S(5),$$

i. e. 82 in toto.

Now from the list of limiting bodies

$$\begin{array}{rl} e_1\, C_{16} & \ldots\ 8\, O\ ,\ 16\, tT \\ P_{tT} & \ldots\ 2\, tT\ ,\ \ 4\, P_3,\ \ 4\, P_6 \\ e_2\, S(5) & \ldots\ 5\, CO,\ 10\, P_3,\ \ 5\, O \\ e_1\, e_3\, S(5) & \ldots\ 5\, tT\ ,\ 10\, P_6,\ 10\, P_3,\ 5\, CO \end{array}$$

of the four different limiting polytopes we can deduce that our polytope is limited by

$$\tfrac{1}{2}(10 \times 8 + 16 \times 5)\, O,\ \ \tfrac{1}{2}(10 \times 16 + 40 \times 2 + 16 \times 5)\, tT,$$
$$\tfrac{1}{2}(16 \times 5 + 16 \times 5)\, CO,\ \ \tfrac{1}{2}(40 \times 4 + 16 \times 10 + 16 \times 10)\, P_3,$$
$$\tfrac{1}{2}(40 \times 4 + 16 \times 10)\, P_6$$

i. e. by 720 polyhedra, viz.

$$80\, O,\ 160\, tT,\ 80\, CO,\ 240\, P_3;\ 160\, P_6.$$

Now finally, according to Euler's rule, the number of faces is 1840. So the result is

$$(480,\ 1680,\ 1840,\ 720,\ 82).$$

This example shows that the method explained *is* sufficient for S_5, as far as the characteristic numbers themselves are concerned. But if we want to extend our knowledge of these *lmpd.* — in relation with the difficulty of realising their lopsided form — by determining the numbers of the *different kinds* of limits the method is insufficient even in S_5 and has to be completed, in one sense or other, with respect to the different kinds of edges and of faces. We shall see that the direct method, which will be explained in the next article, furnishes this complement at least expense.

95. Here once more the direct method in view is based on the

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[1]) This (42200) with edges $2\sqrt{2}$ is similar to (21100) with edges $\sqrt{2}$, i.e. to $e_2\, S(5)$. Likewise (5331–1) leads by (64420) to (32210) or (32110), i. e. to $e_1\, e_3\, S(5)$.

distinction of the different kinds of limits $(l)_p$ by what we have called formerly "unextended" symbols. If we take care to exclude always the petrified syllables we can formulate the method in:

THEOREM LVII. "We obtain the unextended symbol of a polytope $(P)_d$ the vertices of which are vertices of the given *hmpd.* of S_n by applying to the n digits of the symbol of coordinates $\frac{1}{2}[a_1\, a_2 \ldots a_{n-1}\, a_n]$ of this polytope one of the three following processes:

1°. Take the last digit a_n, first with the *positive* and afterwards with the *negative* sign, and place for both cases between pairs of round brackets either one group of $d+1$ digits, or two groups containing together $d+2$ digits, or three groups containing together $d+3$ digits, etc., omitting the digits not included.

2°. Place before $\frac{1}{2}[11]$ of the remaining digits $a_1, a_2, \ldots, a_{n-2}$ between pairs of round brackets either one group of d digits, or two groups containing together $d+1$ digits, etc., omitting the digits not included — and the syllable with one digit for $d=1$.

3°. Place before $\frac{1}{2}[a_{n-k+1}\, a_{n-k+2} \ldots a_{n-1}\, a_n]$, where $k=3, 4, \ldots, d$ successively, between pairs of round brackets either one group of $d-k+1$ of the remaining $n-k$ digits, or two groups containing together $d-k+2$ of these digits, etc., omitting the digits not included — and the syllable with one digit for $d=k$."

"In each of these cases the $(P)_d$ obtained will be a *limiting* polytope of *hmpd.*, if the syllables between round brackets satisfy the two following conditions:

a) each syllable with middle digits exhausts these digits of the symbol of the given *hmpd.*,

b) no two syllables without middle digits have the same end digits."

The proof of this theorem, forming an adaption of theorem XXX to the special character of the *hmpd.*, embodied in the $\frac{1}{2}$ before the square brackets of their symbol, can be copied from that of theorem XXX and theorem XXX'.

We apply it to two definite examples, one in $S(5)$, the other in S_6.

Case $\frac{1}{2}$ **[5 5 3 1 1]**. —

If we place before a vertical stroke the limits deduced from 55311 and after it the *different* ones furnished by 5531 — 1, we get

$(l)_1 \ldots (53)_2, (31)_2, \frac{1}{2}[11]_1$	
$(l)_2 \ldots (553)_1, (531)_4, (311)_1, (53)\frac{1}{2}[11]_2$	$(31-1)_2$
$(l)_3 \ldots (5531)_2, (5311)_2, (553)\frac{1}{2}[11]_1, \frac{1}{2}[311]_1$	$(531-1)_4$
$(l)_4 \ldots (55311)_1, \frac{1}{2}[5311]_2$	$(5531-1)_2$

where the small subscripts at the right indicate the number of limits concurring in any vertex [1]). So we find through any vertex

five edges,

two p_3, two p_4, six p_6,

one P_3, five tT, four tO,

one $(55311) = ce_1\, e_2\, S(5)$, two $(5531{-}1) = e_1\, e_2\, S(5)$,

two $\frac{1}{2}[5311] = ce_1\, e_2\, C_{16}$,

and this gives in a transparent way in toto

$$\frac{5.480}{2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{ i. e. } 1200\ (l)_1,$$

$$\frac{2.480}{3} = 320\, p_3 \quad , \frac{2.480}{4} = 240\, p_4 \quad , \frac{6.480}{6} = 480\, p_6 \ldots\ldots \quad ,, \quad 1040\ (l)_2,$$

$$\frac{480}{6} = 80\, P_3 \quad , \frac{5.480}{12} = 200\, tT \quad , \frac{4.480}{24} = 80\, tO \ldots\ldots \quad ,, \quad 360\ (l)_3,$$

$$\frac{480}{30} = 16\, ce_1\, e_2\, S(5), \frac{2.480}{60} = 16\, e_1\, e_2\, S(5), \frac{2.480}{96} = 10\, ce_1\, e_2\, C_{16} \quad ,, \quad 42\ (l)_4.$$

So the result is

$$(480,\ 1200,\ 1040,\ 360,\ 42)$$

in accordance with the law of Euler.

Case $\frac{1}{2}$ **[755311]**. —

In the same way we find here the table:

$(l)_1$	$(75)_2, (53)_2, (31)_2, \frac{1}{2}[11]_1$	
$(l)_2$	$(755)_1, (75)(53)_2, (75)(31)_4, (553)_1, (531)_4, (311)_1, (75)\frac{1}{2}[11]_2, (53)\frac{1}{2}[11]_1$	$(31{-}1)_2$
$(l)_3$	$(7553)_1, (755)(31)_2, (75)(531)_4, (75)(311)_2, (5531)_2, (5311)_2, (755)\frac{1}{2}[11]_1, (75)(53)\frac{1}{2}[11]_2, (553)\frac{1}{2}[11]_1, \frac{1}{2}[311]_1$	$(75)(31{-}1)_4, (531{-}1)_4$
$(l)_4$	$(75531)_2, (755)(311)_1, (75)(5311)_2, (55311)_1, (7553)\frac{1}{2}[11]_1, (75)\frac{1}{2}[311]_2, \frac{1}{2}[5311]_2$	$(755)(31{-}1)_2, (75)(531{-}1)_4, (5531{-}1)_2$
$(l)_5$	$(755311)_1, (755)\frac{1}{2}[311]_1, (75)\frac{1}{2}[5311]_2, \frac{1}{2}[55311]_1$	$(75531{-}1)_2$

So we find through any vertex

seven edges,

three p_3, ten p_4, six p_6,

one CO, five tT, six P_3, eight P_6, two C, four tO,

two (32110), one (22100), two (32100), four P_{tT}, four P_{tO},

one P_{CO}, one (3 ; 3), two (3 ; 6), two $\frac{1}{2}[5311]$,

one (322100), one $(p_3\,; tT)$, two $P_{\frac{1}{2}[5311]}$, one $\frac{1}{2}[55311]$,

two (432110);

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[1]) So (531) is to bear the subscript 4, as the 5 may be related either to x_2 or to x_3 and the 1 either to x_4 or to x_5; so (31—1) is to admit the subscript 2, as the three digits may apply either to $+x_3, +x_4, -x_5$ or to $+x_3, -x_4, +x_5$, etc.

as the number of vertices is $2^5.6!$ divided by 2^2, i. e. 5760, we get in toto

$$\frac{7.5760}{2} \ldots\ldots\ldots\ldots \text{i. e. } 20160(l)_1,$$

$$\frac{3.5760}{3}=5760p_3, \frac{10.5760}{4}=14400p_4, \frac{6.5760}{6}=5760p_6 \quad ,, \quad 25920(l)_2,$$

$$\frac{5760}{12}=480CO, \frac{5.5760}{12}=2400tT, \frac{6.5760}{6}=5760P_3,$$

$$\frac{8.5760}{12}=3840P_6, \frac{2.5760}{8}=1440C,$$

$$\frac{4.5760}{24}=960tO \ldots\ldots\ldots\ldots \quad ,, \quad 14880(l)_3,$$

$$\frac{2.5760}{60}=192e_1e_3S(5), \frac{5760}{30}=192\,ce_1e_2S(5),$$

$$\frac{2.5760}{60}=192e_1e_2S(5), \frac{4.5760}{24}=960P_{tT},$$

$$\frac{4.5760}{48}=480P_{tO}, \frac{5760}{24}=240P_{CO},$$

$$\frac{5760}{9}=640(3;3), \frac{2.5760}{18}=640(3;6),$$

$$\frac{2.5760}{96}=120ce_1e_2C_{16} \ldots\ldots\ldots\ldots \quad ,, \quad 3656(l)_4,$$

$$\frac{5760}{180}=32e_2e_3S(6), \frac{5760}{36}=160(p_3;tT),$$

$$\frac{2.5760}{192}=60P_{ce_1e_2C_{16}}, \frac{5760}{480}=12\,e_2e_3HM_5,$$

$$\frac{2.5760}{360}=32\,e_1e_2e_4S(6) \ldots\ldots\ldots\ldots \quad ,, \quad 296(l)_5.$$

So the result is, in accordance with the law of Euler,

$$(5760,\ 20160,\ 25920,\ 14880,\ 3656,\ 296).$$

The results obtained in this way are tabulated in Tables VIII and IX.

Table VIII, concerned with the *hmpd.* in S_3, S_4, S_5, has been divided vertically into six main parts, giving respectively the expansion symbol, the symbol of coordinates, the symbol of characteristic numbers, the faces, the limiting polyhedra, the limiting polytopes. The part of the faces is split up into three columns successively related to triangular, square, hexagonal faces; likewise that of the limiting bodies is split up into seven columns corresponding to the seven possibilities T, O, P_3, tT, CO, P_6, tO. Of the two numbers given in any case the first always indicates the total

number of the limits, the second that of the limits concurring in any vertex. But in the sixth part, making its appearance for $n = 5$, the arrangement is an other one, the character of the limiting polytopes and their total number having interchanged places; so in any case the total number appears at the head of the column and the character at the first of the two horizontal places in the column. So the polytope $e_4 HM_5 = \frac{1}{2}[31111]$ with the characteristic numbers 80, 400, 720, 480, 82 is limited by $480 p_3$ and $240 p_4$ of which 18 and 12 respectively meet in any vertex, by $240 T$ and $240 P_3$ of which 12 and 18 respectively meet in any vertex, and by $10 C_{16}$, $40 P_T$, $16 S(5)$ and $16 e_3 S(5)$ of which 1, 4, 1 and 4 respectively meet in any vertex.

Table IX, concerned with the *hmpd.* in S_6, has been divided in the same way into seven main parts. It will be clear without farther explanation; only we are bound to add that in the first column of the sixth part 2 „ means that $2 C_{16}$ is to be taken 60 times and that in this part and the next the numbers of limits concurring at any vertex have been omitted.

96. We insert a few remarks about the character of the limits.

Faces. We find only p_3, p_4, p_6.

Limiting bodies. The set of limiting bodies obtained for $n = 5$ is completed by the addition of C for $n = 6$.

Limiting polytopes. In general the limiting polytopes are

1°. Simplex forms, deduced from $S(n)$, $S(n-1)$, ..., $S(3)$,

2°. Half measure polytope forms, deduced from HM_{n-1}, HM_{n-2}, ..., HM_5,

3°. Prismotopes the constituents of which are simplex forms, deduced from $S(n-1)$, ..., $S(3)$, and at most one half measure polytope form, deduced from HM_{n-2}, ..., HM_5.

This general result shows that the list of limiting bodies is complete for $n = 6$. Moreover that the list of fourdimensional limits will be complete for $n = 8$, as the case $n = 8$ brings C_8 for the first time, etc.

In order to show how theorem LVII works we give the list of the limits $(P)_6$ of the tendimensional form $\frac{1}{2}$ **[9775533311]**: (9775533), — (9775) (5333), (9775) (533) (31), — (977) (55333), (977)(5533)(31),(977)(553)(331),—(97)(755333),(97)(75533)(31), (97)(7553)(331),(97)(755)(3331),(97)(75)(53331),(97)(75)(533)(311), (97)(75)(533)(31—1),(97)(75)(53)(3311),(97)(75)(53)(331—1),— (7755333), — (775533) (31), — (77553) (331), — (7755) (3331), — (775)(53331),(775)(533)(311),(775)(533)(31—1),(775)(53)(3311),

(775)(53)(331—1),—(7553331),—(75533)(311),(75533)(31—1),—(7533) (3311), (7553) (331—1), (755) (33311), (755) (3331—1), (75)(533311), (75)(53331—1),—(5533311),—(553331—1),——(977553) $\frac{1}{2}$[11], — (9775) (533) $\frac{1}{2}$[11], — (977) (5533) $\frac{1}{2}$[11], — (97) (75533) $\frac{1}{2}$[11], (97) (75) (5333) $\frac{1}{2}$[11], — (775533)$\frac{1}{2}$[11], — (775) (5333) $\frac{1}{2}$[11], — (755333) $\frac{1}{2}$[11], —— (9775) $\frac{1}{2}$[311], — (97) (755) $\frac{1}{2}$[311], (97) (75) (53) $\frac{1}{2}$[311], — (7755) $\frac{1}{2}$[311], — (775) (53) $\frac{1}{2}$[311], — (7553) $\frac{1}{2}$[311], — (75) (533) $\frac{1}{2}$[311], —— (977)$\frac{1}{2}$[3311],—(97)(75)$\frac{1}{2}$[3311],—(775)$\frac{1}{2}$[3311],—(755)$\frac{1}{2}$[3311],—(75)(53)$\frac{1}{2}$[3311],—(553)$\frac{1}{2}$[3311],——(97)$\frac{1}{2}$[33311],—(75)$\frac{1}{2}$[33311], —— $\frac{1}{2}$[533311].

C. *Extension number and truncation fractions.*

97. THEOREM LVIII. "The polytopes $\frac{1}{2}[a_1 a_2 \ldots a_{n-1} a_n]$ of S_n, all with edges $2\sqrt{2}$, can be found by means of a regular extension of the measure polytope $M_n^{(2)}$ followed by a regular truncation at the two groups of non adjacent vertices of M_n, either with or without truncation at the limiting $(l)_3$, or at the limiting $(l)_3$ and $(l)_4$, or at the limiting $(l)_3$, $(l)_4$ and $(l)_5$, etc. or at the limiting $(l)_3$, $(l)_4$, $(l)_5$, etc. and $(l)_{n-2}$."

This theorem is an immediate consequence of the character of the equations of the spaces S_{n-1} bearing the limits $(l)_{n-1}$ of the *hmpd.*

The extension number is once more the largest digit of the symbol of coordinates, i. e. a_1; so here it is always *odd.*

On account of the lopsidedness of the *hmpd.* we measure the amount of truncation on the corresponding *half* diameter limited at the centre O of the polytope. So in the case $\frac{1}{2}[775533311]$ the truncation corresponding to the space S_8 with the equation $\sum\limits_1^5 x_i = 27$, i. e. the truncation at the limits M_4 of M_9 extended, is $\frac{PQ}{PO}$, if P is the centre of the M_4 and Q the point of intersection of OP and the indicated space S_8. As $\sum\limits_1^5 x_i$ is 35 for M_9 extended we find $\frac{PQ}{PO} = \frac{PO - QO}{PO} = \frac{35-27}{35} = \frac{8}{35}$. So the truncation fraction is $\frac{8}{35}$ in this case.

This case shows clearly that *in general* the fraction number admits as denominator the product of the extension number by the number of coordinates figuring in the equation of the truncating space. So reducing this denominator to the extension number the numerator

itself becomes in general a fraction. Therefore it is impossible to introduce here the notion of truncation integer.

The following list contains the truncation fractions for the *hmpd.* in S_3, S_4, S_5, S_6; here τ_0, τ_0', τ_3, τ_4 represent successively the two truncations at the vertices and the truncations at the limits $(l)_3$, $(l)_4$.

		τ_0	τ_0'	τ_3
S_3	$\frac{1}{2}[311]$	$\frac{4}{9}$	$\frac{2}{3}$	
S_4	$\frac{1}{2}[3311]$	$\frac{1}{3}$	$\frac{1}{2}$	
	$\frac{1}{2}[3111]$	$\frac{1}{2}$	$\frac{2}{3}$	
	$\frac{1}{2}[5311]$	$\frac{1}{2}$	$\frac{3}{5}$	
S_5	$\frac{1}{2}[33311]$	$\frac{4}{15}$	$\frac{2}{5}$	
	$\frac{1}{2}[33111]$	$\frac{2}{5}$	$\frac{8}{15}$	
	$\frac{1}{2}[31111]$	$\frac{8}{15}$	$\frac{2}{3}$	$\frac{1}{3}$
	$\frac{1}{2}[55311]$	$\frac{2}{5}$	$\frac{12}{25}$	
	$\frac{1}{2}[53311]$	$\frac{12}{25}$	$\frac{14}{25}$	$\frac{1}{5}$
	$\frac{1}{2}[53111]$	$\frac{14}{25}$	$\frac{16}{25}$	$\frac{1}{5}$
	$\frac{1}{2}[75311]$	$\frac{18}{35}$	$\frac{4}{7}$	$\frac{1}{7}$

		τ_0	τ_0'	τ_3	τ_4
S_6	$\frac{1}{2}[333311]$	$\frac{2}{9}$	$\frac{1}{3}$		
	$\frac{1}{2}[333111]$	$\frac{1}{3}$	$\frac{4}{9}$		
	$\frac{1}{2}[331111]$	$\frac{4}{9}$	$\frac{5}{9}$	$\frac{2}{9}$	
	$\frac{1}{2}[311111]$	$\frac{5}{9}$	$\frac{2}{3}$	$\frac{4}{9}$	$\frac{1}{3}$
	$\frac{1}{2}[555311]$	$\frac{1}{3}$	$\frac{2}{5}$		
	$\frac{1}{2}[553311]$	$\frac{2}{5}$	$\frac{7}{15}$	$\frac{2}{15}$	
	$\frac{1}{2}[533311]$	$\frac{7}{15}$	$\frac{8}{15}$	$\frac{4}{15}$	$\frac{1}{5}$
	$\frac{1}{2}[553111]$	$\frac{7}{15}$	$\frac{8}{15}$	$\frac{2}{15}$	
	$\frac{1}{2}[533111]$	$\frac{8}{15}$	$\frac{3}{5}$	$\frac{4}{15}$	$\frac{1}{5}$
	$\frac{1}{2}[531111]$	$\frac{3}{5}$	$\frac{2}{3}$	$\frac{2}{5}$	$\frac{1}{5}$
	$\frac{1}{2}[775311]$	$\frac{3}{7}$	$\frac{10}{21}$	$\frac{2}{21}$	
	$\frac{1}{2}[755311]$	$\frac{10}{21}$	$\frac{11}{21}$	$\frac{4}{21}$	$\frac{1}{7}$
	$\frac{1}{2}[753311]$	$\frac{11}{21}$	$\frac{4}{7}$	$\frac{2}{7}$	$\frac{1}{7}$
	$\frac{1}{2}[753111]$	$\frac{4}{7}$	$\frac{13}{21}$	$\frac{2}{7}$	$\frac{1}{7}$
	$\frac{1}{2}[975311]$	$\frac{14}{27}$	$\frac{5}{9}$	$\frac{2}{9}$	$\frac{1}{9}$

D. *Expansion and contraction symbols.*

98. For $k = 2, 3, \ldots, n-2, n-1$ any limit $M_k^{(2)}$ of the $M_n^{(2)}$ from which the $HM_n(2\sqrt{2})$ has been deduced bears a limit of $HM_n(2\sqrt{2})$, this limit $\frac{1}{2}[\overbrace{11\ldots1}^{k}]$ being a $HM_k(2\sqrt{2})$ and therefore an $(l)_k$ for $k = 3, 4, \ldots, n-1$, but an edge for $k = 2$. Now we will define the expansion e_k of $HM_n(2\sqrt{2})$ — for $k = 2, 3, \ldots, n-2$ — as the influence of the motion of the limits $\frac{1}{2}[\overbrace{11\ldots1}^{k}]$ contained in the limits $M_k^{(2)}$ of $M_n^{(2)}$ caused by a translational motion of these limits $M_k^{(2)}$ and what they contain, to equal distances away from the centre O of $M_n^{(2)}$, each $M_k^{(2)}$ moving in the direction of the line OM joining O to its centre M, these $M_k^{(2)}$ remaining equipollent to their original position, the motion being extended over such a distance that the two new positions of any vertex which was common to two $M_k^{(2)}$ shall be separated by the length $2\sqrt{2}$. In order to justify this definition we have to show for what reasons we deviate here from the custom followed until now: to bring the operation e_k in relation with the limits $(l)_k$ of the polytope itself.

For the deviation indicated we have two reasons. The first is of a didactic cast: it is easier to imagine the motions of the limits of $M_n^{(2)}$ than those of $HM_n(2\sqrt{2})$. But the second is of more importance: "if the limits of $HM_n(2\sqrt{2})$ are carried away by the limits of the circumscribed $M_n^{(2)}$ which contain them, these latter limits being moved out in the ordinary way, we get precisely those expansion operations which lead to the whole set of polytopes *hmpd.* of S_n." This advantage is twofold. In the first place: the only expansion of $HM_n(2\sqrt{2})$ which has no equivalent under the e_k applied to $M_n^{(2)}$, i. e. the expansion according to the *faces,* is excluded, and this is right, for we will show afterwards that this expansion is either impossible or it leads to a polytope which can be derived from $M_n^{(4)}$. But, what is still more, by adhering to the limits of $M_n^{(2)}$ we are never at a loss with respect to the question to which group of limits of $HM_n(2\sqrt{2})$ the expansion is to be applied. So in the case of S_5 the HM_5 admits as limiting bodies tetrahedra only, but they are of two different kinds, i. e. we must distinguish between a T common to two C_{16} and a T common to a cell C_{16} and a cell Cr_5; so, of these two groups the first must undergo the operation e_3, if we wish to apply it, as a T common to two C_{16} is contained in the cube common to the two adjacent eightcells bearing the two C_{16}. Moreover we will prove afterwards that the contraction operation always leads to forms deducible from $M_n^{(4)}$; so we have to consider here the operations e_k only.

On the other hand we do not deny that the new definition has a drawback with respect to the operation of expansion according to the edges of $HM_n(2\sqrt{2})$, a difference in the notation making its appearance there. According to HM_n itself this operation ought to be called $e_1\, HM_n$; nevertheless we propose to indicate it by the symbol $e_2\, HM_n$. This is still more annoying in S_3 and S_4 where we have $e_2\, HM_3 = e_1\, T = tT$ and $e_2\, HM_4 = e_1\, C_{16}$ (see the small table at the end of art. 91). But still we reckon the advantages so prevailing that we do not mind of accepting this small disadvantage into the bargain, the more so as it is easily held under control.

Starting from the new definition we prove:

THEOREM LIX. "The expansion e_k, $(k = 2, 3, \ldots, n - 2)$, applied to $HM_n(2\sqrt{2})$ changes the symbol of coordinates $\frac{1}{2}[\overbrace{11\ldots1}^{n}]$ of that polytope into $\frac{1}{2}[\overbrace{33\ldots3}^{n-k}\,\overbrace{11\ldots1}^{k}]$."

Proof. If we move the limit $HM_k(2\sqrt{2})$ represented by

$$x_1 = x_2 = \ldots = x_{n-k} = 1,\; x_{n-k+1}, x_{n-k+2}, \ldots, x_n = \tfrac{1}{2}[\overbrace{11\ldots1}^{k}]$$

in the direction of the line joining O to its centre M, for which

$$x_1 = x_2 = \ldots = x_{n-k} = 1,\ x_{n-k+1} = x_{n-k+2} = \ldots = x_n = 0,$$

to a λ times larger distance from O we get a new position of this $HM_k^{(2\sqrt{2})}$ characterized by

$$x_1 = x_2 = \ldots = x_{n-k} = \lambda,\ x_{n-k+1}, x_{n-k+2}, \ldots, x_n = \tfrac{1}{2}[\overbrace{11\ldots1}^{k}],$$

in which it is a limit $HM_k^{(2\sqrt{2})}$ of the new polytope $\frac{1}{2}[\overbrace{\lambda\lambda\ldots\lambda}^{n-k}\overbrace{11\ldots1}^{k}]$. According to theorem LVI this new polytope has edges of the same length if and only if we put $\lambda = 3$. This proves the theorem and leads moreover to the result:

Theorem LX. "In the expansion e_k the limits $HM_k^{(2\sqrt{2})}$ of $HM_n^{(2\sqrt{2})}$ are moved away from the centre to a distance always three times the original distance."

This result is also an immediate consequence of the fact that the largest digit 3 of the symbol of the new polytope is the extension number.

Remark. We may express the influence of the operation e_k on the symbol $\frac{1}{2}[\overbrace{11\ldots1}^{n}]$ by saying that it creates an interval 2 between the $n-k^{th}$ and the $n-k+1^{st}$ digit. This is in accordance with the remark inserted at the end of art. 58. In moving out the limits M_k of M_n the distance to be described in order to give the new edges a length $2\sqrt{2}$ is $\sqrt{2}$ times the distance to be described in order to give these edges a length 2; so the interval created which was $\sqrt{2}$ in the case of $M_n^{(2)}$ must be $\sqrt{2}$ times $\sqrt{2}$, i. e. 2 in the case of $HM_n^{(2\sqrt{2})}$.

Theorem LXI. "The influence of any number of expansions $e_k, e_l, e_m, \ldots$ of $HM_n^{(2\sqrt{2})}$ on its symbol $\frac{1}{2}[\overbrace{11\ldots1}^{n}]$ is found by adding together the influences of each of the expansions taken separately.

The proof of this theorem can be copied from art. 59. It leads immediately to:

Theorem LXII. "The operation e_k can still be applied to any expansion form deduced from $HM^{(2\sqrt{2})}$ in the symbol of coordinates of which the $n-k^{th}$ and the $n-k+1^{st}$ digits, i. e. the k^{th} and the $k+1^{st}$ digits counted from the end, are equal."

So in the case $\frac{1}{2}[9775533311]$ we have an $e_2 e_5 e_7 e_9 HM^2_{10}$.

99. We have to come back to the face expansion of the *hmpd.* and to their contraction.

The faces of the polytope $\frac{1}{2}[a_1 a_2 \ldots a_{n-2}\, 1\ 1]$ replacing the faces

(11—1) of $HM_n^{(2\sqrt{2})}$ are represented by (31—1) for $a_{n-2}=3$ and by (11—1) for $a_{n-2}=1$. So we treat these two cases together by considering the face.

$$x_i = a_i\ (i = 1, 2, \ldots, n-3)\ , \qquad x_{n-2}, x_{n-1}, x_n = (a_{n-2}\ 1 — 1)$$

with the centre

$$x_i = a_i\ (i = 1, 2, \ldots, n-3)\ , \quad 3\,x_{n-2} = 3\,x_{n-1} = 3\,x_n = a_{n-2}.$$

By moving this face away from the centre O to a distance λ times as large its centre is transported to the point

$$x_i = \lambda a_i\ (i = 1, 2, \ldots, n-3)\ , \ 3\,x_{n-2} = 3\,x_{n-1} = 3\,x_n = \lambda a_{n-2}.$$

So the new position of the face leads to a new polytope $\frac{1}{2}[\lambda a_1, \lambda a_2, \ldots \lambda a_{n-3}, \ldots]$. As the length $2\sqrt{2}$ of the sides of this face is maintained and the length of the edge $(\lambda a_k, \lambda a_{k+1})$ is $2\lambda\sqrt{2}$ if a_k and a_{k+1} are unequal, we only can arrive for $\lambda \neq 1$ at a polytope all the edges of which have the same length $2\sqrt{2}$ if all the digits $a_1, a_2, \ldots, a_{n-3}$ are equal, i. e. in the four cases

$$\tfrac{1}{2}[\overline{11\ldots1}^{\,n}],\ \tfrac{1}{2}[\overline{33\ldots3}^{\,n-3}111],\ \tfrac{1}{2}[\overline{33\ldots3}^{\,n-2}11],\ \tfrac{1}{2}[\overline{55\ldots5}^{\,n-3}311].$$

In these cases the face becomes

$$\left.\begin{array}{llll} x_i = \lambda, & (i=1, 2, \ldots, n-3), & x_{n-2}, x_{n-1}, x_n = & \left(\frac{\lambda+2}{3}, \frac{\lambda+2}{3}, \frac{\lambda-4}{3}\right) \\ x_i = 3\lambda, & ,, & ,, \quad = & \quad ,, \\ x_i = 3\lambda, & ,, & ,, \quad = & (\lambda+2, \quad \lambda \quad, \lambda-2) \\ x_i = 5\lambda, & ,, & ,, \quad = & \quad ,, \end{array}\right\},$$

furnishing for the edge (a_{n-3}, a_{n-2}) of the new polytope the four symbols

$$\left(\lambda, \frac{\lambda+2}{3}\right),\ \left(3\lambda, \frac{\lambda+2}{3}\right),\ (3\lambda, \lambda+2),\ (5\lambda, \lambda+2).$$

So, if ε represents either 2 or 0 we have in these four cases

$$\lambda = \tfrac{3}{2}\varepsilon + 1,\ 8\lambda = 3\varepsilon + 2,\ 2\lambda = \varepsilon + 2,\ 4\lambda = \varepsilon + 2;$$

so the values of λ different from unity are respectively

$$4 \quad , \quad \tfrac{1}{4} \quad , \quad 2 \quad , \quad \tfrac{1}{2} \quad ,$$

of which the integer values are the only available ones. So the face expansion can be applied to $\frac{1}{2}[\overline{11\ldots1}^{\,n}]$ giving $[\overline{44\ldots4}^{\,n-3}220]$ and to $\frac{1}{2}[\overline{33\ldots3}^{\,n-2}11]$ giving $[\overline{66\ldots6}^{\,n-3}420]$, i. e. in both available cases measure polytope forms deducible from $M_n^{(4)}$. Therefore we can disregard altogether the expansion of the *hmpd.* according to their own

faces and take into consideration the expansions e_k, $(k=2,3,\ldots n-2)$, of the $M_n^{(2)}$ only.

We now pass to the contraction. A motion of the limits of vertex import of $\frac{1}{2}[a_1\, a_2\, \ldots\, a_{n-1}\, a_n]$, i. e. of $(a_1\, a_2\, \ldots\, a_{n-1}\, a_n)$ towards the centre gives $(a_1-\lambda,\, a_2-\lambda,\, \ldots,\, a_{n-1}-\lambda,\, a_n-\lambda)$. So the only new form we can get is $(a_1-1,\, a_2-1,\, \ldots,\, a_{n-2}-1,\, 0,\, 0)$, i. e. a form deducible from $M_n^{(4)}$, etc.

100. We conclude this part by proving the following theorems, which will be useful in the next:

THEOREM LXIII. "The limits of truncation import of $e_{k_1} e_{k_2} \ldots e_{k_{p-1}} e_{k_p} HM_n^{(2\sqrt{2})}$ are $e_{k_1-1} e_{k_2-1} \ldots e_{k_{p-1}-1} e_{k_p-1} S(n)^{2(\sqrt{2})}$."

According to the preceding theorem we have

$$e_{k_1} e_{k_2} \ldots e_{k_{p-1}} e_{k_p} HM_n = \tfrac{1}{2}[\overset{n-k_p}{\overline{2p+1}},\ \overset{k_p-k_{p-1}}{\overline{2p-1}},\ \ldots,\ \overset{k_2-k_1}{\overline{33\ldots3}},\ \overset{k_1}{\overline{11\ldots1}}].$$

So the limits of truncation import are

$$(\overset{n-k_p}{\overline{2p+1}},\ \overset{k_p-k_{p-1}}{\overline{2p-1}},\ \ldots,\ \overset{k_2-k_1}{\overline{33\ldots3}},\ \overset{k_1-1}{\overline{11\ldots1}},\ -1),$$

i. e.

$$(\overset{n-k_p}{\overline{2p+2}},\ \overset{k_p-k_{p-1}}{\overline{2p}},\ \ldots,\ \overset{k_2-k_1}{\overline{44\ldots4}},\ \overset{k_1-1}{\overline{22\ldots2}},\ 0),$$

or reversed

$$-(2p+2,\ \overset{k_1-1}{\overline{2p}},\ \overset{k_2-k_1}{\overline{2p-2}},\ \ldots\ldots,\ \overset{k_p-k_{p-1}}{\overline{22\ldots2}},\ \overset{n-k_p}{\overline{00\ldots0}}),$$

i. e. $-\ e_{k_1-1} e_{k_2-1} \ldots e_{k_{p-1}-1} e_{k_p-1} S(n)^{(2\sqrt{2})}$.

THEOREM LXIV. "The number k_1 of the units figuring in the symbol of coordinates $\frac{1}{2}[a_n\, a_{n-1} \ldots\ \overset{k_1}{\overline{11\ldots1}}]$ of an *hmpd.* in S_n indicates how many limits of truncation import pass through any vertex."

The number of vertices of $\frac{1}{2}[\overset{n-k_p}{\overline{2p+1}},\ \overset{k_p-k_{p-1}}{\overline{2p-1}},\ \ldots,\ \overset{k_2-k_1}{\overline{33\ldots3}},\ \overset{k_1}{\overline{11\ldots1}}]$, respectively of its limits $(\overset{n-k_p}{\overline{2p+1}},\ \overset{k_p-k_{p-1}}{\overline{2p-1}},\ \ldots,\ \overset{k_2-k_1}{\overline{33\ldots3}},\ \overset{k_1-1}{\overline{11\ldots1}},\ -1)$ of truncation import is represented by

$$\frac{2^{n-1}.\ n!}{(n-k_p)!\,(k_p-k_{p-1})!\ldots(k_2-k_1)!\,k_1!},\ \text{resp.}\ \frac{n!}{(n-k_p)!\,(k_p-k_{p-1})!\ldots(k_2-k_1)!\,(k_1-1)!}.$$

So the 2^{n-1} limits of truncation import admit together a number of vertices equal to k_1 times that of the *hmpd.* itself.

E. *Nets of polytopes.*

101. Let us consider the net $N(M_n^2)$ and suppose that it is composed of alternate white and black $M_n^{(2)}$, so that any two $M_n^{(2)}$ with a common limiting $M_{n-1}^{(2)}$ differ in colour. Let us imagine that each white $M_n^{(2)}$ is split up into an inscribed positive HM_n $(= + \frac{1}{2} [\overline{11\,..\,1}^{\,n}])$ and 2^{n-1} pyramids on regular simplexes $S(n)(2\sqrt{2})$ the vertex edges of which have a length 2 and meet at right angles, and that in the same way each black M_n is split up into an inscribed negative HM_n $(= - \frac{1}{2} [\overline{11\,..\,1}^{\,n}])$ and 2^{n-1} pyramids. Then it is clear that a space filling of S_n is formed by three groups of polytopes, two groups of HM_n, i. e. a group of positive ones and a group of negative ones, and one group of cross polytopes $[\overline{200\,..\,0}^{\,n-1}]$, each of which has for centre a vertex of the net $N(M_n^2)$ not belonging to an HM_n and is generated by the addition of 2^n of the equal pyramids. This net, which may be represented by the symbol $N(\pm HM_n, Cr_n)$, forms our starting point here. It is our aim to deduce from this simple net several other ones the constituents of which are forms derived from the regular polytopes and *hmpd.*, partaking with each other of the properties of admitting one kind of vertices and one length of edge, by considering in the application of the expansion operations either the two sets of half measure polytopes as independent and the set of cross polytopes as dependent variables, or reversely.

Any HM_n of the original net $N(\pm HM_n, Cr_n)$ is limited by HM_{n-1} of $(l)_{n-1}$ import and by simplexes $S(n)$ of truncation import; by each HM_{n-1} it is in contact with an HM_n of the other kind, by each $S(n)$ with a Cr_n. We now follow two polytopes HM_n, Cr_n in $S(n)$ contact through any group of expansion operations leading to a new net, by which operations HM_n and its $S(n)$ pass into $(P)_n$ and $(Q)_{n-1}$ and likewise Cr_n and its $S(n)$ into $(P)_n$ and $(Q)'_{n-1}$. Then it is evident that $(Q)_{n-1}$ and $(Q)'_{n-1}$ must coincide, as the application of the operation e_n with respect to the group of Cr_n origin on one hand and the group of HM_n origin on the other would lead to a net with two different kinds of vertices, those of the group of Cr_n origin and those of the group of HM_n origin. This coincidence dominates the *hmpd.* nets, as it creates a very close relation between the two chief constituents. If we denote by the symbol $e_n\ HM_n$ the separation of the two groups of HM_n from each other by the intercalation of prisms on their original

limits, the relation between the two chief constituents of an *hmpd.* net can be thrown into the following form:

THEOREM LXV. "In the *hmpd.* nets the constituent of HM_n origin unequivocally determines that of Cr_n origin and vice versa. If the former is $e_{k_1}\, e_{k_2} \ldots e_{k_{p-1}}\, e_{k_p}\, HM_n$, the latter is represented by $e_{k_1-1}\, e_{k_2-1} \ldots e_{k_{p-1}-1}\, e_{k_p-1}\, Cr_n$."

We divide the proof of this theorem in two parts. In the first part we suppose k_p different from n, in the second we trace the influence of the occurrence of $e_n\, HM_n$.

Let the set of operations to be applied to the Cr_n, in order to obtain a polytope able to form an *hmpd.* net with $e_{k_1}\, e_k \ldots e_{k_{p-1}}\, e_{k_p}\, HM_n$, be represented by $e_{k'_1}\, e_{k'_2} \ldots e_{k'_{q-1}}\, e_{k'_q}$. Then according to the results obtained in the preceding section the limiting $S(n)^{(2\sqrt{2})}$ of Cr_n is transformed into $e_{k'_1}\, e_{k'_2} \ldots e_{k'_{q-1}}\, e_{k'_q}\, S(n)^{(2\sqrt{2})}$, whilst on the other hand the $S(n)^{(2\sqrt{2})}$ of HM_n is transformed into

$$- e_{k_1-1}\, e_{k_2-1} \ldots e_{k_{p-1}-1}\, e_{k_p-1}\, S(n)^{(2\sqrt{2})}.$$

As the negative sign of the second symbol is accounted for by the position of the two polytopes at different sides of the common limit deduced from $S(n)$ the coincidence requires that we have

$$k'_1 = k_1 - 1,\ k'_2 = k_2 - 1,\ \ldots,\ k'_{q-1} = k_{p-1} - 1,\ k'_q = k_p - 1,$$

as the theorem states.

We now suppose that the operation e_n is added to the set of e_k expansions to be applied to the HM_n, i. e. that we drive the two groups of HM_n apart by prisms. Then the enlargement of the side $H_+ H_-$ of the triangle $CH_+ H_-$ (fig. 18), formed by the centres C, H_+, H_- of any triplet of constituents of different kind in mutually $(l)_{n-1}$ contact, caused by the intercalation of the prism implies enlargement of the two other sides, as the triangle must remain similar to itself. This enlargement of CH_+ and CH_- cannot be effected by the application of the operation e_n between the two constituents of different form (see pag. 90); so it must be caused by application of the operation e_{n-1} to the polytopes of Cr_n origin. In other words: the theorem to be proved also holds for the case that e_n occurs under the operations e_k to be applied to the HM_n groups.

Moreover from theorem LXIV we deduce:

THEOREM LXVI. "The totality of the vertices of any *hmpd.* net can always be represented by means of one net symbol, viz. that corresponding to the constituent of Cr_n origin."

We still remark that the number of *hmpd.* nets in S_n is 2^{n-1}.

For we can start either from HM_n as it is, or from one of the $(n-1)_1$ forms $e_{k_1} HM_n$, or from one of the $(n-1)_2$ forms $e_{k_1} e_{k_2} HM_n$, etc., giving altogether

$$1 + (n-1)_1 + (n-1)_2 + \ldots\ldots + (n-1)_{n-1} = (1+1)^{n-1} = 2^{n-1}$$

possibilities. These nets must all be new for $n > 4$, if they prove to exist. On the other hand a preparatory study of the cases $n = 3$ and $n = 4$ will show that $n = 3$ furnishes nothing new, whilst $n = 4$ produces four new cases only.

102. *Hmpd. nets in* S_3. — If we interprete the net of T and O as $N(\pm HM_3, Cr_3)$ the four cases we meet here are

1 .... HM_3, Cr_3	3 .... $e_3\, HM_3$, $e_2\, Cr_3$
2 .... $e_2\, HM_3$, $e_1\, Cr_3$	4 .... $e_2\, e_3\, HM_3$, $e_1\, e_2\, Cr_3$

or in other form

1 .... T, O 12	3 .... T, RCO, .. C 19
2 .... tT, tO, .. CO 24	4 .... tT, tCO, .. tC 23

Here the third constituents CO, C, tC are polyhedra filling gaps, whilst the numbers 12, 24, 19, 23 refer to the stereoscopic diagrams of Andreini. Compare also Table III of M[rs]. Stott's memoir.

Let us pass now to the deduction of the coordinate symbols of these four nets. To that end we have to start in the first case from a T and an O in face contact — and in the other cases from what these polyhedra have become — and to calculate by means of the distance of their centres the periodic term which is to figure in the symbol. We therefore elucidate the mutual position of the two polyhedra in face contact in fig. 19, in projection on to a plane normal to one of the three diameters of the O group. But for clearness' sake we have represented in each of the four cases the T and the O — or what they have become — lying apart; in order to re-establish the real state we have to move the T parallel to itself so as to bring the invisible shadowed face of T indicated by dotted lines in contact with the visible shadowed face of O, i. e. $A'B'$ into coincidence with AB. As we want only the net symbol with respect to the group of O, the origin of the system $O(XYZ)$ of coordinates has been chosen in the centre of the O of the diagram.

The simple diagrams of fig. 19 show an easier way leading to the knowledge of the periodic term of the net symbol. Indeed, in each of the four cases the O — or what it has become — is in

contact by the edge AB with an other polyhedron congruent to it. In other words: if the coordinates x, y of the centre M of AB are p, the centres of the O group are represented by the frame $[2a_1 p, 2a_2 p, 2a_3 p]$ under the conditions a_1, a_2, a_3 integer and $\sum_1^3 a_i$ even, i. e. $2p$ is the *period* of the net. So, as the p has in the four cases successively the values $1, 3, 1+\sqrt{2}, 3+\sqrt{2}$ we find for the four net symbols under the stated conditions

$$
\begin{array}{ll}
1 \,.. & [2a_1 + \mathbf{2}, \; 2a_2 + \mathbf{0}, \; 2a_3 + \mathbf{0}], \\
2 \,.. & [6a_1 + \mathbf{4}, \; 6a_2 + \mathbf{2}, \; 6a_3 + \mathbf{0}], \\
3 \,.. & [2(1+\sqrt{2})a_1 + \mathbf{2} + \sqrt{\mathbf{2}}, \; 2(1+\sqrt{2})a_2 + \sqrt{\mathbf{2}}, \; 2(1+\sqrt{2})a_3 + \sqrt{\mathbf{2}}], \\
4 \,.. & [2(3+\sqrt{2})a_1 + \mathbf{4} + \sqrt{\mathbf{2}}, \; 2(3+\sqrt{2})a_2 + \mathbf{2} + \sqrt{\mathbf{2}}, \; 2(3+\sqrt{2})a_3 + \sqrt{\mathbf{2}}].
\end{array}
$$

Though we pursue the study of these threedimensional nets merely from a didactic point of view it is not necessary to deduce from these net symbols of the O group the net symbols of the two T groups. All we want is to show how the third constituents CO, C, tC can be found. Therefore we give here the net symbols of the two T groups in the form:

$$
\begin{array}{ll}
1 \,.. & \pm\frac{1}{2}[2a_1 \pm 1 + \mathbf{1}, \; 2a_2 \pm 1 + \mathbf{1}, \; 2a_3 \pm 1 + \mathbf{1}], \\
2 \,.. & \pm\frac{1}{2}[3(2a_1 \pm 1) + \mathbf{3}, \; 3(2a_2 \pm 1) + \mathbf{1}, \; 3(2a_3 \pm 1) + \mathbf{1}], \\
3 \,.. & \pm\frac{1}{2}[(2a_1 \pm 1)(1+\sqrt{2}) + \mathbf{1}, \; (2a_2 \pm 1)(1+\sqrt{2}) + \mathbf{1}, \; (2a_3 \pm 1)(1+\sqrt{2}) + \mathbf{1}], \\
4 \,.. & \pm\frac{1}{2}[(2a_1 \pm 1)(3+\sqrt{2}) + \mathbf{3}, \; (2a_2 \pm 1)(3+\sqrt{2}) + \mathbf{1}, \; (2a_3 \pm 1)(3+\sqrt{2}) + \mathbf{1}],
\end{array}
$$

where the double sign refers to the two groups $\pm HM_3$ and the conditions about the a_i and their sum remain the same.

As the polyhedra of the O group remain in contact by faces with those of the two T groups and by edges with each other we have only to look out for new polyhedra filling vertex gaps which make their appearance in the second, third and fourth cases on account of the truncation of the polyhedra of the O group at the vertices. Though all the *vertices* of these new constituents are contained in the net, the second and the fourth cases show that it may happen that some of the *faces* of these new bodies have to be furnished by the polyhedra of the T groups. At any rate we have to determine the new constituent by starting from an octahedron vertex and deducing from the net symbol the vertices at minimum distance from that point.

We treat further each of the four cases by itself.

Case (O, T). — In this case there is no third constituent. Nevertheless we deduce from the net symbol of group O given above that the vertices of all the CO represented by $[2a_1 + \mathbf{2}, 2a_2 + \mathbf{2}, 2a_3 + \mathbf{0}]$, $\sum_1^3 a_i$ odd, are vertices of the net. But these CO are no constituents of the net; for the centre of the CO corresponding to any set of

integers a_i satisfying the condition $\sum\limits_1^3 a_i$ odd is the point $2\,a_1, 2\,a_2, 2\,a_3$, and for $\sum\limits_1^3 a_i$ odd this centre itself is a vertex of the net, i. e. these CO overlap.

Case (tO, tT). — As we have $p = 3$ the point 2, 0, 0, originally common to the central O and an other in vertex contact with it, is carried away from the origin to thrice the distance and arrives at 6, 0, 0. So with respect to this centre of a new constituent as new origin the original net symbol becomes $[6(a_1 - 1) + \mathbf{4}, 6a_2 + \mathbf{2}, 6a_3 + \mathbf{0}]$, Σa_i even, i. e. $[6\,a_1 + \mathbf{4}, 6\,a_2 + \mathbf{2}, 6\,a_3 + \mathbf{0}]$, Σa_i odd. Now the supposition $a_1 = -1, a_2 = a_3 = 0$ gives the square $-2\,[2, 0]$ and so the six suppositions $a_1, a_2, a_3 = [1\,0\,0]$ give the six squares of the $[2, 2, 0]$, i. e. of the CO. The eight triangles of this CO are furnished by tT, four of each group. So by putting $a_1 = 0$, $a_2 = -1$, $a_3 = -1$ in the net symbol of the group of positive tT we get $\frac{1}{2}\,[3 + \mathbf{3}, -3 + \mathbf{1}, -3 + \mathbf{1}]$, i.e. reduced to the new origin 6, 0, 0 the symmetrical form $\frac{1}{2}\,[-3 + \mathbf{3}, -3 + \mathbf{1}, -3 + \mathbf{1}]$, the triangle $(0, -2, -2)$ of which is a face of the CO found above.

Case (RCO, T). — Here we have $p = 1 + \sqrt{2}$ and the centre of the new constituent becomes $2\,(1 + \sqrt{2})$, 0, 0. So the net symbol with respect to that new origin is

$$[2\,(1 + \sqrt{2})\,a_1 + \mathbf{2} + \sqrt{\mathbf{2}},\ 2\,(1 + \sqrt{2})\,a_2 + \sqrt{\mathbf{2}},\ 2\,(1 + \sqrt{2})\,a_3 + \sqrt{\mathbf{2}}],\ \sum_1^3 a_1 \text{ odd.}$$

Here the six suppositions $a_1, a_2, a_3 = [100]$ give the six limiting squares of the cube $[\sqrt{2}, \sqrt{2}, \sqrt{2}]$.

Case (tCO, tT). — Here $p = 3 + \sqrt{2}$ and therefore $2\,(3 + \sqrt{2})$, 0, 0 is the new origin, leading to the new form

$$[2\,(3 + \sqrt{2})\,a_1 + \mathbf{4} + \sqrt{\mathbf{2}},\ 2\,(3 + \sqrt{2})\,a_2 + \mathbf{2} + \sqrt{\mathbf{2}},\ 2\,(3 + \sqrt{2})\,a_3 + \sqrt{\mathbf{2}}],\ \sum_1^3 a_i \text{ odd}$$

of the net symbol. Here the same suppositions give the six limiting octagons of the tC represented by $[2 + \sqrt{2}, 2 + \sqrt{2}, \sqrt{2}]$. By putting $a_1 = 0$, $a_2 = -1$, $a_3 = -1$ in the net symbol of the group of positive tT we get here

$$\tfrac{1}{2}\,[3 + \sqrt{2} + \mathbf{3}, -(3 + \sqrt{2}) + \mathbf{1}, -(3 + \sqrt{2}) + \mathbf{1}],$$

or with respect to the new origin

$$\tfrac{1}{2}\,[-(3 + \sqrt{2}) + \mathbf{3}, -(3 + \sqrt{2}) + \mathbf{1}, -(3 + \sqrt{2}) + \mathbf{1}],$$

the triangle $(-\sqrt{2}, -2-\sqrt{2}, -2-\sqrt{2})$ of which is a face of the tC.

Remark. The p introduced above is not to be confounded with the extension number of the octahedron group which according

to the rule connected with the sum of the digits would be $1, 3, 1 + \frac{3}{2}\sqrt{2}, 3 + \frac{3}{2}\sqrt{2}$ in the four cases.

103. The four cases of *hmpd.* nets in S_3 considered above agree in this that the third constituent is the contraction form of the constituent of octahedron origin. Indeed the contraction forms of O, tO, RCO, tCO are respectively a vertex, CO, C, tC. This fact is too general to be accidental, we will show why it *must* be so.

Therefore we recur to theorem LXVI. As all the vertices of the net figure in the net symbol of the octahedron group — which implies as we already remarked that all the vertices of the new constituent are contained in the net symbol —, the faces which that new constituent has in common with the adjacent polyhedra of the octahedron group must define that new polyhedron. Now in the original net (O, T) any vertex V is a point of concurrence of six O, the centres of which are the opposite vertices V'_i of the six edges of the net of cubes from which (O, T) has been deduced. So the six faces of contact of the new constituent with the six polyhedra of octahedron origin lie in planes normal to the lines OV'_i, in the centres of these faces, lying at equal distance from O. These simple considerations lead to three possibilities compatible with the condition that the new constituent must admit vertices of the same kind and edges of the same length: either the new constituent is equal to the constituent of octahedron origin, or the new one is the contraction form of the other, or the other is the contraction form of the new one. But the first and the last suppositions are to be rejected. For the first would bring equality between the two kinds of limits of the constituent of tetrahedron origin which have been called original limits and limits of truncation import, whilst the last is inadmissible as the constituent of octahedron origin is no contraction form.

We now prove that the preceding result holds for any *hmpd.* net in S_n. If once for all we distinguish for short the constituent of HM_n origin as the first and that of Cr_n origin as the second we can extend theorem LXV by proving:

THEOREM LXVII. "Any *hmpd.* net has three different constituents, none of which is a prism. The third is the contraction form of the second. So, if the first is $e_{k_1} e_{k_2} \ldots e_{k_{p-1}} e_{k_p} HM_n$ and therefore the second $e_{k_1-1} e_{k_2-1} \ldots e_{k_{p-1}-1} e_{k_p-1} Cr_n$, the third is $ce_{k_1-1} e_{k_2-1} \ldots e_{k_{p-1}-1} e_{k_p-1} Cr_n$. In this form of the statement each of the three unequivocally determines the two others."

Proof. In the original $N(HM_n, Cr_n)$ any two Cr_n in contact are either in edge contact, or in vertex contact, in other words the contact of the highest order between two Cr_n of the net is edge contact. Now this contact of the highest order can only be annihilated by separation of the Cr_n i.e. by applying the expansion e_n to them. As this operation is excluded (as leading to a net with two kinds of vertices) the edge contact between the polytopes of Cr_n origin is maintained, though it is changed in character by the operations e_k, $1 < k < n$, contact by *edge* being replaced by contact of an $(l)_{n-1}$ limit of *edge import.* This proves in the first place that there is only room for one new constituent different from a prism, viz. a new polytope with respect to the Cr_n of vertex import; vertex contact being annihilated in any net deduced from $N(HM_n, Cr_n)$, this third constituent *always* makes its appearance. Now we have only to prove still that this third constituent is the contraction form of the second; we prove this in two different ways, in the first place by considering the contact with the second, in the second place by considering the contact with the first constituent.

According to theorem LXVI here also the third constituent is determined by the limits $(l)_{n-1}$ of contact with the $2n$ adjacent polytopes of Cr_n origin, the centres of which are the vertices of a cross polytope with the centre of the vertex gap as centre. So, here also, *if* the $2n$ limits $(l)_{n-1}$ are to determine a polytope with vertices of one kind and edges of one length, there are three possibilities: either the third constituent is equal to the second, or it is the contraction form of the second, or it has the second for contraction form. Here also the first and the last suppositions are inadmissible for the reasons indicated in the case $n = 3$. So the theorem is proved.

We add the following second proof, which we consider even more convincing, as a confirmation of the result obtained. In the notation of the problem the limit of vertex import of $e_{k_1} e_{k_2} \ldots e_{k_{p-1}} e_{k_p} HM_n$ is — compare the proof of theorem LXIII — represented by

$$(\overline{2p+1}^{\,n-k_p},\ \overline{2p-1}^{\,k_p-k_{p-1}},\ \ldots,\ \overline{33\ldots3}^{\,k_2-k_1},\ \overline{11\ldots1}^{\,k_1}),$$

i. e.

$$(\overline{\ \ 2p\ \ }^{\,n-k_p},\ \overline{2p-2}^{\,k_p-k_{p-1}},\ \ldots,\ \overline{22\ldots2}^{\,k_2-k_1},\ \overline{00\ldots0}^{\,k_1}),$$

or reversed

$$-(\overline{\ \ 2p\ \ }^{\,k_1},\ \overline{2p-2}^{\,k_2-k_1},\ \ldots,\ \overline{22\ldots2}^{\,k_p-k_{p-1}},\ \overline{00\ldots0}^{\,n-k_p}),$$

i. e. $-ce_{k_1-1} e_{k_2-1} \ldots e_{k_{p-1}-1} e_{k_p-1} S(n)(2\sqrt{2})$. So the limit $(l)_{n-1}$ of

highest import of the third constituent is the contraction form of the corresponding limit of the second constituent, i. e. the third constituent itself is the contraction form of the second. Or shorter still: by the *reversion* of the symbols the transition from $(a_1\, a_2 \ldots a_{n-2}\; 1{-}1)$ to $(a_1\, a_2 \ldots a_{n-2}\; 1\; 1)$ manifests itself by the diminution of the *first* digit by 2, i. e. by the operation of contraction, leading to the result mentioned in the theorem.

Remark. There is a characteristic difference between the three groups of nets — a) the simplex nets, b) the measure polytope nets, c) the half measure polytope nets — as to the character of the constituents. As we have seen in the preceding sections the simplex nets admit exclusively principal constituents, i. e. neither prisms nor prismotopes, whilst the measure polytope nets admit only *two* principal constituents with exception of the original net of measure polytopes. Now in the case of the *hmpd.* net we always find *three* principal constituents with exception of the original net (HM_n, Cr_n); as soon as two of the three constituents become equal to each other we fall back on a measure polytope net. This only happens for $n > 3$ in S_4, as we shall see in the next article.

104. *Hmpd. nets in S_4.* — Here we have to examine the eight cases:

1 . . . . HM_4, Cr_4		5 . . . . . $e_2\, e_3\, HM_4$, $e_1\, e_2\, Cr_4$
2 . . . . $e_2\, HM_4$, $e_1\, Cr_4$		6 . . . . . $e_2\, e_4\, HM_4$, $e_1\, e_3\, Cr_4$
3 . . . . $e_3\, HM_4$, $e_2\, Cr_4$		7 . . . . . $e_3\, e_4\, HM_4$, $e_2\, e_3\, Cr_4$
4 . . . . $e_4\, HM_4$, $e_3\, Cr_4$		8 . . . $e_2\, e_3\, e_4\, HM_4$, $e_1\, e_2\, e_3\, Cr_4$

Of these eight cases only four are new. The first is $N(C_{16})$, the three equal groups of C_{16} being the groups of $+HM_4$, $-HM_4$, Cr_4. The second case is $e_1\, N(C_{16})$; as $e_2\, HM_4 = e_1\, Cr_4$ we find only two principal constituents. The third case is $ce_2\, N(C_{16})$; as $e_3\, HM_4 = ce_2\, Cr_4$, the third constituent is equal to the first. Finally the fifth case is $ce_1\, e_2\, N(C_{16})$; as $e_2\, e_3\, HM_4 = ce_1\, e_2\, Cr_4$, here also the third constituent is equal to the first. In the four remaining cases the three chief constituents are different; so these cases are new. We represent them in the following small table

$$[e_4\, HM_4,\ e_3\, Cr_4,\ ce_3\, Cr_4,\ P_T],\ 2(1+\sqrt{2})\, a_i\, [2+\sqrt{2},\ \sqrt{2},\ \sqrt{2},\ \sqrt{2}],$$
$$[e_2\, e_4\, HM_4,\ e_1\, e_3\, Cr_4,\ ce_1\, e_3\, Cr_4,\ P_{tT}],\ 2(3+\sqrt{2})\, a_i\, [4+\sqrt{2},\ 2+\sqrt{2},\ \sqrt{2},\ \sqrt{2}],$$
$$[e_3\, e_4\, HM_4,\ e_2\, e_3\, Cr_4,\ ce_2\, e_3\, Cr_4,\ P_T],\ 2(3+\sqrt{2})\, a_i\, [4+\sqrt{2},\ 2+\sqrt{2},\ 2+\sqrt{2},\ \sqrt{2}],$$
$$[e_2\, e_3\, e_4\, HM_4,\ e_1\, e_2\, e_3\, Cr_4,\ ce_1\, e_2\, e_3\, Cr_4,\ P_{tT}],\ 2(5+\sqrt{2})\, a_i\, [6+\sqrt{2},\ 4+\sqrt{2},\ 2+\sqrt{2},\ \sqrt{2}],$$

enumerating the quadruplets of constituents and in condensed form the net symbols; in latter symbols the immovable parts of the digits are placed before the square brackets, whilst the sum of the four integers a_i is always even.

In order to get a better insight into the constitution of the fourdimensional *hmpd.* nets we tabulate the contact between the different constituents. To that end we introduce first a short notation with respect to the nets themselves and to their constituents and the threedimensional limits of these. We denote the *hmpd.* nets in S_4 by the collective symbol NH_4 and distinguish them mutually from each other by putting before that symbol the system of expansion operations applied to the second constituent Cr_4; so the four nets found above are $e_3\,NH_4$, $e_1\,e_3\,NH_4$, $e_2\,e_3\,NH_4$, $e_1\,e_2\,e_3\,NH_4$. Moreover we indicate the four constituents of each net, i. e. the three principal ones taken in the order of succession assumed in theorem LXVII and the prism, by A, B, C, D and we represent their different limits $(l)_3$ by means of subscripts in connexion with their import; so A_3, A_t, A_0 will represent the limits of body, truncation, vertex import of A, whilst $B_i\,(i = 3, 2, 1, 0)$ and $C_k\,(k = 3, 2, 0)$ will represent the limits of $(l)_i$ import of B and of $(l)_k$ import of C, and D_3, D_2, D_0 will stand for the bases of D and the upright limits $(l)_3$ of that prism which correspond to the faces of face import and of vertex import of the bases. So we find the following small table, where the numbers under the columns show how many $(l)_3$ of each kind each polytope admits:

Net	A_3	A_t	A_0	B_3	B_2	B_1	B^0	C_3	C_2	C_0	D_3	D_2	D_0
$e_3\,NH_4$	T	T	—	T	P_3	P_4	C	—	—	C	T	P_3	—
$e_1\,e_3$ "	tT	tT	O	tT	P_6	P_4	RCO	O	P_3	RCO	tT	P_6	P_3
$e_2\,e_3$ "	T	CO	T	CO	P_3	P_8	tC	T	—	tC	T	P_3	—
$e_1\,e_2\,e_3$ "	tT	tO	tT	tO	P_6	P_8	tCO	tT	P_3	tCO	tT	P_6	P_3
	8	8	8	16	32	24	8	16	32	8	2	4	4

This table shows that the contact between the four different constituents is the same in the four nets, i. e. that we have in general

$$A_3 = D_3,\ A_t = B_3,\ A_0 = C_3,\ B_2 = D_2,\ B_0 = C_0,\ C_2 = D_0,$$

whilst B is in contact by its limits B_1 of edge import with other polytopes B, this transformed edge contact being preserved. So the different threedimensional limits cover each other two by two.

The contact between the different constituents can also be deduced from the following small table in which we repeat the constituents of the net in an other form:

Net	C	B	A	D
$e_3\,NH_4$	$[1111]\sqrt{2}$	$[1'111]\sqrt{2}$	$\frac{1}{2}[1111]$	$\frac{1}{2}[111][1]\sqrt{2}$
$e_1\,e_3$,,	$[1'1'11]$,,	$[2'1'11]$,,	,, $[3311]$	,, $[311][1]$,,
$e_2\,e_3$,,	$[1'1'1'1]$,,	$[2'1'1'1]$,,	,, $[3111]$	,, $[111][1]$,,
$e_1\,e_2\,e_3$,,	$[2'2'1'1]$,,	$[3'2'1'1]$,,	,, $[5311]$	,, $[311][1]$,,

So from this table we deduce $A_3 = D_3$ by remarking that the digits of the first syllable of D are the last three digits of the unique syllable of A; in order to facilitate comparison of A and D we have reversed the order of A, B, C.

So we find $A_t = B_3$ (or rather $A_t = - B_3$) as we get the same form by placing the four digits of A between round brackets after having taken the last unit with the negative sign and by placing the digits of B, multiplied by $\sqrt{2}$, between round brackets; etc.

105. Before passing to the case $n = 5$ we will put the last two small tables of the preceding article on duty as to the general results they may suggest for $n > 4$.

We begin by fixing our attention on the extreme case of the relation between the two constituents A and C, being governed in the case $e_3\,NH_4$ by a vertex only. Here A_0, the limit of vertex import of A, is still a vertex; so we have to accept for C the polytope deduced from Cr_4 which admits as limit C_3 of body import a vertex and this is the eightcell $ce_3\,Cr_4$. The same remark holds for $e_2\,NH_3$ already, i.e. for the third of the four cases treated in art. 103.

But the first of our two tables, i. e. the table of contacts, suggests a remark of much wider scope. We deduce it from the fact that each constituent with three kinds of limits $(l)_3$ is in contact with the three others, whilst the only one with four different kinds of limits $(l)_3$ is in contact with the three others and with itself.

This fact suggests that in space S_n we will want in all n different constituents $A, B, C, \ldots$, of which B only admits at most n different limits $(l)_{n-1}$ and all the others at most $n - 1$. We have used this suggestion as working hypothesis and found by its help the sixteen *hmpd.* nets of S_5; this was an easy task: as theorem LXVII gives the three principal constituents A, B, C and the prism D can be deduced from them, the table of contacts shows immediately which limits $(l)_4$ remain uncovered and these limits reveal the character of the fifth constituent. [1])

But there is an other method of deducing the new constituent, much more capable of being extended to S_n, viz the determination of their coordinate symbols by transformation of the net symbol to

[1]) It may seem in accordance with this suggestion that in the cases $e_2\,NH_3$ and $e_1\,e_2\,NH_3$ of S_3 we have found no fourth constituent i.e. no prism, though they require the operation e_3 with respect to the two groups of HM_3 of different orientation, driving these groups asunder. But this not appearing of the prism is rather due to the fact that two adjacent HM_3 of different orientation are in contact by an edge only instead of by a face, so that the separation intercalates a square instead of a prism.

new origins. We introduce this method by remarking that the addition of the second syllable $[1]\sqrt{2}$ of the symbol of the prism D in the last table of the preceding article has a deeper meaning than might be supposed: in this form the coordinate symbol of D is derived from the net symbol, and by examining how this process runs in S_4 we easily hit upon its generalization for S_n, if necessary by the assistance of the knowledge of the fifth constituent in S_5 found in the manner described above. So we indicate for any net in S_4 how the coordinate symbol of the constituents can be derived from the net symbol.

In fig. 20 we represent by $O\,(X_1\,X_2\,X_3\,X_4)$ the system of coordinates and by the shaded pentagon with the axis of symmetry OM a fourth part of the section of the plane $O(X_1\,X_2)$ with the central polytope B. Then OP_0 is the "period" p of the net and the point P_1 of OX_1 lying at twice that distance from O is the centre of an adjacent polytope C filling a vertex gap, whilst P_2 with the coordinates $2p, 2p, 0, 0$ is the centre of an other polytope B in contact with the central one by a polyhedron of edge import. Moreover P_3 is the point $2p, 2p, 2p, 0$ and P_4 the point all the coordinates of which are $2p$; of these P_3 corresponds in character with P_1, and P_4 with O and P_2. So the midpoint Q_4 of OP_4 must be the centre of a polytope in threedimensional contact of body import with the two polytopes B with the centres O and P_4, i.e. of a polytope A. On the other hand the midpoint Q_3 of OP_3 must be the centre of the prism interposed between the two polytopes A of different orientation with the centres Q_4, latter point being the image of Q_4 with respect to the space $x_4 = 0$ as mirror, as these polytopes are derived from the two HM_4 of the original net (HM_4, Cr_4) which were in body contact in that space $x_4 = 0$.

In this manner we find in general for all the cases in S_4 for the coordinates of the centres of the adjacent polytopes

$2p$,	0 ,	0 ,	0 ,	in	the	case	of	C ,	
$2p$,	$2p$,	0 ,	0 ,	,,	,,	,,	,,	an other	B ,
p ,	p ,	p ,	p ,	,,	,,	,,	,,	A ,	
p ,	p ,	p ,	0 ,	,,	,,	,,	,,	D ,	

whilst the upright edges of the prism D are parallel to the axis OX_4.

Now we consider the case $e_1\,e_2\,e_3\,\boldsymbol{NH}_4$ in order to show how the process runs. Here we have $p = 5 + \sqrt{2}$, whilst the central B is represented by $[6 + \sqrt{2}, 4 + \sqrt{2}, 2 + \sqrt{2}, \sqrt{2}]$. So we obtain C, A, D successivey as follows:

$$\begin{array}{rcccc} & 6+\sqrt{2}\ , & 4+\sqrt{2}\ , & 2+\sqrt{2}\ , & \sqrt{2} \\ & 10+2\sqrt{2}\ , & 0\ , & 0\ , & 0 \\ \hline & -(4+\sqrt{2})\ , & 4+\sqrt{2}\ , & 2+\sqrt{2}\ , & \sqrt{2} \end{array} \text{ subtr.}$$

furnishing the polytope $[4+\sqrt{2}, 4+\sqrt{2}, 2+\sqrt{2}, \sqrt{2}]$, i.e. C;

$$\begin{array}{cccc} 6+\sqrt{2}\ , & 4+\sqrt{2}\ , & 2+\sqrt{2}\ , & \sqrt{2} \\ 5+\sqrt{2}\ , & 5+\sqrt{2}\ , & 5+\sqrt{2}\ , & 5+\sqrt{2} \\ \hline 1\ , & -1\ , & -3\ , & -5 \end{array} \text{ subtr.}$$

leading to the polytope $-\frac{1}{2}[5311]$, i. e. A;

$$\begin{array}{cccc} 6+\sqrt{2}\ , & 4+\sqrt{2}\ , & 2+\sqrt{2}\ , & [\sqrt{2}] \\ 5+\sqrt{2}\ , & 5+\sqrt{2}\ , & 5+\sqrt{2}\ , & 0 \\ \hline 1\ , & -1\ , & -3\ , & [\sqrt{2}] \end{array} \text{ subtr.}$$

giving finally the polytope $\frac{1}{2}[311][\sqrt{2}]$, i. e. D.

This will be clear, if we only add one word about the factor $\frac{1}{2}$ before the symbols of A and D, viz. that we want this factor in order to have symbols representing polytopes with one kind of vertex and one length of edge.

106. *Hmpd. nets in* S_5. — We have determined the sixteen *hmpd.* nets of S_5 by means of the two methods given in outline in the preceding article.

The results of the first method are put on record in Table X. This table is divided by vertical lines into eight parts; of these the first contains the symbol of the nets, the last two their constituents and the five others the limits $(l)_4$ of each of the five constituents A, B, C, D, E. In the construction of this table we started from theorem LXVII enabling us to register in the last part but one in the columns with the superscripts A, B, C the character of the three principal constituents and to add under D, in the cases where e_4 appears amongst the expansion symbols of the net, the prisms on the polytopes of polytope import of A as bases. After having finished this task we have inscribed in the columns with the headings $A_4, A_t, \ldots D_t, D_0$ the limits $(l)_4$ of these constituents A, B, C, D, taken from the tables given in the preceding sections of this memoir; this will be clear if we add the remark that the notation D_3, D_t, D_0 for the limits $(l)_4$ of D differing from the bases D_4 has been chosen in accordance with the consideration of these bases as deduced from HM_4. This second task having been performed we can formulate the contact between the constituents A, B, C, D; we find generally:

$$A_4 = A_4 \text{ (if } e_4 \text{ is \textit{absent}) and } A_4 = D_4 \text{ (if } e_4 \text{ is \textit{present}}),$$
$$A_t = B_4\ ,\ A_0 = C_4\ ,\ B_3 = D_3\ ,\ B_1 = B_1\ ,\ B_0 = C_0\ ,\ C_3 = D_0.$$

So A_3, B_2, C_2, D_t remain uncovered, i. e. have still to be covered by limits $(l)_4$ of E. We represent these limits $(l)_4$ of E by E_a, E_b, E_c, E_d, indicating by the subscripts the constituents with which they are in $(l)_4$ contact and repeat these limits in the column with the headings E_a, E_b, E_c, E_d. Finally from these limits we deduce the constituent E itself, see the last column of the seventh part of the table. We remark that this fifth constituent is a prismotope, the two components of which are HM_3 (or $e_2\, HM_3$) and p_4 (or p_8); it presents itself if and only if either e_3, or e_4, or both operations are present.

In applying the second method to S_5 we have to extend the $M_4^{(2p)}$ of fig. 20 with the broken line $O\,P_1\,P_2\,P_3\,P_4$ of edges leading from O to the opposite vertex P_4 into an $M_5^{(2p)}$ with $O\,P_1\,P_2\,P_3\,P_4\,P_5$ as corresponding broken line of edges from O to the opposite vertex P. If we represent the midpoints of OP_5, OP_4, OP_3 respectively by Q_5, Q_4, Q_3 we find for the new origins leading to the constituents C, A, D, E the points P_1, Q_5, Q_4, Q_3 with the coordinates

$$\left.\begin{array}{ccccc} 2p, & 0, & 0, & 0, & 0 \\ p, & p, & p, & p, & p \\ p, & p, & p, & p, & 0 \\ p, & p, & p, & 0, & 0 \end{array}\right\} .$$

So in the case $\boldsymbol{e}_1\,\boldsymbol{e}_3\,\boldsymbol{e}_4\,\boldsymbol{N}\boldsymbol{H}_5$, i. e. $[3'2'1'1'1]\sqrt{2}$ with $p = 5 + \sqrt{2}$ the constituents A, D, E are obtained by the three processes

$$\begin{array}{ccccc} 6+\sqrt{2}\,, & 4+\sqrt{2}\,, & 2+\sqrt{2}\,, & 2+\sqrt{2}\,, & \sqrt{2} \\ 5+\sqrt{2}\,, & 5+\sqrt{2}\,, & 5+\sqrt{2}\,, & 5+\sqrt{2}\,, & 5+\sqrt{2} \\ \hline 1\,, & -1\,, & -3\,, & -3\,, & -5 \end{array}\ \text{subtr.}$$

$$\begin{array}{ccccc} 6+\sqrt{2}\,, & 4+\sqrt{2}\,, & 2+\sqrt{2}\,, & 2+\sqrt{2}\,, & [\sqrt{2}] \\ 5+\sqrt{2}\,, & 5+\sqrt{2}\,, & 5+\sqrt{2}\,, & 5+\sqrt{2}\,, & 0 \\ \hline 1\,, & -1\,, & -3\,, & -3\,, & [\sqrt{2}] \end{array}\ \text{subtr.}$$

$$\begin{array}{ccccc} 6+\sqrt{2}\,, & 4+\sqrt{2}\,, & 2+\sqrt{2}\,, & [2+\sqrt{2}\,, & \sqrt{2}] \\ 5+\sqrt{2}\,, & 5+\sqrt{2}\,, & 5+\sqrt{2}\,, & 0\,, & 0 \\ \hline 1\,, & -1\,, & -3\,, & [2+\sqrt{2}\,, & \sqrt{2}] \end{array}\ \text{subtr.}$$

giving respectively $\frac{1}{2}[53311]$, $\frac{1}{2}[3311][1]\sqrt{2}$, $\frac{1}{2}[311][1'1]\sqrt{2}$. The results obtained in this way are collected in Table XI. To this we have only to add a few remarks.

The processes used just now show clearly why the syllables $\frac{1}{2}[3311]$ and $\frac{1}{2}[311]$ of D and E must correspond in digits with

the last digits of $\frac{1}{2}[53311]$, the symbol of A, and likewise why the other syllables $[1]\sqrt{2}$ and $[1'1]\sqrt{2}$ must correspond in the same manner with either of the symbols $[3'2'1'1'1]$ and $[2'2'1'1'1]$ of B and C. Also why D must be a prism and E a prismotope, in connexion with the faculty of inverting the signs of $\sqrt{2}$ in the case of D, and of $2+\sqrt{2}$ and $\sqrt{2}$ in the case of E, these inversion having no influence whatever on the distance of the vertices obtained of the new origin which is to be the centre of the gap filling polytope.

Moreover the processes themselves indicate under which circumstances the prism D and the prismotope E present themselves. If the symbol of B winds up in zero the second syllable of the symbol of D is $[0]$, i. e, the prism is lacking; but we know from theorem XXXV that the last digit of the symbol of B *is* zero, if the operation e_4 has not been applied to B. Likewise, if the last two digits of the symbol of B are zero, the second syllable of E is $[0, 0]$, i. e. there is no prismotope E, and the last two digits of the symbol of B *are* zero, if neither e_3 nor e_4 has been applied to B.

Finally it is evident why we cannot add a fourth process to the three considered ones and subtract $5+\sqrt{2}$, $5+\sqrt{2}$, 0, 0, 0. For then we would get $1, -1, [2+\sqrt{2}, 2+\sqrt{2}, \sqrt{2}]$, leading to $\frac{1}{2}[11][1'1'1]\sqrt{2}$, i. e. — as $\frac{1}{2}[11]$ is an edge instead of a face — to a limiting body and not to a limit $(l)_4$.

107. *Hmpd. nets in S_n.* — It is easy to see how the processes of the preceding article must be extended to S_n, as the algorithm always remains the same and the number of the subtractions has to be augmented until only three digits of the subtrahend differ from zero. So, if we indicate by $A^{(k)}$ the constituent obtained by the subtraction of $\overbrace{pp..p}^{n-k}\ \overbrace{00..0}^{k}$ we can formulate the general result in the following theorem:

THEOREM LXVIII. — "In any net deduced from NH_n we find, besides the three *principal* constituents A, B, C always present, under certain circumstances one or more prismotopes $A^{(k)}$ for $k = 1, 2, \ldots, n-3$, which may be called *accidental* constituents. The prismotope $A^{(k)}$ presents itself if — and only if — one or more of the expansions $e_{n-k}, e_{n-k+1}, e_{n-1}$ have contributed to the transformation of Cr_n into B; the two syllables of its symbol are the last $n-k$ digits of A between square brackets preceded by $\frac{1}{2}$ and the last k digits of B between square brackets."

So we find in the case $B = [5'4'4'3'3'2'2'2'1'1]\sqrt{2}$ of S_{10}:

$$\begin{aligned}
C &= \quad [4'4'4'3'3'2'2'2'1'1]\sqrt{2} \\
A &= \tfrac{1}{2}[9\,7\,5\,5\,5\,3\,3\,1\,1\,1] \\
A^{(1)} &= \;,,[7\,5\,5\,5\,3\,3\,1\,1\,1]\;[1]\sqrt{2} \\
A^{(2)} &= \;,,[5\,5\,5\,3\,3\,1\,1\,1]\;[1'1]\;,, \\
A^{(3)} &= \;,,[5\,5\,3\,3\,1\,1\,1]\;[2'1'1]\;,, \\
A^{(4)} &= \;,,[5\,3\,3\,1\,1\,1]\;[2'2'1'1]\;,, \\
A^{(5)} &= \;,,[3\,3\,1\,1\,1]\;[2'2'2'1'1]\;,, \\
A^{(6)} &= \;,,[3\,1\,1\,1]\;[3'2'2'2'1'1]\;,, \\
A^{(7)} &= \;,,[1\,1\,1]\;[3'3'2'2'2'1'1]\;,,
\end{aligned}$$

By applying this theorem we find immediately the sixteen nets of S_5, as they have been registered in the eighth part of Table X with the heading "constituents in an other notation". Moreover Table XI gives the corresponding results for the 32 nets of S_6.

F. *Polarity.*

108. By polarizing an n-dimensional *hmpd.* with respect to a concentric spherical space (with ∞^{n-1} points) as polarisator we get a new polytope admitting one kind of limit $(l)_{n-1}$ and equal dispacial angles, to which corresponds the inversed symbol of characteristic numbers of the original polytope. Moreover, if $\frac{1}{2}[a_1, a_2, \ldots, a_{n-1}, a_n]$ is the coordinate symbol of the original *hmpd.*, this symbol also represents the limiting spaces S_{n-1} of the new polytope in space coordinates.

The fact that there is no *hmpd.* proper in S_3 and S_4 implies the corresponding fact with respect to the new forms. So, if by the subscript s is indicated that space coordinates are meant, we have: $\frac{1}{2}[\;111]_s = (4, 6, 4) = T$, $\frac{1}{2}[311]_s = (8, 18, 12) = T$ with pyramids on the faces,
$\frac{1}{2}[1111]_s = (16, 32, 24, 8) = M_4$, $\frac{1}{2}[3311]_s = (24, 96, 120, 48) = M_4$ with pyramids on the cubes, etc.

109. Theorem LXIX. "Any *hmpd.* in S_n has the property that the vertices V_i adjacent to any arbitrary vertex V lie in the same space S_{n-1} normal to the line joining this vertex V to the centre O of the polytope. The system of the spaces S_{n-1} corresponding in this way to the different vertices of the *hmpd.* include an other polytope, the reciprocal polar of the original polytope with respect to a certain concentric spherical space, unless the chosen

hmpd. be the cross polytope HM_4 of S_4 in which case all the spaces S_3 pass through the centre."

The simple geometrical proof of this theorem can be copied from that of theorem XL (see art. 66).

110. We have to add a single word about the reciprocation of the *hmpd.* nets. The results obtained here run parallel to those of art. 68.

In general the system of vertices found by polarizing an *hmpd.* net is the combination of several groups of limits $M_k^{(2p)}$ of the measure polytopes of the net $N(M_n^{(2p)})$, p being the *period*. These groups are formed by the centres of the constituents B, C, A, $A^{(1)}, \ldots, A^{(n-3)}$, i. e.

for B the *even* vertices of $N(M_n^{(2p)})$, represented by $[2pa_1, \ldots, 2pa_n]$, $\sum_1^n a_i$ even,
" C " *odd* " " " , " " " , $\sum_1^n a_i$ odd,
" A " centres of the M_n of $N(M_n^{(2p)})$,
" $A^{(1)}$ " " " " limiting M_{n-1} of the M_n of $N(M_n^{(2p)})$,
" $A^{(2)}$ " " " " " M_{n-2} " " " " " ,
. . . . ,
" $A^{(n-3)}$ " " " " " M_3 " " " " " .

In the case of the net NH_n itself only the first and the third group are present; so in S_4 we find then the net $N(C_{24})$. In all other cases we have to deal with at least three groups, the first three. As we already remarked in art. 68 an other paper, also destined to complement art. 39, will contain more ample developments about these reciprocal nets.

G. *Symmetry, considerations of the theory of groups, regularity.*

111. We first determine the spaces of symmetry Sy_{n-1} of HM_n itself and afterwards those of any *hmpd.* derived from it.

Case of HM_n. — We have to investigate here how the reasoning which led us to the spaces of symmetry of the measure polytope is affected by the alternate truncation.

In the case of M_n we found two possibilities under which the space S_{n-1} bisecting orthogonally the join $A_1 A_2$ of two vertices A_1, A_2 is a space Sy_{n-1} of the polytope, i. e. that $A_1 A_2$ is either an edge or the diagonal of a face; in the first case we got the n spaces $x_i = 0$, in the second the $n(n-1)$ spaces $x_i \pm x_k = 0$. Now on the one hand it is immediately evident that the alternate trun-

cation behaves itself differently with respect to these two groups of spaces: it destroys the symmetry property of the first and preserves that of the second. But on the other hand we have to examine whether the alternate truncation does not enervate the force of the argument by means of which we excluded the cases that $A_1 A_2$ was a diagonal of a limiting M_k of the M_n for $k > 2$, i. e. that the projections of the two regular simplexes $S(k)$ of the vertices of M_k adjacent to A_1 and to A_2 on the space normal to $A_1 A_2$ are of opposite orientation. Indeed this argumentation has to be revised, as the two simplexes $S(k)$ disappear altogether by applying the truncation and are replaced as groups of vertices of HM_k adjacent to A_1 and to A_2 by the two sets of $\frac{1}{2} k (k-1)$ vertices of M_k lying in the following layers S_{n-1} normal to $A_1 A_2$. But the two polytopes [1]) determined by these groups of vertices are neither central symmetric and maintain the property of the differently orientated projections, unless they coincide in the space S_{n-1} normally bisecting $A_1 A_2$ for $k = 4$. So any space orthogonally bisecting a diagonal of a limiting sixteencell of HM_n is an Sy_{n-1} and therefore HM_n also admits two groups of spaces Sy_{n-1}, the spaces $x_i \pm x_k = 0$ and the spaces $x_i \pm x_k \pm x_l \pm x_m = 0$. The number of the former is always $n(n-1)$, whilst that of the latter is $\frac{1}{3} n(n-1)(n-2)(n-3)$ for $n > 4$ and four for $n = 4$.

Case of the hmpd. derived from HM_n. — From the structure of the *hmpd.* it is immediately evident that a space S_{n-1} is an Sy_{n-1} for an *hmpd.* if and only if it is an Sy_{n-1} for the HM_n from which the *hmpd.* has been dirived. So we have proved the

THEOREM LXX. "Any *hmpd.* of S_n admits two groups of spaces Sy_{n-1}, viz. the $n(n-1)$ spaces $x_i \pm x_k = 0$ and the $\frac{1}{3} n(n-1)(n-2)(n-3)$ spaces $x_i \pm x_k \pm x_l \pm x_m = 0$".

112. From theorem XLIII we deduce:

THEOREM LXXI. "The order of the group of anallagmatic displacements of HM_n and of the *hmpd.* derived from it is $2^{n-2} n!$ for $n > 4$".

"The order of the extended group of anallagmatic displacements of these polytopes, reflexions with respect to spaces Sy_{n-1} included, is $2^{n-1} n!$. In this extended group the first group of order $2^{n-1} n!$ forms a perfect subgroup".

[1]) Compare for these polytopes: "The sections of the measure polytope M_n of space Sp_n with a central space Sp_{n-1} perpendicular to a diagonal", *Proceedings* of Amsterdam, vol. X, p. 495.

The proof of this theorem is to be based on the remark that the order of the group must be half of that of theorem XLIII on account of the alternate truncation.

113. As to the application of Elte's scale of regularity we have to use theorem XLIV. We illustrate this, sticking to the original scale, by the following examples.

a). *Example* $\frac{1}{2}[\mathbf{11111}]$. Here we find one kind of edge, one kind of face, but two kinds of limiting tetrahedra, viz. tetrahedra of body import and tetrahedra of truncation import. So the contributions to the numerator are 1 from each of the three groups of vertices, edges, faces, and $\frac{1}{2}$ from the limiting bodies. So the fraction is $\frac{3+\frac{1}{2}}{5}=\frac{7}{10}$.

b). *Example* $\frac{1}{2}[\mathbf{553111}]$. Here we find three different groups of edges $(5, 3), (3, 1), \frac{1}{2}[1, 1]$. So the fraction is $\frac{1+\frac{1}{2}}{6}=\frac{1}{4}$

c). *Example* $\boldsymbol{N(HM_5, Cr_5)}$. This simple net admits one kind of edge, one kind of face, but two kinds of limiting tetrahedra, as a tetrahedron of body import of HM_5 is common to four HM_5, a tetrahedron of truncation import to two HM_5 and one Cr_5. So we find $\frac{3+\frac{1}{2}}{6}=\frac{7}{12}$.

d). *Example* $\boldsymbol{e_1 e_2 NH_6}$. Here we have to deal with three groups of constituents represented with their frames in the table

$$\begin{array}{lllllllll}
B\ldots [321000]\,2\ldots & (2p_1 & , 2p_2 & , 2p_3 & , 2p_4 & , 2p_5 & , 2p_6 &)\,5, & \Sigma p \text{ even},\\
C\ldots [221000]\,2\ldots & (2p_1 & , 2p_2 & , 2p_3 & , 2p_4 & , 2p_5 & , 2p_6 &)\,5, & \Sigma p \text{ odd},\\
A\ldots\frac{1}{2}[555311]\ldots\ldots & (2p_1+1, & 2p_2+1, & 2p_3+1, & 2p_4+1, & 2p_5+1, & 2p_6+1 &)\,5. &
\end{array}$$

So through the vertex 6, 4, 2, 0, 0, 0 pass

$$\begin{array}{rllllll}
[& \mathbf{6}, & \mathbf{4}, & \mathbf{2}, & \mathbf{0}, & \mathbf{0}, & \mathbf{0}]\ldots\ldots B_1\\
[& 10+\mathbf{4}, & 10+\mathbf{6}, & \mathbf{2}, & \mathbf{0}, & \mathbf{0}, & \mathbf{0}]\ldots\ldots B_2\\
[& 10+\mathbf{4}, & \mathbf{4}, & \mathbf{2}, & \mathbf{0}, & \mathbf{0}, & \mathbf{0}]\ldots\ldots C\\
-\frac{1}{2}[& 5+\mathbf{1}, & 5-\mathbf{1}, & 5-\mathbf{3}, & 5-\mathbf{5}, & 5-\mathbf{5}, & 5-\mathbf{5}]\ldots\ldots A_1\\
\frac{1}{2}[& 5+\mathbf{1}, & 5-\mathbf{1}, & 5-\mathbf{3}, & (-5+\mathbf{5}, & 5-\mathbf{5}, & 5-\mathbf{5})]\ldots A_2, A_3, A_4\\
-\frac{1}{2}[& 5+\mathbf{1}, & 5-\mathbf{1}, & 5-\mathbf{3}, & (-5+\mathbf{5}, & -5+\mathbf{5}, & 5-\mathbf{5})]\ldots A_5, A_6, A_7\\
\frac{1}{2}[& 5+\mathbf{1}, & 5-\mathbf{1}, & 5-\mathbf{3}, & -5+\mathbf{5}, & -5+\mathbf{5}, & -5+\mathbf{5}]\ldots\ldots A_8.
\end{array}$$

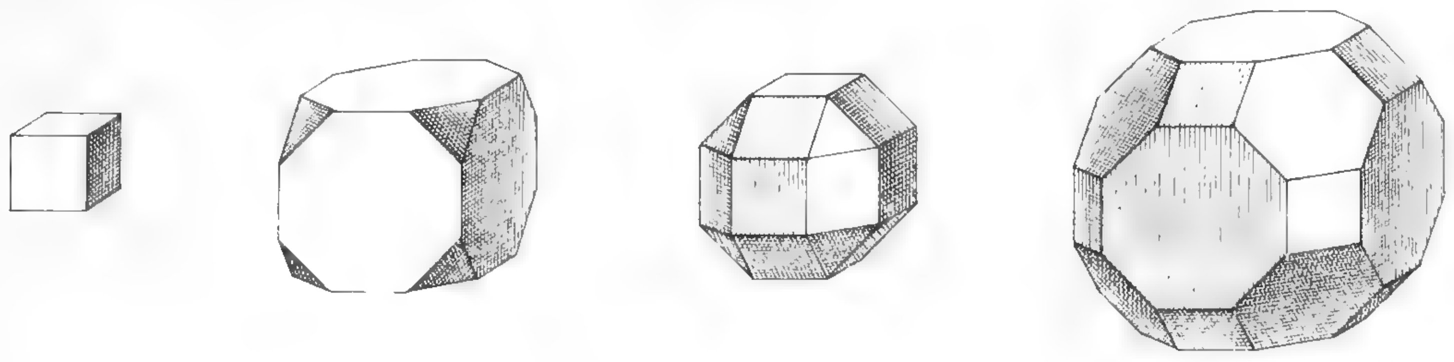
Now the edge (64)2000 belongs to all these polytopes with exception of C, 6(42)000 belongs to all with exception of B_2, whilst 64(20)00 belongs to seven only. So we find three kinds of edges and the fraction is $\frac{3}{14}$.

Remark. Only the HM_n itself admits a regularity fraction $\frac{7}{2n}$, all the *hmpd.* derived from it a fraction $\frac{3}{2n}$.

As a rule the net NH_n admits the fraction $\frac{7}{2(n+1)}$ and a net derived from it $\frac{3}{2(n+1)}$.

Groningen, December, 1912.

Fig. 13.



a. b c. d.

Fig. 14.

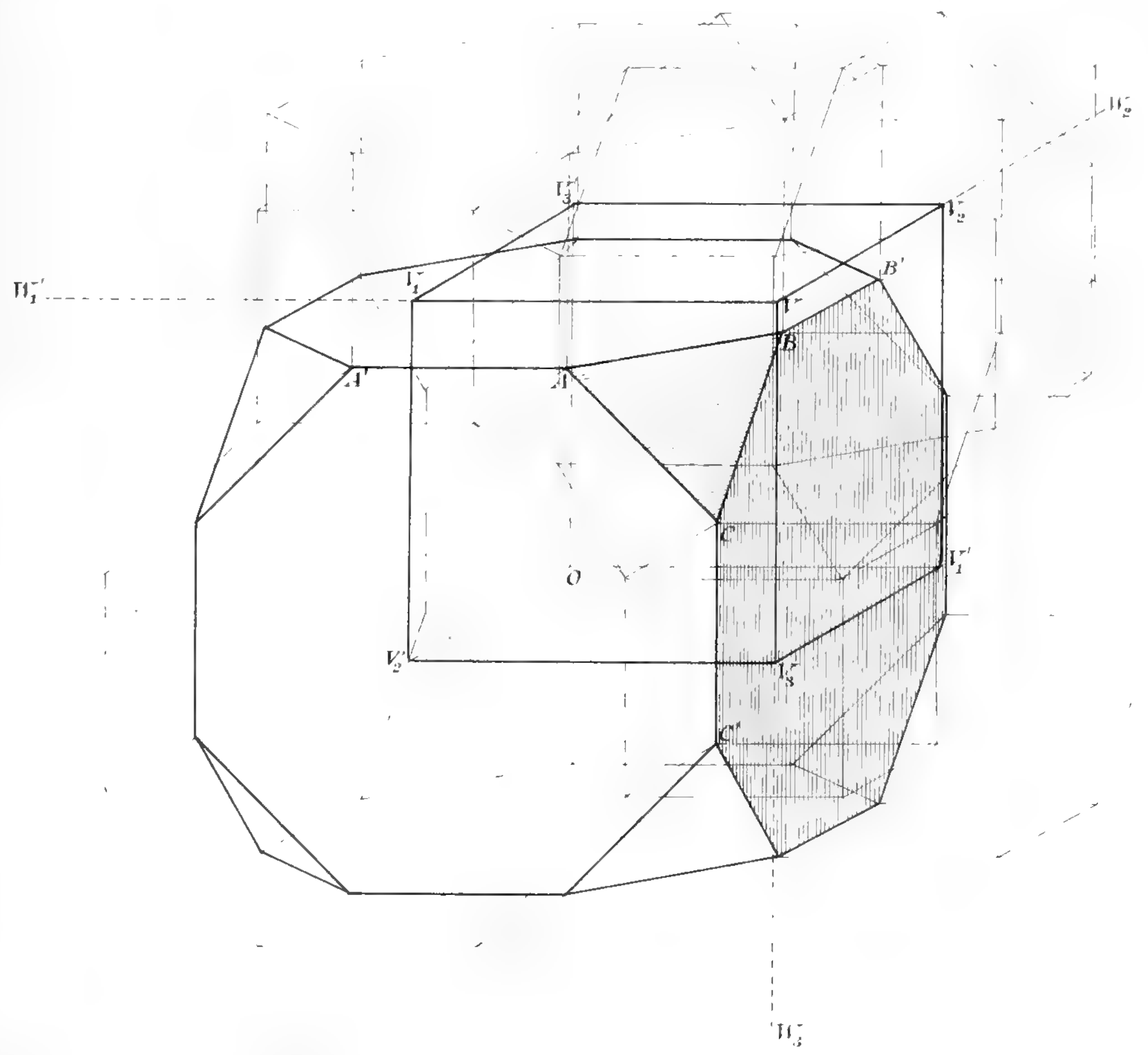


Fig. 15

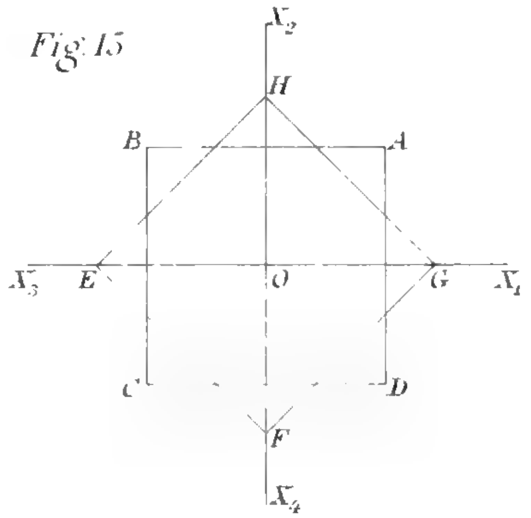


Fig. 16.

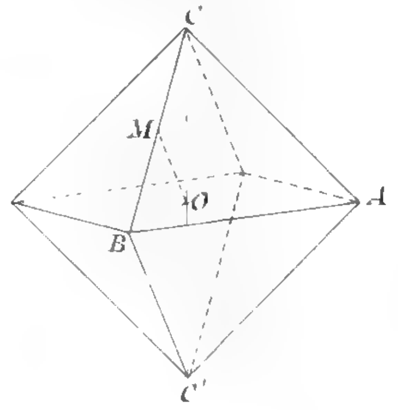
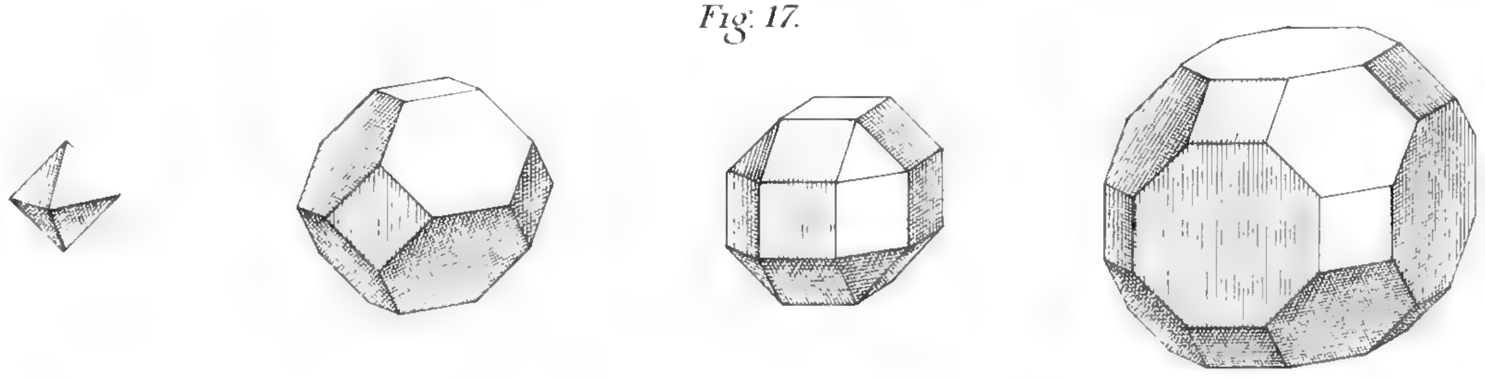


Fig. 17.



a. b. c. d

Fig. 20.

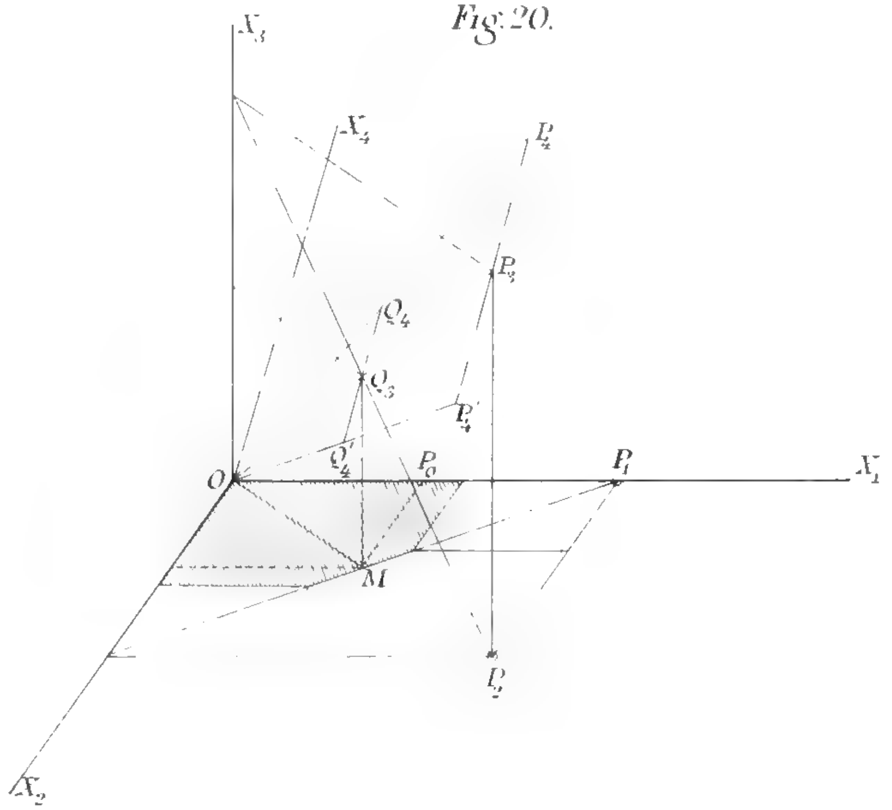


Fig. 18.

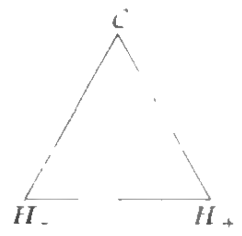
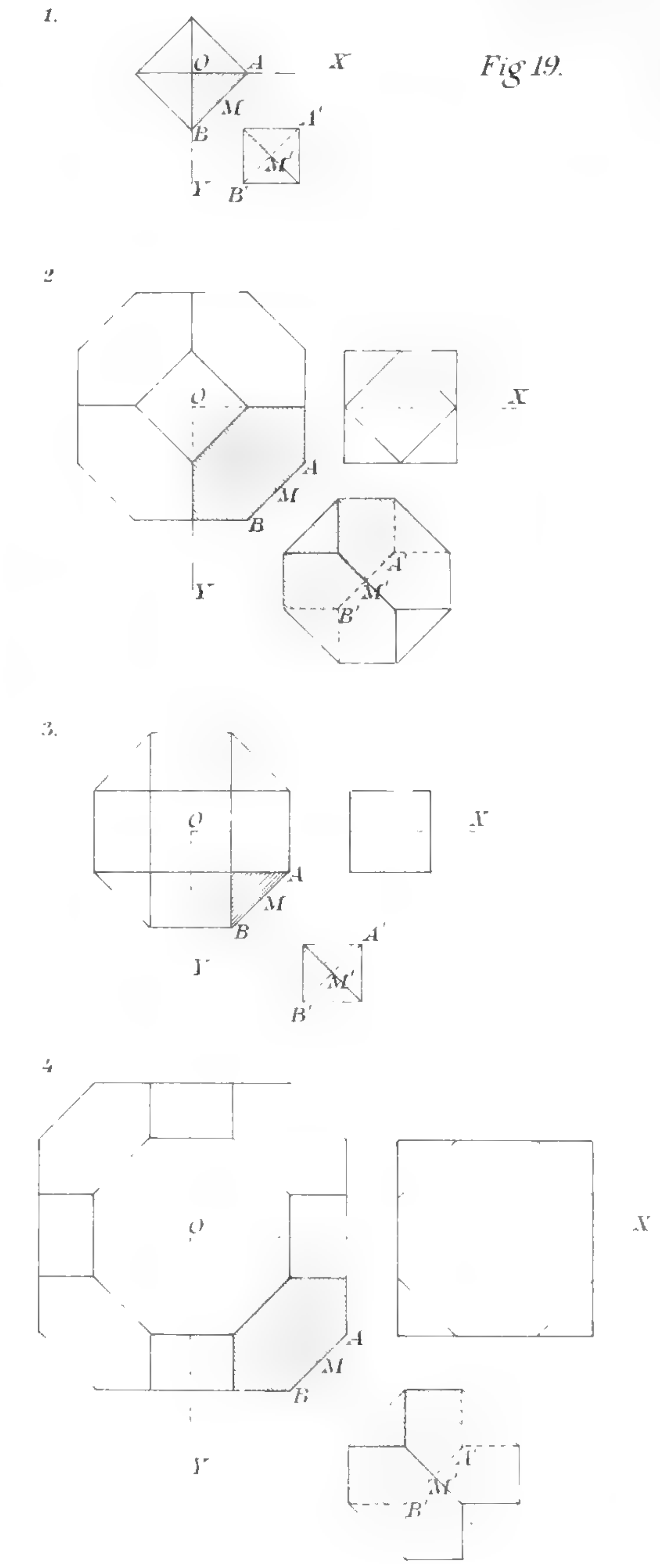


Fig. 19.



LIST OF POLYTOPES DEDUCED FROM MEASURE POLYTOPE AND CROSS POLYTOPE.

Table IV.

$n = 3$

						6	12	8				
C	$=$	$ce_2\, O$	$[1\ 1\ 1]$	$(8_1\ 12_1\ 6_1)$	1	p_4	—	—	1			
$e_1\, C = tC$	$=$	$ce_1\, e_2\, O$	$[1'\, 1'\, 1]$	$(24_1\ 36_2\ 14_2)$	$\frac{1}{2}$	p_8	—	p_3	$1'$	1		
$e_2\, C = RCO$	$=$	$e_2\, O$	$[1'\, 1\ 1]$	$(24_1\ 48_2\ 26_3)$	$\frac{1}{2}$	p_4	p_4	p_3	$1'$	2, 1		
$e_1\, e_2\, C = tCO$	$=$	$e_1\, e_2\, O$	$[2'\, 1'\, 1]$	$(48_1\ 72_3\ 26_3)$	$\frac{1}{2}$	p_8	p_4	p_6	$2'$	3, 1		
$ce_1\, C = CO$	$=$	$ce_1\, O$	$[110]\sqrt{2}$	$(12_1\ 24_1\ 14_2)$	$\frac{2}{3}$	p_4	—	p_3	$1''$	1	2	1
$ce_2\, C = O$	$=$	O	$[100]\sqrt{2}$	$(6_1\ 12_1\ 8_1)$	1	—	—	p_3	$1''$	2, 1	1	
$ce_1\, e_2\, C = tO$	$=$	$e_1\, O$	$[210]\sqrt{2}$	$(24_1\ 36_2\ 14_2)$	$\frac{1}{2}$	p_4	—	p_6	$2''$	3, 1	3	1

$n = 4$

						8	24	32	16				
C_8	$=$	$ce_3\, C_{16}$	$[1\ 1\ 1\ 1]$	$(16_1\ 32_1\ 24_1\ 8_1)$	1	C	—	—	—	1			
$e_1\, C_8$	$=$	$ce_2\, e_3\, C_{16}$	$[1'\, 1'\, 1'\, 1]$	$(64_1\ 128_2\ 88_2\ 24_2)$	$\frac{3}{8}$	tC	—	—	T	$1'$	1		
$e_2\, C_8$	$=$	$ce_1\, e_3\, C_{16}$	$[1'\, 1'\, 1\ 1]$	$(96_1\ 288_2\ 248_4\ 56_3)$	$\frac{3}{8}$	RCO	—	P_3	O	$1'$	2, 1		
$e_3\, C_8$	$=$	$e_3\, C_{16}$	$[1'\, 1\ 1\ 1]$	$(64_1\ 192_2\ 208_3\ 80_4)$	$\frac{3}{8}$	C	P_4	P_3	T	$1'$	3, 2, 1		
$e_1\, e_2\, C_8$	$=$	$ce_1\, e_2\, e_3\, C_{16}$	$[2'\, 2'\, 1'\, 1]$	$(192_1\ 384_3\ 248_4\ 56_3)$	$\frac{3}{8}$	tCO	—	P_3	tT	$2'$	3, 1		
$e_1\, e_3\, C_8$	$=$	$e_2\, e_3\, C_{16}$	$[2'\, 1'\, 1'\, 1]$	$(192_1\ 480_3\ 368_5\ 80_4)$	$\frac{3}{8}$	tC	P_8	P_3	CO	$2'$	4, 2, 1		
$e_2\, e_3\, C_8$	$=$	$e_1\, e_3\, C_{16}$	$[2'\, 1'\, 1\ 1]$	$(192_1\ 480_3\ 368_5\ 80_4)$	$\frac{3}{8}$	RCO	P_4	P_6	tT	$2'$	5, 3, 1		
$e_1\, e_2\, e_3\, C_8$	$=$	$e_1\, e_2\, e_3\, C_{16}$	$[3'\, 2'\, 1'\, 1]$	$(384_1\ 768_4\ 464_6\ 80_4)$	$\frac{3}{8}$	tCO	P_8	P_6	tO	$3'$	6, 3, 1		
$ce_1\, C_8$	$=$	$c\, e_2\, C_{16}$	$[1110]\sqrt{2}$	$(32_1\ 96_1\ 88_2\ 24_2)$	$\frac{1}{2}$	CO	—	—	T	$1''$	1	3	2, 1
$ce_2\, C_8$	$= C_{24} =$	$ce_1\, C_{16}$	$[1100]\sqrt{2}$	$(24_1\ 96_1\ 96_1\ 24_1)$	1	O	—	—	O	$1''$	2, 1	2	1
$ce_3\, C_8$	$=$	C_{16}	$[1000]\sqrt{2}$	$(8_1\ 24_1\ 32_1\ 16_1)$	1	—	—	—	T	$1''$	3, 2, 1	1	
$ce_1\, e_2\, C_8$	$=$	$ce_1\, e_2\, C_{16}$	$[2210]\sqrt{2}$	$(96_1\ 192_2\ 120_3\ 24_2)$	$\frac{3}{8}$	tO	—	—	tT	$2''$	3, 1	5	3, 1
$ce_1\, e_3\, C_8$	$=$	$e_2\, C_{16}$	$[2110]\sqrt{2}$	$(96_1\ 288_2\ 240_4\ 48_3)$	$\frac{3}{8}$	CO	P_4	—	CO	$2''$	4, 2, 1	4	2, 1
$ce_2\, e_3\, C_8$	$=$	$e_1\, C_{16}$	$[2100]\sqrt{2}$	$(48_1\ 120_2\ 96_2\ 24_2)$	$\frac{3}{8}$	O	—	—	tT	$2''$	5, 3, 1	3	1
$ce_1\, e_2\, e_3\, C_8$	$=$	$e_1\, e_2\, C_{16}$	$[3210]\sqrt{2}$	$(192_1\ 384_3\ 240_4\ 48_3)$	$\frac{3}{8}$	tO	P_4	—	tO	$3''$	6, 3, 1	6	3, 1

$n = 5$

						10	40	80	80	32				
C_{10}	$=$	$ce_4\, C_{32}$	$[1\ 1\ 1\ 1\ 1]$	$(32_1\ 80_1\ 80_1\ 40_1\ 10_1)$	1	C_8	—	—	—	—	1			
$e_1\, C_{10}$	$=$	$ce_3\, e_4\, C_{32}$	$[1'\, 1'\, 1'\, 1'\, 1]$	$(160_1\ 400_2\ 400_2\ 200_2\ 42_2)$	$\frac{3}{10}$	$e_1\, C_8$	—	—	—	$S(5)$	$1'$	1		
$e_2\, C_{10}$	$=$	$ce_2\, e_4\, C_{32}$	$[1'\, 1'\, 1'\, 1\ 1]$	$(320_1\ 1280_2\ 1520_4\ 680_4\ 122_3)$	$\frac{3}{10}$	$e_2\, C_8$	—	—	P_T	$ce_1\, S(5)$	$1'$	2, 1		
$e_3\, C_{10}$	$=$	$ce_1\, e_4\, C_{32}$	$[1'\, 1'\, 1\ 1\ 1]$	$(320_1\ 1440_2\ 2160_4\ 1240_6\ 202_4)$	$\frac{3}{10}$	$e_3\, C_8$	—	$(4\,;\,3)$	P_O	$ce_1\, S(5)$	$1'$	3, 2, 1		
$e_4\, C_{10}$	$=$	$e_4\, C_{32}$	$[1'\, 1\ 1\ 1\ 1]$	$(160_1\ 640_2\ 1040_3\ 800_4\ 242_5)$	$\frac{3}{10}$	C_8	P_C	$(4\,;\,3)$	P_T	$S(5)$	$1'$	4, 3, 2, 1		
$e_1\, e_2\, C_{10}$	$=$	$ce_2\, e_3\, e_4\, C_{32}$	$[2'\, 2'\, 2'\, 1'\, 1]$	$(640_1\ 1600_3\ 1520_4\ 680_4\ 122_3)$	$\frac{3}{10}$	$e_1\, e_2\, C_8$	—	—	P_T	$e_1\, S(5)$	$2'$	3, 1		
$e_1\, e_3\, C_{10}$	$=$	$ce_1\, e_3\, e_4\, C_{32}$	$[2'\, 2'\, 1'\, 1'\, 1]$	$(960_1\ 3360_3\ 3760_6\ 1560_7\ 202_4)$	$\frac{3}{10}$	$e_1\, e_3\, C_8$	—	$(8\,;\,3)$	P_O	$e_2\, S(5)$	$2'$	4, 2, 1		
$e_1\, e_4\, C_{10}$	$=$	$e_3\, e_4\, C_{32}$	$[2'\, 1'\, 1'\, 1'\, 1]$	$(640_1\ 2240_3\ 2960_5\ 1600_7\ 242_5)$	$\frac{3}{10}$	$e_1\, C_8$	P_{tC}	$(8\,;\,3)$	P_T	$e_3\, S(5)$	$2'$	5, 3, 2, 1		
$e_2\, e_3\, C_{10}$	$=$	$ce_1\, e_2\, e_4\, C_{32}$	$[2'\, 2'\, 1'\, 1\ 1]$	$(960_1\ 2880_3\ 2960_6\ 1240_6\ 202_4)$	$\frac{3}{10}$	$e_2\, e_3\, C_8$	—	$(4\,;\,3)$	P_{tT}	$ce_1\, e_2\, S(5)$	$2'$	5, 3, 1		
$e_2\, e_4\, C_{10}$	$=$	$e_2\, e_4\, C_{32}$	$[2'\, 1'\, 1'\, 1\ 1]$	$(960_1\ 3840_3\ 4720_7\ 2080_9\ 242_5)$	$\frac{3}{10}$	$e_2\, C_8$	P_{RCO}	$(4\,;\,3)$	P_{CO}	$e_2\, S(5)$	$2'$	6, 4, 2, 1		
$e_3\, e_4\, C_{10}$	$=$	$e_1\, e_4\, C_{32}$	$[2'\, 1'\, 1\ 1\ 1]$	$(640_1\ 2240_3\ 2880_5\ 1520_7\ 242_5)$	$\frac{3}{10}$	$e_3\, C_8$	P_C	$(4\,;\,6)$	P_{tT}	$e_1\, S(5)$	$2'$	7, 5, 3, 1		
$e_1\, e_2\, e_3\, C_{10}$	$=$	$ce_1\, e_2\, e_3\, e_4\, C_{32}$	$[3'\, 3'\, 2'\, 1'\, 1]$	$(1920_1\ 4800_4\ 4240_7\ 1560_7\ 202_4)$	$\frac{3}{10}$	$e_1\, e_2\, e_3\, C_8$	—	$(8\,;\,3)$	P_{tT}	$e_1\, e_2\, S(5)$	$3'$	6, 3, 1		
$e_1\, e_2\, e_4\, C_{10}$	$=$	$e_2\, e_3\, e_4\, C_{32}$	$[3'\, 2'\, 2'\, 1'\, 1]$	$(1920_1\ 5760_4\ 6000_8\ 2400_8\ 242_5)$	$\frac{3}{10}$	$e_1\, e_2\, C_8$	P_{tCO}	$(8\,;\,3)$	P_{CO}	$e_1\, e_3\, S(5)$	$3'$	7, 4, 2, 1		
$e_1\, e_3\, e_4\, C_{10}$	$=$	$e_1\, e_3\, e_4\, C_{32}$	$[3'\, 2'\, 1'\, 1'\, 1]$	$(1920_1\ 5760_4\ 5760_8\ 2160_9\ 242_5)$	$\frac{3}{10}$	$e_1\, e_3\, C_8$	P_{tC}	$(8\,;\,6)$	P_{tT}	$e_2\, e_3\, S(5)$	$3'$	8, 5, 3, 1		
$e_2\, e_3\, e_4\, C_{10}$	$=$	$e_1\, e_2\, e_4\, C_{32}$	$[3'\, 2'\, 1'\, 1\ 1]$	$(1920_1\ 5760_4\ 5920_8\ 2320_9\ 242_5)$	$\frac{3}{10}$	$e_2\, e_3\, C_8$	P_{RCO}	$(4\,;\,6)$	P_{tO}	$e_1\, e_2\, S(5)$	$3'$	9, 6, 3, 1		
$e_1\, e_2\, e_3\, e_4\, C_{10}$	$=$	$e_1\, e_2\, e_3\, e_4\, C_{32}$	$[4'\, 3'\, 2'\, 1'\, 1]$	$(3840_1\ 9600_5\ 8160_{10}\ 2640_{10}\ 242_5)$	$\frac{3}{10}$	$e_1\, e_2\, e_3\, C_8$	P_{tCO}	$(8\,;\,6)$	P_{tO}	$e_1\, e_2\, e_3\, S(5)$	$4'$	10, 6, 3, 1		
$ce_1\, C_{10}$	$=$	$ce_3\, C_{32}$	$[11110]\sqrt{2}$	$(80_1\ 320_1\ 400_2\ 200_2\ 42_2)$	$\frac{2}{5}$	$ce_1\, C_8$	—	—	—	$S(5)$	$1''$	1	4	3, 2, 1
$ce_2\, C_{10}$	$=$	$ce_2\, C_{32}$	$[11100]\sqrt{2}$	$(80_1\ 480_1\ 640_2\ 280_3\ 42_2)$	$\frac{1}{2}$	$ce_2\, C_8$	—	—	—	$ce_1\, S(5)$	$1''$	2, 1	3	2, 1
$ce_3\, C_{10}$	$=$	$ce_1\, C_{32}$	$[11000]\sqrt{2}$	$(40_1\ 240_1\ 400_2\ 240_2\ 42_2)$	$\frac{1}{2}$	$ce_3\, C_8$	—	—	—	$ce_1\, S(5)$	$1''$	3, 2, 1	2	1
$ce_4\, C_{10}$	$=$	C_{32}	$[10000]\sqrt{2}$	$(10_1\ 40_1\ 80_1\ 80_1\ 32_1)$	1	—	—	—	—	$S(5)$	$1''$	4, 3, 2, 1	1	
$ce_1\, e_2\, C_{10}$	$=$	$ce_2\, e_3\, C_{32}$	$[22210]\sqrt{2}$	$(320_1\ 800_2\ 720_3\ 280_3\ 42_2)$	$\frac{3}{10}$	$ce_1\, e_2\, C_8$	—	—	—	$e_1\, S(5)$	$2''$	3, 1	7	5, 3, 1
$ce_1\, e_3\, C_{10}$	$=$	$ce_1\, e_3\, C_{32}$	$[22110]\sqrt{2}$	$(480_1\ 1920_2\ 2160_5\ 840_5\ 122_3)$	$\frac{3}{10}$	$ce_1\, e_3\, C_8$	—	$(4\,;\,3)$	—	$e_2\, S(5)$	$2''$	4, 2, 1	6	4, 2, 1
$ce_1\, e_4\, C_{10}$	$=$	$e_3\, C_{32}$	$[21110]\sqrt{2}$	$(320_1\ 1440_2\ 2160_4\ 1200_6\ 162_4)$	$\frac{3}{10}$	$ce_1\, C_8$	P_{CO}	$(4\,;\,3)$	—	$e_3\, S(5)$	$2''$	5, 3, 2, 1	5	3, 2, 1
$ce_2\, e_3\, C_{10}$	$=$	$ce_1\, e_2\, C_{32}$	$[22100]\sqrt{2}$	$(160_1\ 480_2\ 560_3\ 280_3\ 42_2)$	$\frac{3}{10}$	$ce_2\, e_3\, C_8$	—	—	—	$ce_1\, e_2\, S(5)$	$2''$	5, 3, 1	5	3, 1
$ce_2\, e_4\, C_{10}$	$=$	$e_2\, C_{32}$	$[21100]\sqrt{2}$	$(240_1\ 1200_2\ 1520_4\ 640_4\ 82_3)$	$\frac{3}{10}$	$ce_2\, C_8$	P_O	—	—	$e_2\, S(5)$	$2''$	6, 4, 2, 1	4	2, 1
$ce_3\, e_4\, C_{10}$	$=$	$e_1\, C_{32}$	$[21000]\sqrt{2}$	$(80_1\ 280_2\ 400_2\ 240_2\ 42_2)$	$\frac{3}{10}$	$ce_3\, C_8$	—	—	—	$e_1\, S(5)$	$2''$	7, 5, 3, 1	3	1
$ce_1\, e_2\, e_3\, C_{10}$	$=$	$ce_1\, e_2\, e_3\, C_{32}$	$[33210]\sqrt{2}$	$(960_1\ 2400_3\ 2160_5\ 840_5\ 122_3)$	$\frac{3}{10}$	$ce_1\, e_2\, e_3\, C_8$	—	$(4\,;\,3)$	—	$e_1\, e_2\, S(5)$	$3''$	6, 3, 1	9	6, 3, 1
$ce_1\, e_2\, e_4\, C_{10}$	$=$	$e_2\, e_3\, C_{32}$	$[32210]\sqrt{2}$	$(960_1\ 2880_3\ 2960_6\ 1200_6\ 162_4)$	$\frac{3}{10}$	$ce_1\, e_2\, C_8$	P_{tO}	$(4\,;\,3)$	—	$e_2\, e_3\, S(5)$	$3''$	7, 4, 2, 1	8	5, 3, 1
$ce_1\, e_3\, e_4\, C_{10}$	$=$	$e_1\, e_3\, C_{32}$	$[32110]\sqrt{2}$	$(960_1\ 3360_3\ 3680_6\ 1440_7\ 162_4)$	$\frac{3}{10}$	$ce_1\, e_3\, C_8$	P_{CO}	$(4\,;\,6)$	—	$e_1\, e_3\, S(5)$	$3''$	8, 5, 3, 1	7	4, 2, 1
$ce_2\, e_3\, e_4\, C_{10}$	$=$	$e_1\, e_2\, C_{32}$	$[32100]\sqrt{2}$	$(480_1\ 1440_3\ 1520_4\ 640_4\ 82_3)$	$\frac{3}{10}$	$ce_2\, e_3\, C_8$	P_O	—	—	$e_1\, e_2\, S(5)$	$3''$	9, 6, 3, 1	6	3, 1
$ce_1\, e_2\, e_3\, e_4\, C_{10}$	$=$	$e_1\, e_2\, e_3\, C_{32}$	$[43210]\sqrt{2}$	$(1920_1\ 4800_4\ 4160_7\ 1440_7\ 162_4)$	$\frac{3}{10}$	$ce\ e_2\, e_3\, C_8$	P_{tO}	$(4\,;\,6)$	—	$e_1\, e_2\, e_3\, S(5)$	$4''$	10, 6, 3, 1	10	6, 3, 1

$n = 3$

	g_3	g_2	g_1	g_0		$p.$	$(l)_1$	$(l)_2$	$(l)_3$			
1	$[1\,1\,1] = C$					$r.$	1	1	1	1		
e_1	$[1'1'1] = tC$			$[1\,0\,0]\sqrt{2} = O$	$ce_2\,e_3$		2	2	2	$\frac{3}{8}$	17	$4_3\,(o,\,2tc_{38}),\ 1_4\,(4tc_{88})$
e_2	$[1'1\,1] = RCO$		$[1][1\,0]\sqrt{2} = P_4$	$[1\,1\,0]\sqrt{2} = CO$	$ce_1\,e_3$		2	2	3	$\frac{3}{8}$	20	$4_3\,(c,\,co_{34},\,rco_{34}),\ 2_3\,(c,\,2rco_{44})$
$e_1\,e_2$	$[2'1'1] = tCO$		$[1][1\,0]\sqrt{2} = P_4$	$[2\,1\,0]\sqrt{2} = tO$	$ce_1\,e_2\,e_3$		3	3	3	$\frac{3}{8}$	21	$1_3\,(c,\,2tco_{48}),\ 2_3\,(c,\,tco_{46},\,to_{46}),\ 1_3\,(2tco_{68},\,to_{66})$
e_3	$[1\,1\,1] = C$	$[1\,1][1] = P_4$	$[1][1\,1] = P_4$	$[1\,1\,1] = C$	e_3	$r.$	1	1	1	1		
$e_1\,e_3$	$[1'1'1] = tC$	$[1'1][1] = P_8$	$[1][1\,1] = P_4$	$[1'1\,1] = RCO$	$e_2\,e_3$		3	3	4	$\frac{3}{8}$	22	$1_4\,(c,\,2p^8_{48},\,tc_{88}),\ 2_3\,(c,\,p^8_{44},\,rco_{34}),\ 2_3\,(p^8_{48},\,tc_{38},\,rco_{34})$
$e_1\,e_2\,e_3$	$[2'1'1] = tCO$	$[1'1][1] = P_8$	$[1][1'1] = P_8$	$[2'1'1] = tCO$	$e_1\,e_2\,e_3$	$s.\,p.$	2	3	2	$\frac{3}{8}$	24^{bis}	$2_3\,(p^8_{48},\,p^8_{44},\,tco_{48}),\ 2_3\,(p^8_{48},\,tco_{68},\,tco_{46})$
ce_1	$[1\,1\,0]\sqrt{2} = CO$			$[1\,0\,0]\sqrt{2} = O$	ce_2		1	2	2	$\frac{1}{2}$	18	$8_3\,(2co_{34},\,o)$
$ce_1\,e_2$	$[2\,1\,0]\sqrt{2} = tO$			$[2\,1\,0]\sqrt{2} = tO$	$ce_1\,e_2$	$s.\,p.$	1	2	1	$\frac{1}{2}$	14	$4_3\,(2to_{46},\,2to_{66})$

$n = 4$

	g_4	g_3	g_2	g_1	g_0		$p.$	$(l)_1$	$(l)_2$	$(l)_3$	$(l)_4$	
1	$[1\,1\,1\,1] = 1$						$r.$	1	1	1	1	1
e_1	$[1'1'1'1] = e_1$				$[1\,0\,0\,0]\sqrt{2} = ce_3 = C_{16}$	$ce_3\,e_4$		2	2	2	2	$\frac{3}{10}$
e_2	$[1'1'1\,1] = e_2$			$[1][1\,0\,0]\sqrt{2} = P_O$	$[1\,1\,0\,0]\sqrt{2} = ce_2 = C_{24}$	$ce_2\,e_4$		2	2	3	3	$\frac{3}{10}$
e_3	$[1'1\,1\,1] = e_3$		$[1\,1][1\,0]\sqrt{2} = (p_4;\,p_4) = C_8$	$[1][1\,1\,0]\sqrt{2} = P_{CO}$	$[1\,1\,1\,0]\sqrt{2} = ce_1$	$ce_1\,e_4$		2	2	4	4	$\frac{3}{10}$
$e_1\,e_2$	$[2'2'1'1] = e_1\,e_2$			$[1][1\,0\,0]\sqrt{2} = P_O$	$[2\,1\,0\,0]\sqrt{2} = ce_2\,e_3$	$ce_2\,e_3\,e_4$		2	4	4	3	$\frac{3}{10}$
$e_1\,e_3$	$[2'1'1'1] = e_1\,e_3$		$[1'1][1\,0]\sqrt{2} = (p_8;\,p_4)$	$[1][1\,1\,0]\sqrt{2} = P_{CO}$	$[2\,1\,1\,0]\sqrt{2} = ce_1\,e_3$	$ce_1\,e_3\,e_4$	$s.\,p.$	2	3	5	4	$\frac{3}{10}$
$e_2\,e_3$	$[2'1'1\,1] = e_2\,e_3$		$[1\,1][1\,0]\sqrt{2} = (p_4;\,p_4) = C_8$	$[1][2\,1\,0]\sqrt{2} = P_{tO}$	$[2\,2\,1\,0]\sqrt{2} = ce_1\,e_2$	$ce_1\,e_2\,e_4$		2	3	5	4	$\frac{3}{10}$
$e_1\,e_2\,e_3$	$[3'2'1'1] = e_1\,e_2\,e_3$		$[1'1][1\,0]\sqrt{2} = (p_8;\,p_4)$	$[1][2\,1\,0]\sqrt{2} = P_{tO}$	$[3\,2\,1\,0]\sqrt{2} = ce_1\,e_2\,e_3$	$ce_1\,e_2\,e_3\,e_4$	$s.\,p.$	2	3	5	4	$\frac{3}{10}$
e_4	$[1\,1\,1\,1] = 1$	$[1\,1\,1][1] = P_C = C_8$	$[1\,1][1\,1] = (p_4;\,p_4) = C_8$	$[1][1\,1\,1] = P_C = C_8$	$[1\,1\,1\,1] = 1$	e_4	$r.$	1	1	1	1	1
$e_1\,e_4$	$[1'1'1'1] = e_1$	$[1'1'1][1] = P_{tC}$	$[1'1][1\,1] = (p_8;\,p_4)$	$[1][1\,1\,1] = P_C = C_8$	$[1'1\,1\,1] = e_3$	$e_3\,e_4$		2	3	5	5	$\frac{3}{10}$
$e_2\,e_4$	$[1'1'1\,1] = e_2$	$[1'1\,1][1] = P_{RCO}$	$[1\,1][1\,1] = (p_4;\,p_4) = C_8$	$[1][1'1\,1] = P_{RCO}$	$[1'1'1\,1] = e_2$	$e_2\,e_4$	$s.\,p.$	2	2	4	3	$\frac{3}{10}$
$e_1\,e_2\,e_4$	$[2'2'1'1] = e_1\,e_2$	$[2'1'1][1] = P_{tCO}$	$[1'1][1\,1] = (p_8;\,p_4)$	$[1][1'1\,1] = P_{RCO}$	$[2'1'1\,1] = e_2\,e_3$	$e_2\,e_3\,e_4$		2	4	7	5	$\frac{3}{10}$
$e_1\,e_3\,e_4$	$[2'1'1'1] = e_1\,e_3$	$[1'1'1][1] = P_{tC}$	$[1'1][1'1] = (p_8;\,p_8)$	$[1][1'1'1] = P_{tC}$	$[2'1'1'1] = e_1\,e_3$	$e_1\,e_3\,e_4$	$s.\,p.$	2	3	4	3	$\frac{3}{10}$
$e_1\,e_2\,e_3\,e_4$	$[3'2'1'1] = e_1\,e_2\,e_3$	$[2'1'1][1] = P_{tCO}$	$[1'1][1'1] = (p_8;\,p_8)$	$[1][2'1'1] = P_{tCO}$	$[3'2'1'1] = e_1\,e_2\,e_3$	$e_1\,e_2\,e_3\,e_4$	$s.\,p.$	2	3	5	3	$\frac{3}{10}$
ce_1	$[1\,1\,1\,0]\sqrt{2} = ce_1$				$[1\,0\,0\,0]\sqrt{2} = ce_3 = C_{16}$	ce_3		1	2	2	2	$\frac{2}{5}$
ce_2	$[1\,1\,0\,0]\sqrt{2} = ce_2 = C_{24}$				$[1\,1\,0\,0]\sqrt{2} = ce_2 = C_{24}$	ce_2	$r.$	1	1	1	1	1
$ce_1\,e_2$	$[2\,2\,1\,0]\sqrt{2} = ce_1\,e_2$				$[2\,1\,0\,0]\sqrt{2} = ce_2\,e_3$	$ce_2\,e_3$		2	3	3	2	$\frac{3}{10}$
$ce_1\,e_3$	$[2\,1\,1\,0]\sqrt{2} = ce_1\,e_3$		$[1\,0]\sqrt{2}[1\,0]\sqrt{2} = (p_4;\,p_4) = C_8$		$[2\,1\,1\,0]\sqrt{2} = ce_1\,e_3$	$ce_1\,e_3$	$s.\,p.$	1	2	2	2	$\frac{2}{5}$
$ce_1\,e_2\,e_3$	$[3\,2\,1\,0]\sqrt{2} = ce_1\,e_2\,e_3$		$[1\,0]\sqrt{2}[1\,0]\sqrt{2} = (p_4;\,p_4) = C_8$		$[3\,2\,1\,0]\sqrt{2} = ce_1\,e_2\,e_3$	$ce_1\,e_2\,e_3$	$s.\,p.$	2	2	2	2	$\frac{3}{10}$

	g_5	g_4	g_3	g	g_1	g_0	$p.$	$(l)_0$	$(l)_1$	$(l)_2$	$(l)_3$	
1	$[1\,1\,1\,1\,1] = M_5$						$r.$					1
e_1	$[1'1'1'1'1] = e_1$,,					$[10000]\sqrt{2} = ce_4 M_5$		1	2			$\frac{1}{4}$
e_2	$[1'1'1'1\,1] = e_2$,,				$[1][1000]\sqrt{2} = P_{ce_3 C_8}$	$[11000]\sqrt{2} = ce_3$,,		1	2			$\frac{1}{4}$
e_3	$[1'1'1\,1\,1] = e_3$,,			$[1\,1][100]\sqrt{2} = (p_4;O)$	$[1][1100]\sqrt{2} = P_{ce_2 C_8}$	$[11100]\sqrt{2} = ce_2$,,		1	2			$\frac{1}{4}$
e_4	$[1'1\,1\,1\,1] = e_4$,,		$[1\,1\,1][10]\sqrt{2} = (C;p_4) = M_5$	$[1\,1][110]\sqrt{2} = (p_4;CO)$	$[1][1110]\sqrt{2} = P_{ce_1 C_8}$	$[11110]\sqrt{2} = ce_1$,,		1	2			$\frac{1}{4}$
$e_1 e_2$	$[2'2'2'1'1] = e_1 e_2$,,				$[1][1000]\sqrt{2} = P_{ce_3 C_8}$	$[21000]\sqrt{2} = ce_3 e_4$,,		1	2			$\frac{1}{4}$
$e_1 e_3$	$[2'2'1'1'1] = e_1 e_3$,,			$[1'1][100]\sqrt{2} = (p_8;O)$	$[1][1100]\sqrt{2} = P_{ce_2 C_8}$	$[21100]\sqrt{2} = ce_2 e_4$,,		1	2			$\frac{1}{4}$
$e_1 e_4$	$[2'1'1'1'1] = e_1 e_4$,,		$[1'1'1][10]\sqrt{2} = (tC;p_4)$	$[1'1][110]\sqrt{2} = (p_8;CO)$	$[1][1110]\sqrt{2} = P_{ce_1 C_8}$	$[21110]\sqrt{2} = ce_1 e_4$,,		1	2			$\frac{1}{4}$
$e_2 e_3$	$[2'2'1'1\,1] = e_2 e_3$,,			$[1\,1][100]\sqrt{2} = (p_4;O)$	$[1][2100]\sqrt{2} = P_{ce_2 e_3 C_8}$	$[22100]\sqrt{2} = ce_2 e_3$,,		1	2			$\frac{1}{4}$
$e_2 e_4$	$[2'1'1'1\,1] = e_2 e_4$,,		$[1'1\,1][10]\sqrt{2} = (RCO;p_4)$	$[1\,1][110]\sqrt{2} = (p_4;CO)$	$[1][2110]\sqrt{2} = P_{ce_1 e_3 C_8}$	$[22110]\sqrt{2} = ce_1 e_3$,,		1	2			$\frac{1}{4}$
$e_3 e_4$	$[2'1'1\,1\,1] = e_3 e_4$,,		$[1\,1\,1][10]\sqrt{2} = (C;p_4) = M_5$	$[1\,1][210]\sqrt{2} = (p_4;tO)$	$[1][2210]\sqrt{2} = P_{ce_1 e_2 C_8}$	$[22210]\sqrt{2} = ce_1 e_2$,,		1	2			$\frac{1}{4}$
$e_1 e_2 e_3$	$[3'3'2'1'1] = e_1 e_2 e_3$,,			$[1'1][100]\sqrt{2} = (p_8;O)$	$[1][2100]\sqrt{2} = P_{ce_2 e_3 C_8}$	$[32100]\sqrt{2} = ce_2 e_3 e_4$,,		1	2			$\frac{1}{4}$
$e_1 e_2 e_4$	$[3'2'2'1'1] = e_1 e_2 e_4$,,		$[2'1'1][10]\sqrt{2} = (tCO;p_4)$	$[1'1][110]\sqrt{2} = (p_8;CO)$	$[1][2110]\sqrt{2} = P_{ce_1 e_3 C_8}$	$[32110]\sqrt{2} = ce_1 e_3 e_4$,,		1	2			$\frac{1}{4}$
$e_1 e_3 e_4$	$[3'2'1'1'1] = e_1 e_3 e_4$,,		$[1'1'1][10]\sqrt{2} = (tC;p_4)$	$[1'1][210]\sqrt{2} = (p_8;tO)$	$[1][2210]\sqrt{2} = P_{ce_1 e_2 C_8}$	$[32210]\sqrt{2} = ce_1 e_2 e_4$,,		1	2			$\frac{1}{4}$
$e_2 e_3 e_4$	$[3'2'1'1\,1] = e_2 e_3 e_4$,,		$[1'1\,1][10]\sqrt{2} = (RCO;p_4)$	$[1\,1][210]\sqrt{2} = (p_4;tO)$	$[1][3210]\sqrt{2} = P_{ce_1 e_2 e_3 C_8}$	$[33210]\sqrt{2} = ce_1 e_2 e_3$,,		1	2			$\frac{1}{4}$
$e_1 e_2 e_3 e_4$	$[4'3'2'1'1] = e_1 e_2 e_3 e_4$,,		$[2'1'1][10]\sqrt{2} = (tCO;p_4)$	$[1'1][210]\sqrt{2} = (p_8;tO)$	$[1][3210]\sqrt{2} = P_{ce_1 e_2 e_3 C_8}$	$[43210]\sqrt{2} = ce_1 e_2 e_3 e_4$,,		1	2			$\frac{1}{4}$
e_5	$[1\,1\,1\,1\,1] = M_5$	$[1\,1\,1\,1][1] = P_{C_8} = M_5$	$[1\,1\,1]\,[1\,1] = (C;p_4) = M_5$	$[1\,1]\,[1\,1\,1] = (p_4;C) = M_5$	$[1]\,[1\,1\,1\,1] = P_{C_8} = M_5$	$[1\,1\,1\,1\,1] = M_5$	$r.$					1
$e_1 e_5$	$[1'1'1'1'1] = e_1$,,	$[1'1'1'1][1] = P_{e_1 C_8}$	$[1'1'1]\,[1\,1] = (tC;p_4)$	$[1'1]\,[1\,1\,1] = (p_8;C)$	$[1]\,[1\,1\,1\,1] = P_{C_8} = M_5$	$[1'1\,1\,1\,1] = e_4$,,		1	2			$\frac{1}{4}$
$e_2 e_5$	$[1'1'1'1\,1] = e_2$,,	$[1'1'1\,1][1] = P_{e_2 C_8}$	$[1'1\,1]\,[1\,1] = (RCO;p_4)$	$[1\,1]\,[1\,1\,1] = (p_4;C) = M_5$	$[1]\,[1'1\,1\,1] = P_{e_3 C_8}$	$[1'1'1\,1\,1] = e_3$,,		1	2			$\frac{1}{4}$
$e_1 e_2 e_5$	$[2'2'2'1'1] = e_1 e_2$,,	$[2'2'1'1][1] = P_{e_1 e_2 C_8}$	$[2'1'1]\,[1\,1] = (tCO;p_4)$	$[1'1]\,[1\,1\,1] = (p_8;C)$	$[1]\,[1'1\,1\,1] = P_{e_3 C_8}$	$[2'1'1\,1\,1] = e_3 e_4$,,		1	2			$\frac{1}{4}$
$e_1 e_3 e_5$	$[2'2'1'1'1] = e_1 e_3$,,	$[2'1'1'1][1] = P_{e_1 e_3 C_8}$	$[1'1'1]\,[1\,1] = (tC;p_4)$	$[1'1]\,[1'1\,1] = (p_8;RCO)$	$[1]\,[1'1'1\,1] = P_{e_2 C_8}$	$[2'1'1'1\,1] = e_2 e_4$,,		1	2			$\frac{1}{4}$
$e_1 e_4 e_5$	$[2'1'1'1'1] = e_1 e_4$,,	$[1'1'1'1][1] = P_{e_1 C_8}$	$[1'1'1]\,[1'1] = (tC;p_8)$	$[1'1]\,[1'1'1] = (p_8;tC)$	$[1]\,[1'1'1'1] = P_{e_1 C_8}$	$[2'1'1'1'1] = e_1 e_4$,,	$s.\,p.$	1	2			$\frac{1}{4}$
$e_2 e_3 e_5$	$[2'2'1'1\,1] = e_2 e_3$,,	$[2'1'1'1][1] = P_{e_2 e_3 C_8}$	$[1'1\,1]\,[1\,1] = (RCO;p_4)$	$[1\,1]\,[1'1\,1] = (p_4;RCO)$	$[1]\,[2'1'1\,1] = P_{e_2 e_3 C_8}$	$[2'2'1'1\,1] = e_2 e_3$,,	$s.\,p.$	1	2			$\frac{1}{4}$
$e_1 e_2 e_3 e_5$	$[3'3'2'1'1] = e_1 e_2 e_3$,,	$[3'2'1'1][1] = P_{e_1 e_2 e_3 C_8}$	$[2'1'1]\,[1\,1] = (tCO;p_4)$	$[1'1]\,[1'1\,1] = (p_8;RCO)$	$[1]\,[2'1'1\,1] = P_{e_2 e_3 C_8}$	$[3'2'1'1\,1] = e_2 e_3 e_4$,,		1	2			$\frac{1}{4}$
$e_1 e_2 e_4 e_5$	$[3'2'2'1'1] = e_1 e_2 e_4$,,	$[2'2'1'1][1] = P_{e_1 e_2 C_8}$	$[2'1'1]\,[1'1] = (tCO;p_8)$	$[1'1]\,[1'1'1] = (p_8;tC)$	$[1]\,[2'1'1'1] = P_{e_1 e_3 C_8}$	$[3'2'1'1'1] = e_1 e_3 e_4$,,		1	2			$\frac{1}{4}$
$e_1 e_2 e_3 e_4 e_5$	$[4'3'2'1'1] = e_1 e_2 e_3 e_4$,,	$[3'2'1'1][1] = P_{e_1 e_2 e_3 C_8}$	$[2'1'1]\,[1'1] = (tCO;p_8)$	$[1'1]\,[2'1'1] = (p_8;tCO)$	$[1]\,[3'2'1'1] = P_{e_1 e_2 e_3 C_8}$	$[4'3'2'1'1] = e_1 e_2 e_3 e_4$,,	$s.\,p.$	1	2			$\frac{1}{4}$
ce_1	$[11110]\sqrt{2} = ce_1 M_5$					$[10000]\sqrt{2} = ce_4 M_5$		1	1	2		$\frac{1}{3}$
ce_2	$[11100]\sqrt{2} = ce_2$,,					$[11000]\sqrt{2} = ce_3$,,		1	1	1	2	$\frac{1}{2}$
$ce_1 e_2$	$[22210]\sqrt{2} = ce_1 e_2$,,					$[21000]\sqrt{2} = ce_3 e_4$,,		1	2			$\frac{1}{4}$
$ce_1 e_3$	$[22110]\sqrt{2} = ce_1 e_3$,,			$[10]\sqrt{2}[100]\sqrt{2} = (p_4;O)$		$[21100]\sqrt{2} = ce_2 e_4$,,		1	2			$\frac{1}{4}$
$ce_1 e_4$	$[21110]\sqrt{2} = ce_1 e_4$,,		$[110]\sqrt{2}[10]\sqrt{2} = (CO;p_4)$	$[10]\sqrt{2}[110]\sqrt{2} = (p_4;CO)$		$[21110]\sqrt{2} = ce_1 e_4$,,	$s.\,p.$	1	1	2		$\frac{1}{3}$
$ce_2 e_3$	$[22100]\sqrt{2} = ce_2 e_3$,,					$[22100]\sqrt{2} = ce_2 e_3$,,	$s.\,p.$	1	1	2		$\frac{1}{3}$
$ce_1 e_2 e_3$	$[33210]\sqrt{2} = ce_1 e_2 e_3$,,			$[10]\sqrt{2}[100]\sqrt{2} = (p_4;O)$		$[32100]\sqrt{2} = ce_2 e_3 e_4$,,		1	2			$\frac{1}{4}$
$ce_1 e_2 e_4$	$[32210]\sqrt{2} = ce_1 e_2 e_4$,,		$[210]\sqrt{2}[10]\sqrt{2} = (tO;p_4)$	$[10]\sqrt{2}[110]\sqrt{2} = (p_4;CO)$		$[32110]\sqrt{2} = ce_1 e_3 e_4$,,		1	2			$\frac{1}{4}$
$ce_1 e_2 e_3 e_4$	$[43210]\sqrt{2} = ce_1 e_2 e_3 e_4$,,		$[210]\sqrt{2}[10]\sqrt{2} = (tO;p_4)$	$[10]\sqrt{2}[210]\sqrt{2} = (p_4;tO)$		$[43210]\sqrt{2} = ce_1 e_2 e_3 e_4$,,	$s.\,p.$	1	2			$\frac{1}{4}$

			g_4		g_3		g_2		g_1		g_0		
1	1	1	$[2,\ 0,\ 0,\ 0]$									ce_4	1
2	e_1	e_1	$[4,\ 2,\ 0,\ 0]$							ce_3	$[2,\ 2,\ 0,\ 0]$	$ce_3 e_4$	$\frac{3}{10}$
3	e_2	e_2	$[4,\ 2,\ 2,\ 0]$					P_c	$[2,\ 0][2,\ 0]$	ce_2	$[4,\ 2,\ 2,\ 0]$	$ce_2 e_4$	$\frac{2}{5}$
4	e_3	e_3	$[2+\sqrt2,\ \sqrt2,\ \sqrt2,\ \sqrt2]$			(3 ; 3)	$(\frac43-\frac13\sqrt2,\ -\frac23-\frac13\sqrt2,\ -\frac23-\frac13\sqrt2)[\sqrt2]$ $(\frac43+\frac23\sqrt2,\ -\frac23+\frac23\sqrt2,\ -\frac23+\frac23\sqrt2)\ 0$	P_{co}	$\frac12[\sqrt2+1,\ \sqrt2-1][\sqrt2,\ \sqrt2]$ $[1,\ 1][2\sqrt2,\ 0]$	ce_1	$[3\sqrt2,\ \sqrt2,\ \sqrt2,\ \sqrt2]$ $[2\sqrt2,\ 2\sqrt2,\ 2\sqrt2,\ 0]$	$ce_1 e_4$	$\frac{3}{10}$
5	$e_1 e_2$	$e_1 e_2$	$[6,\ 4,\ 2,\ 0]$					P_c	$[2,\ 0][2,\ 0]$	$ce_2 e_3$	$[6,\ 4,\ 2,\ 0]$	$ce_2 e_3 e_4$	$\frac{3}{10}$
6	$e_1 e_3$	$e_1 e_3$	$[4+\sqrt2,\ 2+\sqrt2,\ \sqrt2,\ \sqrt2]$			(3 ; 6)	$(2-\frac13\sqrt2,\ -\frac13\sqrt2,\ -2-\frac13\sqrt2)[\sqrt2]$ $(2+\frac23\sqrt2,\ \frac23\sqrt2,\ -2+\frac23\sqrt2)\ 0$	P_{co}	$\frac12[\sqrt2+1,\ \sqrt2-1][\sqrt2,\ \sqrt2]$ $[1,\ 1][2\sqrt2,\ 0]$	$ce_1 e_3$	$[2+3\sqrt2,\ 2+\sqrt2,\ \sqrt2,\ \sqrt2]$ $[2+2\sqrt2,\ 2+2\sqrt2,\ 2\sqrt2,\ 0]$	$ce_1 e_3 e_4$	$\frac{3}{10}$
7	$e_2 e_3$	$e_2 e_3$	$[4+\sqrt2,\ 2+\sqrt2,\ 2+\sqrt2,\ \sqrt2]$			(3 ; 3)	$(4+\sqrt2,\ 2+\sqrt2,\ 2+\sqrt2)[\sqrt2]$ $(4+2\sqrt2,\ 2+2\sqrt2,\ 2+2\sqrt2)\ 0$	P_o	$\frac12[2+\sqrt2,\ \sqrt2][2+\sqrt2,\ \sqrt2]$ $[2,\ 0][2+2\sqrt2,\ 0]$	$ce_1 e_2$	$[4+3\sqrt2,\ 2+\sqrt2,\ 2+\sqrt2,\ \sqrt2]$ $[4+2\sqrt2,\ 2+2\sqrt2,\ 2+2\sqrt2,\ 0]$	$ce_1 e_2 e_4$	$\frac{3}{10}$
8	$e_1 e_2 e_3$	$e_1 e_2 e_3$	$[6+\sqrt2,\ 4+\sqrt2,\ 2+\sqrt2,\ \sqrt2]$			(3 ; 6)	$(2-\frac13\sqrt2,\ -\frac13\sqrt2,\ -2-\frac13\sqrt2)[\sqrt2]$ $(2+\frac23\sqrt2,\ \frac23\sqrt2,\ -2+\frac23\sqrt2)\ 0$	P_o	$\frac12[2+\sqrt2,\ \sqrt2][2+\sqrt2,\ \sqrt2]$ $[2,\ 0][2+2\sqrt2,\ 0]$	$ce_1 e_2 e_3$	$[6+3\sqrt2,\ 4+\sqrt2,\ 2+\sqrt2,\ \sqrt2]$ $[6+2\sqrt2,\ 4+2\sqrt2,\ 2+2\sqrt2,\ 0]$	$ce_1 e_2 e_3 e_4$	$\frac{3}{10}$
9	e_4	1	$[2,\ 0,\ 0,\ 0]$	P_r	$(\frac32-\frac12\sqrt2,\ -\frac12-\frac12\sqrt2,\ -\frac12-\frac12\sqrt2,\ -\frac12-\frac12\sqrt2)$ $(\frac32+\frac12\sqrt2,\ -\frac12+\frac12\sqrt2,\ -\frac12+\frac12\sqrt2,\ -\frac12+\frac12\sqrt2)$	(3 ; 3)	$(\frac43-\frac23\sqrt2,\ -\frac23-\frac23\sqrt2,\ -\frac23-\frac23\sqrt2)\ 0$ $(\frac43+\frac13\sqrt2,\ -\frac23+\frac13\sqrt2,\ -\frac23+\frac13\sqrt2)[\sqrt2]$	P_o	$\frac12[\sqrt2+1,\ \sqrt2-1]\ [0,\ 0]$ $(1,\ -1)\ [\sqrt2,\ \sqrt2]$	1	$[2\sqrt2,\ 0,\ 0,\ 0]$ $[\sqrt2,\ \sqrt2,\ \sqrt2,\ \sqrt2]$	e_4	$\frac{3}{10}$
10	$e_1 e_4$	e_1	$[4,\ 2,\ 0,\ 0]$	P_{cr}	$(\frac52-\frac12\sqrt2,\ \frac12-\frac12\sqrt2,\ -\frac32-\frac12\sqrt2,\ -\frac32-\frac12\sqrt2)$ $(\frac52+\frac12\sqrt2,\ \frac12+\frac12\sqrt2,\ -\frac32+\frac12\sqrt2,\ -\frac32+\frac12\sqrt2)$	(3 ; 6)	$(2-\frac23\sqrt2,\ -\frac23\sqrt2,\ -2-\frac23\sqrt2)\ 0$ $(2+\frac13\sqrt2,\ \frac13\sqrt2,\ -2+\frac13\sqrt2)[\sqrt2]$	P_o	$\frac12[\sqrt2+1,\ \sqrt2-1]\ [0,\ 0]$ $(1,\ -1)\ [\sqrt2,\ \sqrt2]$	e_3	$[2+2\sqrt2,\ 2,\ 0,\ 0]$ $[2+\sqrt2,\ 2+\sqrt2,\ \sqrt2,\ \sqrt2]$	$e_3 e_4$	$\frac{3}{10}$
11	$e_2 e_4$	e_2	$[4,\ 2,\ 2,\ 0]$	P_{co}	$(2-\frac12\sqrt2,\ -\frac12\sqrt2,\ -\frac12\sqrt2,\ -2-\frac12\sqrt2)$ $(2+\frac12\sqrt2,\ \frac12\sqrt2,\ \frac12\sqrt2,\ -2+\frac12\sqrt2)$	(3 ; 3)	$(\frac43-\frac23\sqrt2,\ \frac23-\frac23\sqrt2,\ -\frac23-\frac23\sqrt2)\ 0$ $(\frac43+\frac13\sqrt2,\ -\frac23+\frac13\sqrt2,\ -\frac23-\frac13\sqrt2)[\sqrt2]$	P_{ro}	$\frac12[2+\sqrt2,\ \sqrt2][2,\ 0]$ $[2,\ 0][2+\sqrt2,\ \sqrt2]$	e_2	$[4+2\sqrt2,\ 2,\ 2,\ 0]$ $[4+\sqrt2,\ 2+\sqrt2,\ 2+\sqrt2,\ \sqrt2]$	$e_2 e_4$	$\frac{3}{10}$
12	$e_3 e_4$	e_3	$[2+\sqrt2,\ \sqrt2,\ \sqrt2,\ \sqrt2]$	P_r	$(\frac32-\frac12\sqrt2,\ -\frac12-\frac12\sqrt2,\ -\frac12-\frac12\sqrt2,\ \frac12-\frac12\sqrt2)$ $(\frac32+\frac12\sqrt2,\ -\frac12+\frac12\sqrt2,\ -\frac12+\frac12\sqrt2,\ -\frac12+\frac12\sqrt2)$	(6 ; 3)	$(\frac43-\sqrt2,\ -\frac23-\sqrt2,\ -\frac23-\sqrt2)[\sqrt2]$ $(\frac43+\sqrt2,\ -\frac23+\sqrt2,\ -\frac23+\sqrt2)[\sqrt2]$ $(\frac43,\ -\frac23,\ -\frac23)[2\sqrt2]$	P_o	$\frac12[2\sqrt2+1,\ 2\sqrt2-1][\sqrt2,\ \sqrt2]$ $(1,\ -1)[3\sqrt2,\ \sqrt2]$ $\frac12[\sqrt2+1,\ \sqrt2-1][2\sqrt2,\ 2\sqrt2]$	e_1	$[5\sqrt2,\ \sqrt2,\ \sqrt2,\ \sqrt2]$ $[3\sqrt2,\ 3\sqrt2,\ 3\sqrt2,\ \sqrt2]$ $[4\sqrt2,\ 2\sqrt2,\ 2\sqrt2,\ 2\sqrt2]$	$e_1 e_4$	$\frac{3}{10}$
13	$e_1 e_2 e_4$	$e_1 e_2$	$[6,\ 4,\ 2,\ 0]$	P_o	$(3-\frac12\sqrt2,\ 1-\frac12\sqrt2,\ -1-\frac12\sqrt2,\ -3-\frac12\sqrt2)$ $(3+\frac12\sqrt2,\ 1+\frac12\sqrt2,\ 1+\frac12\sqrt2,\ -3+\frac12\sqrt2)$	(3 ; 6)	$(2-\frac23\sqrt2,\ -\frac23\sqrt2,\ -2-\frac23\sqrt2)\ 0$ $(2+\frac13\sqrt2,\ \frac13\sqrt2,\ -2+\frac13\sqrt2)[\sqrt2]$	P_{ro}	$\frac12[2+\sqrt2,\ \sqrt2][2,\ 0]$ $[2,\ 0][2+\sqrt2,\ \sqrt2]$	$e_2 e_3$	$[6+2\sqrt2,\ 4,\ 2,\ 0]$ $[6+\sqrt2,\ 4+\sqrt2,\ 2+\sqrt2,\ \sqrt2]$	$e_2 e_3 e_4$	$\frac{3}{10}$
14	$e_1 e_3 e_4$	$e_1 e_3$	$[4+\sqrt2,\ 2+\sqrt2,\ \sqrt2,\ \sqrt2]$	P_{cr}	$(\frac52-\frac12\sqrt2,\ \frac12-\frac12\sqrt2,\ -\frac32-\frac12\sqrt2,\ -\frac32-\frac12\sqrt2)$ $(\frac52+\frac12\sqrt2,\ \frac12+\frac12\sqrt2,\ -\frac32+\frac12\sqrt2,\ \frac32+\frac12\sqrt2)$	(6 ; 6)	$(2-\sqrt2,\ -\sqrt2,\ -2-\sqrt2)[\sqrt2]$ $(2+\sqrt2,\ \sqrt2,\ -2+\sqrt2)[\sqrt2]$ $(2,\ 0,\ -2)[2\sqrt2]$	P_o	$\frac12[2\sqrt2+1,\ 2\sqrt2-1][\sqrt2,\ \sqrt2]$ $(1,\ 1)[3\sqrt2,\ \sqrt2]$ $\frac12[\sqrt2+1,\ \sqrt2-1][2\sqrt2,\ 2\sqrt2]$	$e_1 e_3$	$[2+5\sqrt2,\ 2+\sqrt2,\ \sqrt2,\ \sqrt2]$ $[2+3\sqrt2,\ 2+3\sqrt2,\ 3\sqrt2,\ \sqrt2]$ $[2+4\sqrt2,\ 2+2\sqrt2,\ 2\sqrt2,\ 2\sqrt2]$	$e_1 e_3 e_4$	$\frac{3}{10}$
15	$e_2 e_3 e_4$	$e_2 e_3$	$[4+\sqrt2,\ 2+\sqrt2,\ 2+\sqrt2,\ \sqrt2]$	P_{co}	$(2-\frac12\sqrt2,\ -\frac12\sqrt2,\ -\frac12\sqrt2,\ -2-\frac12\sqrt2)$ $(2+\frac12\sqrt2,\ \frac12\sqrt2,\ \frac12\sqrt2,\ -2+\frac12\sqrt2)$	(6 ; 3)	$(\frac43-\sqrt2,\ -\frac23-\sqrt2,\ -\frac23-\sqrt2)[\sqrt2]$ $(\frac43+\sqrt2,\ -\frac23+\sqrt2,\ -\frac23+\sqrt2)[\sqrt2]$ $(\frac43,\ -\frac23,\ -\frac23)\ 0$	P_{co}	$\frac12[2+2\sqrt2,\ 2\sqrt2][2+\sqrt2,\ \sqrt2]$ $[2,\ 0][2+3\sqrt2,\ \sqrt2]$ $\frac12[2+\sqrt2,\ \sqrt2][2+2\sqrt2,\ 2\sqrt2]$	$e_1 e_2$	$[4+5\sqrt2,\ 2+\sqrt2,\ 2+\sqrt2,\ \sqrt2]$ $[4+3\sqrt2,\ 2+3\sqrt2,\ 2+3\sqrt2,\ \sqrt2]$ $[4+4\sqrt2,\ 2+2\sqrt2,\ 2+2\sqrt2,\ 2\sqrt2]$	$e_1 e_2 e_4$	$\frac{3}{10}$
16	$e_1 e_2 e_3 e_4$	$e_1 e_2 e_3$	$[6+\sqrt2,\ 4+\sqrt2,\ 2+\sqrt2,\ \sqrt2]$	P_o	$(3-\frac12\sqrt2,\ 1-\frac12\sqrt2,\ -1-\frac12\sqrt2,\ -3-\frac12\sqrt2)$ $(3+\frac12\sqrt2,\ 1+\frac12\sqrt2,\ -1+\frac12\sqrt2,\ -3+\frac12\sqrt2)$	(6 ; 6)	$(2-\sqrt2,\ -\sqrt2,\ -2-\sqrt2)[\sqrt2]$ $(2+\sqrt2,\ \sqrt2,\ -2+\sqrt2)[\sqrt2]$ $(2,\ 0,\ -2)[2\sqrt2]$	P_{co}	$\frac12[2+2\sqrt2,\ 2\sqrt2][2+\sqrt2,\ \sqrt2]$ $[2,\ 0][2+3\sqrt2,\ \sqrt2]$ $\frac12[2+\sqrt2,\ \sqrt2][2+2\sqrt2,\ 2\sqrt2]$	$e_1 e_2 e_3$	$[6+5\sqrt2,\ 4+\sqrt2,\ 2+\sqrt2,\ \sqrt2]$ $[6+3\sqrt2,\ 4+3\sqrt2,\ 2+3\sqrt2,\ \sqrt2]$ $[6+4\sqrt2,\ 4+2\sqrt2,\ 2+2\sqrt2,\ 2\sqrt2]$	$e_1 e_2 e_3 e_4$	$\frac{3}{10}$
17	ce_1	ce_1	$[2,\ 2,\ 0,\ 0]$							ce_3	$[2,\ 2,\ 0,\ 0]$	ce_3	1
18	ce_2	ce_2	$[2,\ 2,\ 2,\ 0]$							ce_2	$[4,\ 2,\ 2,\ 0]$	ce_2	$\frac{2}{5}$
19	ce_3	ce_3	$[2,\ 2,\ 2,\ 2]$							ce_1	$[3\sqrt2,\ \sqrt2,\ \sqrt2,\ \sqrt2]$ $[2\sqrt2,\ 2\sqrt2,\ 2\sqrt2,\ 0]$	ce_1	$\frac{2}{5}$
20	$ce_1 e_2$	$ce_1 e_2$	$[4,\ 4,\ 2,\ 0]$							$ce_2 e_3$	$[6,\ 4,\ 2,\ 0]$	$ce_2 e_3$	$\frac{3}{10}$
21	$ce_1 e_3$	$ce_1 e_3$	$[2+\sqrt2,\ 2+\sqrt2,\ \sqrt2,\ \sqrt2]$			(3 ; 3)	$(\frac43-\frac13\sqrt2,\ \frac23-\frac13\sqrt2,\ -\frac43-\frac13\sqrt2)[\sqrt2]$ $(\frac43+\frac23\sqrt2,\ \frac23+\frac23\sqrt2,\ -\frac43+\frac23\sqrt2)\ 0$			$ce_1 e_3$	$[2+3\sqrt2,\ 2+\sqrt2,\ \sqrt2,\ \sqrt2]$ $[2+2\sqrt2,\ 2+2\sqrt2,\ 2\sqrt2,\ 0]$	$ce_1 e_3$	$\frac{3}{10}$
22	$ce_2 e_3$	$ce_2 e_3$	$[2+\sqrt2,\ 2+\sqrt2,\ 2+\sqrt2,\ \sqrt2]$							$ce_1 e_2$	$[4+3\sqrt2,\ 2+\sqrt2,\ 2+\sqrt2,\ \sqrt2]$ $[4+2\sqrt2,\ 2+2\sqrt2,\ 2+2\sqrt2,\ 0]$	$ce_1 e_2$	$\frac{3}{10}$
23	$ce_1 e_2 e_3$	$ce_1 e_2 e_3$	$[4+\sqrt2,\ 4+\sqrt2,\ 2+\sqrt2,\ \sqrt2]$			(3 ; 3)	$(\frac23-\frac13\sqrt2,\ \frac23-\frac13\sqrt2,\ -\frac43-\frac13\sqrt2)[\sqrt2]$ $(\frac23+\frac23\sqrt2,\ \frac23+\frac23\sqrt2,\ -\frac43+\frac23\sqrt2)\ 0$			$ce_1 e_2 e_3$	$[6+3\sqrt2,\ 4+\sqrt2,\ 2+\sqrt2,\ \sqrt2]$ $[6+2\sqrt2,\ 4+2\sqrt2,\ 2+2\sqrt2,\ 0]$	$ce_1 e_2 e_3$	$\frac{3}{10}$
24	ce_4									1	$[2\sqrt2,\ 0,\ 0,\ 0]$ $[\sqrt2,\ \sqrt2,\ \sqrt2,\ \sqrt2]$	1	1
25	$ce_1 e_4$	ce_1	$[2,\ 2,\ 0,\ 0]$	P_o	$(1-\frac12\sqrt2,\ 1-\frac12\sqrt2,\ -1-\frac12\sqrt2,\ -1-\frac12\sqrt2)$ $(1+\frac12\sqrt2,\ 1+\frac12\sqrt2,\ -1+\frac12\sqrt2,\ -1+\frac12\sqrt2)$	(3 ; 3)	$(\frac23+\frac13\sqrt2,\ \frac23+\frac13\sqrt2,\ -\frac43+\frac13\sqrt2)[\sqrt2]$ $(\frac23-\frac23\sqrt2,\ \frac23-\frac23\sqrt2,\ -\frac43-\frac23\sqrt2)\ 0$			e_3	$[2+2\sqrt2,\ 2,\ 0,\ 0]$ $[2+\sqrt2,\ 2+\sqrt2,\ \sqrt2,\ \sqrt2]$	e_3	$\frac{3}{10}$
26	$ce_2 e_4$	ce_2	$[2,\ 2,\ 2,\ 0]$	P_r	$(\frac12-\frac12\sqrt2,\ \frac12-\frac12\sqrt2,\ \frac12-\frac12\sqrt2,\ -\frac32-\frac12\sqrt2)$ $(\frac12+\frac12\sqrt2,\ \frac12+\frac12\sqrt2,\ \frac12+\frac12\sqrt2,\ -\frac32+\frac12\sqrt2)$					e_2	$[4+2\sqrt2,\ 2,\ 2,\ 0]$ $[4+\sqrt2,\ 2+\sqrt2,\ 2+\sqrt2,\ \sqrt2]$	e_2	$\frac{3}{10}$
27	$ce_3 e_4$	ce_3	$[2,\ 2,\ 2,\ 2]$							e_1	$[5\sqrt2,\ \sqrt2,\ \sqrt2,\ \sqrt2]$ $[3\sqrt2,\ 3\sqrt2,\ 3\sqrt2,\ \sqrt2]$ $[4\sqrt2,\ 2\sqrt2,\ 2\sqrt2,\ 2\sqrt2]$	e_1	$\frac{2}{5}$
28	$ce_1 e_2 e_4$	$ce_1 e_2$	$[4,\ 4,\ 2,\ 0]$	P_{cr}	$(\frac32-\frac12\sqrt2,\ \frac32-\frac12\sqrt2,\ -\frac12-\frac12\sqrt2,\ -\frac52-\frac12\sqrt2)$ $(\frac32+\frac12\sqrt2,\ \frac32+\frac12\sqrt2,\ -\frac12+\frac12\sqrt2,\ -\frac52+\frac12\sqrt2)$	(3 ; 3)	$(\frac83+\frac13\sqrt2,\ \frac23+\frac13\sqrt2,\ -\frac43+\frac13\sqrt2)[\sqrt2]$ $(\frac83-\frac23\sqrt2,\ \frac23-\frac23\sqrt2,\ -\frac43-\frac23\sqrt2)\ 0$			$e_2 e_3$	$[6+2\sqrt2,\ 4,\ 2,\ 0]$ $[6+\sqrt2,\ 4+\sqrt2,\ 2+\sqrt2,\ \sqrt2]$	$e_2 e_3$	$\frac{3}{10}$
29	$ce_1 e_3 e_4$	$ce_1 e_3$	$[2+\sqrt2,\ 2+\sqrt2,\ \sqrt2,\ \sqrt2]$	P_o	$(1-\frac12\sqrt2,\ 1-\frac12\sqrt2,\ 1-\frac12\sqrt2,\ -1-\frac12\sqrt2)$ $(1+\frac12\sqrt2,\ 1+\frac12\sqrt2,\ 1+\frac12\sqrt2,\ -1+\frac12\sqrt2)$	(6 ; 3)	$(\frac23-\sqrt2,\ \frac23-\sqrt2,\ -\frac43-\sqrt2)[\sqrt2]$ $(\frac23+\sqrt2,\ \frac23+\sqrt2,\ -\frac43+\sqrt2)[\sqrt2]$ $(\frac23,\ \frac23,\ -\frac43)[2\sqrt2]$			$e_1 e_3$	$[2+5\sqrt2,\ 2+\sqrt2,\ \sqrt2,\ \sqrt2]$ $[2+3\sqrt2,\ 2+3\sqrt2,\ 3\sqrt2,\ \sqrt2]$ $[2+4\sqrt2,\ 2+2\sqrt2,\ 2\sqrt2,\ 2\sqrt2]$	$e_1 e_3$	$\frac{3}{10}$
30	$ce_2 e_3 e_4$	$ce_2 e_3$	$[2+\sqrt2,\ 2+\sqrt2,\ 2+\sqrt2,\ \sqrt2]$	P_r	$(\frac12-\frac12\sqrt2,\ \frac12-\frac12\sqrt2,\ \frac12-\frac12\sqrt2,\ \frac32-\frac12\sqrt2)$ $(\frac12+\frac12\sqrt2,\ \frac12+\frac12\sqrt2,\ \frac12+\frac12\sqrt2,\ \frac32+\frac12\sqrt2)$					$e_1 e_2$	$[4+5\sqrt2,\ 2+\sqrt2,\ 2+\sqrt2,\ \sqrt2]$ $[4+3\sqrt2,\ 2+3\sqrt2,\ 2+3\sqrt2,\ \sqrt2]$ $[4+4\sqrt2,\ 2+2\sqrt2,\ 2+2\sqrt2,\ 2\sqrt2]$	$e_1 e_2$	$\frac{3}{10}$
31	$ce_1 e_2 e_3 e_4$	$ce_1 e_2 e_3$	$[4+\sqrt2,\ 4+\sqrt2,\ 2+\sqrt2,\ \sqrt2]$	P_{cr}	$(\frac32-\frac12\sqrt2,\ \frac32-\frac12\sqrt2,\ \frac12-\frac12\sqrt2,\ -\frac32-\frac12\sqrt2)$ $(\frac32+\frac12\sqrt2,\ \frac32+\frac12\sqrt2,\ \frac12+\frac12\sqrt2,\ -\frac32+\frac12\sqrt2)$	(6 ; 3)	$(\frac23-\sqrt2,\ \frac23-\sqrt2,\ -\frac43-\sqrt2)[\sqrt2]$ $(\frac23+\sqrt2,\ \frac23+\sqrt2,\ -\frac43+\sqrt2)[\sqrt2]$ $(\frac23,\ \frac23,\ -\frac43)[2\sqrt2]$			$e_1 e_2 e_3$	$[6+5\sqrt2,\ 4+\sqrt2,\ 2+\sqrt2,\ \sqrt2]$ $[6+3\sqrt2,\ 4+3\sqrt2,\ 2+3\sqrt2,\ \sqrt2]$ $[6+4\sqrt2,\ 4+2\sqrt2,\ 2+2\sqrt2,\ 2\sqrt2]$	$e_1 e_2 e_3$	$\frac{3}{10}$

			Faces.						$n = 3$ Limiting polyhedra.														$n = 4$ Limiting polytopes.								
	Symbol of coordinates	Characteristic numbers.	p_3		p_4		p_6		T		O		P_3		tT		CO		P_6		tO		10		40		16		16		
HM_3																															
1 ,,	$\frac{1}{2}[1\ 1\ 1] = T$	(4, 6, 4)	4	3																											
e_2 ,,	$\frac{1}{2}[3\ 1\ 1] = tT$	(12, 18, 8)	4	1			4	2																							
HM_4																															
1 ,,	$\frac{1}{2}[1\ 1\ 1\ 1] = C_{16}$	(8, 24, 32, 16)	32	12					16	8																					
e_2 ,,	$\frac{1}{2}[3\ 3\ 1\ 1] = e_1$,,	(48, 120, 96, 24)	64	4			32	4			8	1			16	4															
e_3 ,,	$\frac{1}{2}[3\ 1\ 1\ 1] = ce_2$,,	(32, 96, 88, 24)	64	6	24	3			16	2							8	3													
$e_2\,e_3$,,	$\frac{1}{2}[5\ 3\ 1\ 1] = ce_1\,e_2$,,	(96, 192, 120, 24)	32	1	24	1	64	4							16	2					8	2									
HM_5																															
1 ,,	$\frac{1}{2}[1\ 1\ 1\ 1\ 1]$	(16, 80, 160, 120, 26)	160	30					120	30													C_{16}	5					$S(5)$	5	
e_2 ,,	$\frac{1}{2}[3\ 3\ 3\ 1\ 1]$	(160, 560, 640, 280, 42)	480	9			160	6	80	2	80	3			120	9							e_1 ,,	3			$c\,e_1\,S(5)$	1	e_1 ,,	2	
e_3 ,,	$\frac{1}{2}[3\ 3\ 1\ 1\ 1]$	(160, 720, 880, 360, 42)	640	12	240	6			120	3	80	3	80	3			80	6					$c\,e_2$,,	2			$c\,e_1$,,	1	e_2 ,,	3	
e_4 ,,	$\frac{1}{2}[3\ 1\ 1\ 1\ 1]$	(80, 400, 720, 480, 82)	480	18	240	12			240	12			240	18									C_{16}	1	P_T	4	$S(5)$	1	e_3 ,,	4	$n = 5$
$e_2\,e_3$,,	$\frac{1}{2}[5\ 5\ 3\ 1\ 1]$	(480, 1200, 1040, 360, 42)	320	2	240	2	480	6					80	1	200	5					80	4	$c\,e_1\,e_2$,,	2			$c\,e_1\,e_2$,,	1	$e_1\,e_2$,,	2	
$e_2\,e_4$,,	$\frac{1}{2}[5\ 3\ 3\ 1\ 1]$	(480, 1680, 1840, 720, 82)	800	5	720	6	320	4			80	1	240	3	160	4	80	2	160	4			e_1 ,,	1	P_{tT}	2	e_2 ,,	1	$e_1\,e_3$,,	2	
$e_3\,e_4$,,	$\frac{1}{2}[5\ 3\ 1\ 1\ 1]$	(320, 1120, 1280, 560, 82)	640	6	480	6	160	3	160	2			160	3	80	3	80	3	80	3			$c\,e_2$,,	1	P_T	1	e_1 ,,	1	$e_2\,e_3$,,	3	
$e_2\,e_3\,e_4$,,	$\frac{1}{2}[7\ 5\ 3\ 1\ 1]$	(960, 2400, 2080, 720, 82)	320	1	960	4	800	5					160	1	160	2			240	3	160	4	$c\,e_1\,e_2$,,	1	P_{tT}	1	$e_1\,e_2$,,	1	$e_1\,e_2\,e_3$,,	2	

HM_6		Symbol of coordinates.	Characteristic numbers.	Faces. p_3	p_4	p_6
1	,,	$\frac{1}{2}[1\ 1\ 1\ 1\ 1\ 1]$	(32, 240, 640, 640, 252, 14)	640 60		
e_2	,,	$\frac{1}{2}[3\ 3\ 3\ 3\ 1\ 1]$	(480, 2160, 3200, 2080, 636, 76)	2560 16		640 8
e_3	,,	$\frac{1}{2}[3\ 3\ 3\ 1\ 1\ 1]$	(640, 3840, 5920, 3520, 876, 76)	4480 21	1440 9	
e_4	,,	$\frac{1}{2}[3\ 3\ 1\ 1\ 1\ 1]$	(480, 3360, 7360, 6240, 1996, 236)	4480 28	2880 24	
e_5	,,	$\frac{1}{2}[3\ 1\ 1\ 1\ 1\ 1]$	(192, 1440, 4000, 4800, 2344, 296)	2560 40	1440 30	
$e_2\ e_3$	,,	$\frac{1}{2}[5\ 5\ 5\ 3\ 1\ 1]$	(1920, 5760, 6560, 3520, 876, 76)	2560 1	1440 3	2560 8
$e_2\ e_4$	,,	$\frac{1}{2}[5\ 5\ 3\ 3\ 1\ 1]$	(2880, 12960, 18240, 10560, 2636, 236,	7680 8	8640 12	1920 4
$e_2\ e_5$	,,	$\frac{1}{2}[5\ 3\ 3\ 3\ 1\ 1]$	(1920, 9600, 16800, 12480, 3656, 296)	7680 12	7200 15	1920 6
$e_3\ e_4$	,,	$\frac{1}{2}[5\ 5\ 3\ 1\ 1\ 1]$	(1920, 7680, 10720, 6720, 1996, 236)	1480 7	1320 9	1920 6
$e_3\ e_5$	,,	$\frac{1}{2}[5\ 3\ 3\ 1\ 1\ 1]$	(1920, 10560, 16960, 11040, 3016, 296)	8320 13	8640 18	
$e_4\ e_5$	,,	$\frac{1}{2}[5\ 3\ 1\ 1\ 1\ 1]$	(960, 5280, 10720, 9120, 3016, 296)	5760 18	4320 18	640 1
$e_2\ e_3\ e_4$	,,	$\frac{1}{2}[7\ 7\ 5\ 3\ 1\ 1]$	(5760, 17280, 19680, 10560, 2636, 236)	3840 2	10080 7	5760 6
$e_2\ e_3\ e_5$	,,	$\frac{1}{2}[7\ 5\ 5\ 3\ 1\ 1]$	(5760, 20160, 25920, 11880, 3656, 296)	5760 3	14400 10	5760 6
$e_2\ e_4\ e_5$	,,	$\frac{1}{2}[7\ 5\ 3\ 3\ 1\ 1]$	(5760, 23040, 31200, 17760, 4136, 296)	9600 5	15840 11	5760 6
$e_3\ e_4\ e_5$	,,	$\frac{1}{2}[7\ 5\ 3\ 1\ 1\ 1]$	(3840, 15360, 21760, 13440, 3196, 296)	7680 6	11520 12	2560 1
$e_2\ e_3\ e_4\ e_5$	,,	$\frac{1}{2}[9\ 7\ 5\ 3\ 1\ 1]$	(11520, 34560, 38400, 19200, 4136, 296)	3840 1	23040 8	11520 6

HM_6	Limiting polyhedra. T	O	P_3	C	tT	CO	P_6	tO
1	640 80							
e_2	960 8	480 6			640 16			
e_3	1120 7	960 9	960 9			480 9		
e_4	1920 16	480 6	3840 48					
e_5	1920 40		2880 90					
$e_2\ e_3$	480 1		960 3		1600 10			480 6
$e_2\ e_4$		960 2	5760 12		960 1	960 1	1920 8	
$e_2\ e_5$	1440 3	960 3	6720 21		1440 9		1920 12	
$e_3\ e_4$	960 2		2880 9		1440 9	480 3	960 6	
$e_3\ e_5$	1440 3	960 3	5760 18	1440 6		1440 9		
$e_4\ e_5$	2880 12		4800 30		480 6		960 12	
$e_2\ e_3\ e_4$			4800 5		1440 3		2880 6	1440 6
$e_2\ e_3\ e_5$			5760 6	1440 2	2400 5	480 1	3840 8	960 4
$e_2\ e_4\ e_5$		960 1	6720 7		2400 5	960 2	6720 14	
$e_3\ e_4\ e_5$	1920 2		5760 9	1440 3	960 3	960 3	1920 6	480 3
$e_2\ e_3\ e_4\ e_5$			3840 2	1440 1	1920 2		9600 10	2400 5

HM_6	Limiting polytopes $(P)_4$. 60	192	192	192	240	240	240	640	640
1	1 C_{16}	1 $S(5)$							
e_2	e_1 ,,	1 ,,	$e_1\ S(5)$	$c\ e_1\ S(5)$					
e_3	$c\ e_2$,,	e_2 ,,	$c\ e_1$,,	$c\ e_1$,,	P_T				
e_4	1 ,,	1 ,,	e_3 ,,	$c\ e_1$,,	2 P_T	P_O		(3 ; 3)	
e_5	2 ,,	1 ,,	1 ,,		5 P_T			(3 ; 3)	
$e_2\ e_3$	$c\ e_1\ e_2$,,	e_1 ,,	$e_1\ e_2$,,	$c\ e_1\ e_2$,,	P_I				
$e_2\ e_4$	e_1 ,,	e_2 ,,	e_2 ,,	$e_1\ e_3$,,	P_O	2 P_{tT}		(3 ; 3)	(3 ; 6)
$e_2\ e_5$	2 e_1 ,,	e_1 ,,	e_3 ,,	$c\ e_1$,,	P_T	2 P_O	4 P_{tT}	(3 , 3)	(3 ; 6)
$e_3\ e_4$	$c\ e_2$,,	e_1 ,,	$e_1\ e_3$,,	$c\ e_1\ e_2$,,	2 P_I	P_{tT}		(3 ; 3)	
$e_3\ e_5$	2 $c\ e_2$,,	e_2 ,,	e_2 ,,	$c\ e_1$,,	4 P_I	3 P_{tO}		(3 ; 3)	
$e_4\ e_5$	2 ,,	1 ,,	e_1 ,,	e_3 ,,	6 P_T	P_{tT}		(6 ; 3)	
$e_2\ e_3\ e_4$	$c\ e_1\ e_2$,,	$e_1\ e_2$,,	$e_1\ e_2$,,	$e_1\ e_2\ e_3$,,	3 P_{tT}			(3 ; 3)	(3 ; 6)
$e_2\ e_3\ e_5$	2 $c\ e_1\ e_2$,,	$e_1\ e_2$,,	$e_1\ e_3$,,	$c\ e_1\ e_2$,,	1 P_{tI}	P_{tO}	2 P_{tO}	(3 ; 3)	(3 ; 6)
$e_2\ e_4\ e_5$	2 e_1 ,,	e_2 ,,	$e_1\ e_3$,,	$e_1\ e_3$,,	2 P_O	7 P_{tT}		(6 ; 3)	(6 , 6)
$e_3\ e_4\ e_5$	2 $c\ e_2$,,	e_1 ,,	$e_1\ e_2$,,	$e_1\ e_3$,,	6 P_{tI}	2 P_{tO}	P_{tO}	(6 3)	
$e_2\ e_3\ e_4\ e_5$	2 $c\ e_1\ e_2$,,	$e_1\ e_2$,,	$e_1\ e_2\ e_3$,,	$e_1\ e_2\ e_3$,,	6 P_{tI}	3 P_{tO}		(6 , 3)	(6 , 6)

HM_6	Limiting polytopes $(P)_5$. 12	60	160	32	32
1	$\frac{1}{2}[1\ 1\ 1\ 1\ 1]$				$S(6)$
e_2	$\frac{1}{2}[3\ 3\ 3\ 1\ 1]$			$c\ e_1\ S(6)$	e_1 ,,
e_3	$\frac{1}{2}[3\ 3\ 1\ 1\ 1]$			$c\ e_2$,,	e_2 ,,
e_4	$\frac{1}{2}[3\ 1\ 1\ 1\ 1]$		$(3\ 3\ 1)\frac{1}{2}[1\ 1\ 1]$	$c\ e_1$,,	e_3 ,,
e_5	$\frac{1}{2}[1\ 1\ 1\ 1\ 1]$	$(31)\frac{1}{2}[1\ 1\ 1\ 1]$	$(3\ 3\ 1)\frac{1}{2}[1\ 1\ 1]$	$S(6)$	e_4 ,,
$e_2\ e_3$	$\frac{1}{2}[5\ 5\ 3\ 1\ 1]$			$c\ e_1\ e_2$,,	$e_1\ e_2$,,
$e_2\ e_4$	$\frac{1}{2}[5\ 3\ 3\ 1\ 1]$		$(5\ 5\ 3)\frac{1}{2}[3\ 1\ 1]$	$c\ e_1\ e_3$,,	$e_1\ e_3$,,
$e_2\ e_5$	$\frac{1}{2}[3\ 3\ 3\ 1\ 1]$	$(53)\frac{1}{2}[3\ 3\ 1\ 1]$	$(5\ 3\ 3)\frac{1}{2}[3\ 1\ 1]$	e_3 ,,	$e_1\ e_4$,,
$e_3\ e_4$	$\frac{1}{2}[5\ 3\ 1\ 1\ 1]$		$(5\ 5\ 3)\frac{1}{2}[1\ 1\ 1]$	$c\ e_1\ e_2$,,	$e_2\ e_3$,,
$e_3\ e_5$	$\frac{1}{2}[3\ 3\ 1\ 1\ 1]$	$(53)\frac{1}{2}[3\ 1\ 1\ 1]$	$(5\ 3\ 3)\frac{1}{2}[1\ 1\ 1]$	e_2 ,,	$e_2\ e_4$,,
$e_4\ e_5$	$\frac{1}{2}[3\ 1\ 1\ 1\ 1]$	$(53)\frac{1}{2}[1\ 1\ 1\ 1]$	$(5\ 3\ 1)\frac{1}{2}[1\ 1\ 1]$	e_1 ,,	$e_3\ e_4$,,
$e_2\ e_3\ e_4$	$\frac{1}{2}[7\ 5\ 3\ 1\ 1]$		$(7\ 7\ 5)\frac{1}{2}[3\ 1\ 1]$	$c\ e_1\ e_2\ e_3$,,	$e_1\ e_2\ e_3$,,
$e_2\ e_3\ e_5$	$\frac{1}{2}[5\ 5\ 3\ 1\ 1]$	$(75)\frac{1}{2}[5\ 3\ 1\ 1]$	$(7\ 5\ 5)\frac{1}{2}[3\ 1\ 1]$	$e_2\ e_3$,,	$e_1\ e_2\ e_4$,,
$e_2\ e_4\ e_5$	$\frac{1}{2}[5\ 3\ 3\ 1\ 1]$	$(75)\frac{1}{2}[3\ 3\ 1\ 1]$	$(7\ 5\ 3)\frac{1}{2}[3\ 1\ 1]$	$e_1\ e_3$,,	$e_1\ e_3\ e_4$,,
$e_3\ e_4\ e_5$	$\frac{1}{2}[5\ 3\ 1\ 1\ 1]$	$(75)\frac{1}{2}[3\ 1\ 1\ 1]$	$(7\ 5\ 3)\frac{1}{2}[1\ 1\ 1]$	$e_1\ e_2$,,	$e_2\ e_3\ e_4$,,
$e_2\ e_3\ e_4\ e_5$	$\frac{1}{2}[7\ 5\ 3\ 1\ 1]$	$(97)\frac{1}{2}[5\ 3\ 1\ 1]$	$(9\ 7\ 5)\frac{1}{2}[3\ 1\ 1]$	$e_1\ e_2\ e_3$,,	$e_1\ e_2\ e_3\ e_4$,,

HMPD. NETS IN S_5.

Table X.

Nets	A_4	A_t	A_3	A_0	B_4	B_3	B_2	B_1	B_0	C_4	C_3	C_2	C_0
NH_5	C_{16}	$S(5)$	—	—	$S(5)$	—	—	—	—	—	—	—	—
e_1 ,,	e_1 ,,	e_1 ,,	—	ce_1 $S(5)$	e_1 ,,	—	—	—	C_{16}	ce_1 $S(5)$	—	—	C_{16}
e_2 ,,	ce_2 ,,	e_2 ,,	—	ce_1 ,,	e_2 ,,	—	—	P_O	ce_1 ,,	ce_1 ,,	—	—	ce_1 ,,
e_3 ,,	C_{16}	e_3 ,,	P_T	$S(5)$	e_3 ,,	—	(4 ; 3)	P_{CO}	ce_2 ,,	$S(5)$	—	—	ce_2 ,,
e_4 ,,	C_{16}	$S(5)$	—	—	$S(5)$	P_T	(4 ; 3)	P_C	ce_3 ,,	—	—	—	ce_3 ,,
e_1e_2 ,,	ce_1e_2 ,,	e_1e_2 ,,	—	ce_1e_2 $S(5)$	e_1e_2 ,,	—	—	P_O	e_1 ,,	ce_1e_2 $S(5)$	—	—	e_1 ,,
e_1e_3 ,,	e_1 ,,	e_1e_3 ,,	P_{tT}	e_2 ,,	e_1e_3 ,,	—	(4 ; 6)	P_{CO}	e_2 ,,	e_2 ,,	—	(4 ; 3)	e_2 ,,
e_1e_4 ,,	e_1 ,,	e_1 ,,	—	ce_1 ,,	e_1 ,,	P_{tT}	(4 ; 6)	P_C	e_3 ,,	ce_1 ,,	P_O	(4 ; 3)	e_3 ,,
e_2e_3 ,,	ce_2 ,,	e_2e_3 ,,	P_T	e_1 ,,	e_2e_3 ,,	—	(4 ; 3)	P_{tO}	ce_1e_2 ,,	e_1 ,,	—	—	ce_1e_2 ,,
e_2e_4 ,,	ce_2 ,,	e_2 ,,	—	ce_1 ,,	e_2 ,,	P_{CO}	(4 ; 3)	P_{RCO}	ce_1e_3 ,,	ce_1 ,,	P_T	—	ce_1e_3 ,,
e_3e_4 ,,	C_{16}	e_3 ,,	P_T	$S(5)$	e_3 ,,	P_T	(8 ; 3)	P_{tC}	ce_2e_3 ,,	$S(5)$	—	—	ce_2e_3 ,,
$e_1e_2e_3$,,	ce_1e_2 ,,	$e_1e_2e_3$,,	P_{tT}	e_1e_2 ,,	$e_1e_2e_3$,,	—	(4 ; 6)	P_{tO}	e_1e_2 ,,	e_1e_2 ,,	—	(4 ; 3)	e_1e_2 ,,
$e_1e_2e_4$,,	ce_1e_2 ,,	e_1e_2 ,,	—	ce_1e_2 ,,	e_1e_2 ,,	[illegible]	(4 ; 6)	P_{RCO}	e_1e_3 ,,	ce_1e_2 ,,	[illegible]	(4 ; 3)	e_1e_3 ,,
$e_1e_3e_4$,,	e_1 ,,	e_1e_3 ,,	P_{tT}	e_2 ,,	e_1e_3 ,,	P_{tT}	(8 ; 6)	P_{tC}	e_2e_3 ,,	e_2 ,,	P_O	(8 ; 3)	e_2e_3 ,,
$e_2e_3e_4$,,	ce_2 ,,	e_2e_3 ,,	P_T	e_1 ,,	e_2e_3 ,,	P_{CO}	(8 ; 3)	P_{tCO}	$ce_1e_2e_3$,,	e_1 ,,	P_T	—	$ce_1e_2e_3$,,
$e_1e_2e_3e_4$,,	ce_1e_2 ,,	$e_1e_2e_3$,,	P_{tT}	e_1e_2 ,,	$e_1e_2e_3$,,	P_{tO}	(8 ; 6)	P_{tCO}	$e_1e_2e_3$,,	e_1e_2 ,,	P_{tT}	(8 ; 3)	$e_1e_2e_3$,,
	10	16	40	16	32	80	80	40	10	32	80	80	10

									CONSTITUENTS.				
Nets	D_4	D_3	D_t	D_0	E_a	E_b	E_c	E_d	A	B	C	base of D	E
NH_5	—	—	—	—	—	—	—	—	HM_5	Cr_5	Cr_5	—	—
e_1 ,,	—	—	—	—	—	—	—	—	e_2 ,,	e_1 ,,	ce_1 ,,	—	—
e_2 ,,	—	—	—	—	—	—	—	—	e_3 ,,	e_2 ,,	ce_2 ,,	—	—
e_3 ,,	—	—	—	—	P_T	(4 ; 3)	—	—	e_4 ,,	e_3 ,,	ce_3 ,,	—	$(T ; p_4)$
e_4 ,,	C_{16}	P_T	P_T	—	—	(4 ; 3)	—	P_T	e_5 ,,	e_4 ,,	ce_4 ,,	½ [1 1 1 1]	$(T ; p_4)$
e_1e_2 ,,	—	—	—	—	—	—	—	—	e_2e_3 ,,	e_1e_2 ,,	ce_1e_2 ,,	—	—
e_1e_3 ,,	—	—	—	—	P_{tT}	(4 ; 6)	(4 ; 3)	—	e_2e_4 ,,	e_1e_3 ,,	ce_1e_3 ,,	—	$(tT ; p_4)$
e_1e_4 ,,	e_1 C_{16}	P_{tT}	P_{tT}	P_O	—	(4 ; 6)	(4 ; 3)	P_{tT}	e_2e_5 ,,	e_1e_4 ,,	ce_1e_4 ,,	½ [3 3 1 1]	$(tT ; p_4)$
e_2e_3 ,,	—	—	—	—	P_T	(4 ; 3)	—	—	e_3e_4 ,,	e_2e_3 ,,	ce_2e_3 ,,	—	$(T ; p_4)$
e_2e_4 ,,	ce_2 C_{16}	P_T	P_{CO}	P_T	—	(4 ; 3)	—	P_T	e_3e_5 ,,	e_2e_4 ,,	ce_2e_4 ,,	½ [3 1 1 1]	$(T ; p_4)$
e_3e_4 ,,	C_{16}	P_T	P_T	—	P_T	(8 ; 3)	—	P_T	e_4e_5 ,,	e_3e_4 ,,	ce_3e_4 ,,	,, [1 1 1 1]	$(T ; p_8)$
$e_1e_2e_3$,,	—	—	—	—	P_{tT}	(4 ; 6)	(4 ; 3)	—	$e_2e_3e_4$,,	$e_1e_2e_3$,,	$ce_1e_2e_3$,,	—	$(tT ; p_4)$
$e_1e_2e_4$,,	ce_1e_2 C_{16}	[illegible]	P_{tO}	[illegible]	—	(4 ; 6)	(4 ; 3)	[illegible]	$e_2e_3e_5$,,	$e_1e_2e_4$,,	$ce_1e_2e_4$,,	½ [5 3 1 1]	$(tT ; p_4)$
$e_1e_3e_4$,,	e_1 ,,	P_{tT}	P_{tT}	P_O	P_{tT}	(8 ; 6)	(8 ; 3)	P_{tT}	$e_2e_4e_5$,,	$e_1e_3e_4$,,	$ce_1e_3e_4$,,	,, [3 3 1 1]	$(tT ; p_8)$
$e_2e_3e_4$,,	ce_2 ,,	P_T	P_{CO}	P_T	P_T	(8 ; 3)	—	P_T	$e_3e_4e_5$,,	$e_2e_3e_4$,,	$ce_2e_3e_4$,,	,, [3 1 1 1]	$(T ; p_8)$
$e_1e_2e_3e_4$,,	ce_1e ,,	P_{tT}	P_{tO}	P_{tT}	P_{tT}	(8 ; 6)	(8 ; 3)	P_{tT}	$e_2e_3e_4e_5$,,	$e_1e_2e_3e_4$,,	$ce_1e_2e_3e_4$,,	,, [5 3 1 1]	$(tT ; p_8)$
	2	8	8	8	4	4	4	4					

	CONSTITUENTS in an other notation.				
Nets	C	B	A	D	E
NH_5	—	[1 0 0 0 0] 2	½ [1 1 1 1 1]	—	—
e_1 ,,	[1 1 0 0 0] 2	[2 1 0 0 0] ,,	,, [3 3 3 1 1]	—	—
e_2 ,,	[1 1 1 0 0] ,,	[2 1 1 0 0] ,,	,, [3 3 1 1 1]	—	—
e_3 ,,	[1 1 1 1 0] ,,	[2 1 1 1 0] ,,	,, [3 1 1 1 1]	—	½ [1 1 1] [1 0] 2
e_4 ,,	[1 1 1 1 1] $\sqrt{2}$	[1' 1 1 1 1] $\sqrt{2}$	,, [1 1 1 1 1]	½ [1 1 1 1] [1] $\sqrt{2}$	,, [1 1 1] [1 1] $\sqrt{2}$
e_1e_2 ,,	[2 2 1 0 0] 2	[3 2 1 0 0] 2	,, [5 5 3 1 1]	—	—
e_1e_3 ,,	[2 2 1 1 0] ,,	[3 2 1 1 0] ,,	,, [5 3 3 1 1]	—	½ [3 1 1] [1 0] 2
e_1e_4 ,,	[1' 1' 1 1 1] $\sqrt{2}$	[2' 1' 1 1 1] $\sqrt{2}$	,, [3 3 3 1 1]	½ [3 3 1 1] [1] $\sqrt{2}$	,, [3 1 1] [1 1] $\sqrt{2}$
e_2e_3 ,,	[2 2 2 1 0] 2	[3 2 2 1 0] 2	,, [5 3 1 1 1]	—	,, [1 1 1] [1 0] 2
e_2e_4 ,,	[1' 1' 1' 1 1] $\sqrt{2}$	[2' 1' 1' 1 1] $\sqrt{2}$	,, [3 3 1 1 1]	½ [3 1 1 1] [1] $\sqrt{2}$	,, [1 1 1] [1 1] $\sqrt{2}$
e_3e_4 ,,	[1' 1' 1' 1' 1] ,,	[2' 1' 1' 1' 1] ,,	,, [3 1 1 1 1]	,, [1 1 1 1] [1] ,,	,, [1 1 1] [1' 1] ,,
$e_1e_2e_3$,,	[3 3 2 1 0] 2	[4 3 2 1 0] 2	,, [7 5 3 1 1]	—	,, [3 1 1] [1 0] 2
$e_1e_2e_4$,,	[2' 2' 1' 1 1] $\sqrt{2}$	[3 2 1 1 1] $\sqrt{2}$	,, [5 5 3 1 1]	½ [5 3 1 1] [1] $\sqrt{2}$	,, [3 1 1] [1 1] $\sqrt{2}$
$e_1e_3e_4$,,	[2' 2' 1' 1' 1] ,,	[3' 2' 1' 1' 1] ,,	,, [5 3 3 1 1]	,, [3 3 1 1] [1] ,,	,, [3 1 1] [1' 1] ,,
$e_2e_3e_4$,,	[2' 2' 2' 1' 1] ,,	[3' 2' 2' 1' 1] ,,	,, [5 3 1 1 1]	,, [3 1 1 1] [1] ,,	,, [1 1 1] [1' 1] ,,
$e_1e_2e_3e_4$,,	[3' 3' 2' 1' 1] ,,	[4' 3' 2' 1' 1] ,,	,, [7 5 3 1 1]	,, [5 3 1 1] [1] ,,	,, [3 1 1] [1' 1] ,,

CONSTITUENTS.

Nets	C	B	A	$A^{(5)}$	$A^{(4)}$	$A^{(3)}$
NH_6	—	[1 0 0 0 0 0] 2	½ [1 1 1 1 1 1]	—	—	—
e_1 ,,	[1 1 0 0 0 0] 2	[2 1 0 0 0 0] ,,	,, [3 3 3 3 1 1]	—	—	—
e_2 ,,	[1 1 1 0 0 0] ,,	[2 1 1 0 0 0] ,,	,, [3 3 3 1 1 1]	—	—	—
e_3 ,,	[1 1 1 1 0 0] ,,	[2 1 1 1 0 0] ,,	,, [3 3 1 1 1 1]	—	—	½ [1 1 1] [1 0 0] 2
e_4 ,,	[1 1 1 1 1 0] ,,	[2 1 1 1 1 0] ,,	,, [3 1 1 1 1 1]	—	½ [1 1 1 1] [1 0] 2	—
e_5 ,,	[1 1 1 1 1 1] √2	[1'1 1 1 1 1] √2	,, [1 1 1 1 1 1]	½ [1 1 1 1 1] [1] √2	—	—
$e_1 e_2$,,	[2 2 1 0 0 0] 2	[3 2 1 0 0 0] 2	,, [5 5 3 3 1 1]	—	—	—
$e_1 e_3$,,	[2 2 1 1 0 0] ,,	[3 2 1 1 0 0] ,,	,, [5 5 3 3 1 1]	—	—	½ [3 1 1] [1 0 0] 2
$e_1 e_4$,,	[2 2 1 1 1 0] ,,	[3 2 1 1 1 0] ,,	,, [5 3 3 3 1 1]	—	½ [3 3 1 1] [1 0] 2	—
$e_1 e_5$,,	[1'1'1 1 1 1] √2	[2'1'1 1 1 1] √2	,, [3 3 3 3 1 1]	½ [3 3 3 1 1] [1] √2	—	—
$e_2 e_3$,,	[2 2 2 1 0 0] 2	[3 2 2 1 0 0] 2	,, [5 5 3 1 1 1]	—	—	½ [1 1 1] [1 0 0] 2
$e_2 e_4$,,	[2 2 2 1 1 0] ,,	[3 2 2 1 1 0] ,,	,, [5 3 3 1 1 1]	—	½ [3 3 1 1] [1 0] 2	—
$e_2 e_5$,,	[1'1'1'1 1 1] √2	[2'1'1'1 1 1] √2	,, [3 3 3 1 1 1]	½ [3 3 1 1 1] [1] √2	—	—
$e_3 e_4$,,	[2 2 2 2 1 0] 2	[3 2 2 2 1 0] 2	,, [5 3 1 1 1 1]	—	½ [1 1 1 1] [1 0] 2	½ [1 1 1] [2 1 0] 2
$e_3 e_5$,,	[1'1'1'1'1 1] √2	[2'1'1'1'1 1] √2	,, [3 3 1 1 1 1]	½ [3 1 1 1 1] [1] √2	—	,, [1 1 1] [1'1 1] √2
$e_4 e_5$,,	[1'1'1'1'1'1] ,,	[2'1'1'1'1'1] ,,	,, [3 1 1 1 1 1]	,, [1 1 1 1 1] [1] ,,	½ [1 1 1 1] [1'1] √2	—
$e_1 e_2 e_3$,,	[3 3 2 1 0 0] 2	[4 3 2 1 0 0] 2	,, [7 7 5 3 1 1]	—	—	½ [3 1 1] [1 0 0] 2
$e_1 e_2 e_4$,,	[3 3 2 1 1 0] ,,	[4 3 2 1 1 0] ,,	,, [7 5 5 3 1 1]	—	½ [5 3 1 1] [1 0] 2	—
$e_1 e_2 e_5$,,	[2'2'1'1 1 1] √2	[3'2'1'1 1 1] √2	,, [5 5 5 3 1 1]	½ [5 5 3 1 1] [1] √2	—	—
$e_1 e_3 e_4$,,	[3 3 2 2 1 0] 2	[4 3 2 2 1 0] 2	,, [7 5 3 3 1 1]	—	½ [3 3 1 1] [1 0] 2	½ [3 1 1] [2 1 0] 2
$e_1 e_3 e_5$,,	[2'2'1'1'1 1] √2	[3'2'1'1'1 1] √2	,, [5 5 3 3 1 1]	½ [5 3 3 1 1] [1] √2	—	,, [3 1 1] [1'1 1] √2
$e_1 e_4 e_5$,,	[2'2'1'1'1'1] ,,	[3'2'1'1'1'1] 2	,, [5 3 3 3 1 1]	,, [3 3 3 1 1] [1] ,,	½ [3 3 1 1] [1'1] √2	—
$e_2 e_3 e_4$,,	[3 3 3 2 1 0] 2	[4 3 3 2 1 0] 2	,, [7 5 3 1 1 1]	—	½ [3 1 1 1] [1 0] 2	½ [1 1 1] [2 1 0] 2
$e_2 e_3 e_5$,,	[2'2'2'1'1 1] √2	[3'2'2'1'1 1] √2	,, [5 5 3 1 1 1]	½ [5 3 1 1 1] [1] √2	—	,, [1 1 1] [1'1 1] √2
$e_2 e_4 e_5$,,	[2'2'2'1'1'1] ,,	[3'2'2'1'1'1] ,,	,, [5 3 3 1 1 1]	,, [3 3 1 1 1] [1] ,,	½ [3 1 1 1] [1'1] √2	—
$e_3 e_4 e_5$,,	[2'2'2'2'1'1] ,,	[3'2'2'2'1'1] ,,	,, [5 3 1 1 1 1]	,, [3 1 1 1 1] [1] ,,	,, [1 1 1 1] [1'1] ,,	½ [1 1 1] [2'1'1] √2
$e_1 e_2 e_3 e_4$,,	[4 4 3 2 1 0] 2	[5 4 3 2 1 0] 2	,, [9 7 5 3 1 1]	—	,, [5 3 1 1] [1 0] 2	,, [3 1 1] [2 1 0] 2
$e_1 e_2 e_3 e_5$,,	[3'3'2'1'1 1] √2	[4'3'2'1'1 1] √2	,, [7 7 5 3 1 1]	½ [7 5 3 1 1] [1] √2	—	,, [3 1 1] [1'1 1] √2
$e_1 e_2 e_4 e_5$,,	[3'3'2'1'1'1] ,,	[4'3'2'1'1'1] ,,	,, [7 5 5 3 1 1]	,, [5 5 3 1 1] [1] ,,	½ [5 3 1 1] [1'1] √2	—
$e_1 e_3 e_4 e_5$,,	[3'3'2'2'1'1] ,,	[4'3'2'2'1'1] ,,	,, [7 5 3 3 1 1]	,, [5 3 3 1 1] [1] ,,	,, [3 3 1 1] [1'1] ,,	½ [3 1 1] [2'1'1] √2
$e_2 e_3 e_4 e_5$,,	[3'3'3'2'1'1] ,,	[4'3'3'2'1'1] ,,	,, [7 5 3 1 1 1]	,, [5 3 1 1 1] [1] ,,	,, [3 1 1 1] [1'1] ,,	,, [1 1 1] [2'1'1] ,,
$e_1 e_2 e_3 e_4 e_5$,,	[4'4'3'2'1'1] ,,	[5'4'3'2'1'1] ,,	,, [9 7 5 3 1 1]	,, [7 5 3 1 1] [1] ,,	,, [5 3 1 1] [1'1] ,,	,, [3 1 1] [2'1'1] ,,

ÉTUDE
SUR LES
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(EERSTE SECTIE).

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(Avec une planche.)

AMSTERDAM,
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Valeur approximative d'une intégrale définie.

§ 1. La plupart du temps la valeur d'une intégrale $\int_a^b \varphi(x)\, dx$ ne peut pas, par intégration, être déterminée d'une façon simple; souvent même la fonction placée sous le signe intégral a une forme si compliquée, qu'il est extrêmement difficile et même parfois tout à fait impossible de l'intégrer. Généralement on ne peut non plus, sans des calculs embarrassants, exécuter le calcul approximatif de l'intégrale en développant sa fonction dans une série. Par contre, on peut toujours aisément, et d'habitude assez vite, et cela avec toute l'exactitude désirable, trouver, au moyen de formules dites d'approximation, une valeur approximative de presque toutes les intégrales.

La valeur de l'intégrale $\int \varphi(x)\, dx$, entre les limites a et b, peut, dans le sens géométrique, être représentée par l'aire de la surface plane $A'\,A\,B\,B'$ [1]) limitée par l'axe X, les ordonnées qui appartiennent aux abscisses $O'A' = a$ et $O'B' = b$ et la courbe dont l'équation est

$$y = \varphi(x) \ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

Si maintenant la fonction sub (1) *peut* être représentée par une série continue, infinie ou non, de la forme

$$y = f(x) = K_0 + K_1\, x + K_2\, x^2 + \text{etc.} \ldots\ldots \quad (2)$$

alors on peut toujours, comme nous le verrons ci-dessous, développer des formules par lesquelles il est donné de calculer des *valeurs approximatives* de l'intégrale définie, sans qu'il soit nécessaire d'intégrer ou de connaître la série elle-même, et, dans ce dernier cas, malgré que les formules d'approximation en question se basent sur cette série.

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[1]) Voir la figure à la fin de cet exposé; les axes des coordonnées sont considérés comme perpendiculaires l'un sur l'autre.

Si la série est convergente, elle peut être représentée, avec un degré illimité d'exactitude, par la série *finie:*

$$y = f(x) = K_0 + K_1 x + K_2 x^2 + K_3 x^3 + \ldots + K_{z-1} x^{z-1} \ldots . \quad (3)$$

qui représente l'équation d'une courbe parabolique passant par z points de la ligne sub (1).

Dans (3), on peut attribuer à z une valeur aussi grande que l'on voudra.

Nous attribuerons, dans cet exposé, à z une grandeur telle que les deux lignes sub (3) et (1) coïncident à extrêmement peu de chose près, de sorte que l'aire de la figure limitée par la ligne sub (3) a assez bien la même grandeur, que celle limitée par la ligne sub (1). Dans ce cas, la première aire peut prendre la place de la seconde.

§ 2. Les séries, dont il est question sub (3), peuvent être divisées en deux classes.

Dans la première classe se ramènent les séries dans lesquelles les coefficients K représentent *uniquement* des valeurs connues et dans lesquelles dans les n premiers, éventuellement dans les $2n$ premiers termes, *aucun* K n'est égal à zéro, c'est-à-dire qu'aucun des termes avec K_0 jusque K_{n-1} inclusivement, éventuellement jusque K_{2n-1} inclusivement, ne manque [1]).

Toutes les autres séries, nous les considérons comme appartenant à la seconde classe.

§ 3. Alors que, en rapport avec la convergence, il suffit, au point de vue théorique, que la série sub (3) converge, afin d'en pouvoir calculer des valeurs approximatives d'une intégrale définie, *la pratique* exige que la série *converge assez fort pour qu'un nombre borné de termes* (n) puissent donner une approximation suffisante.

L'approximation doit donc, lorsque le nombre des termes de la série augmente ou du moins lorsque cette augmentation est quelque peu considérable, s'améliorer d'une façon *constante.*

Par exemple, dans la série

$$\frac{x}{a+1} + \frac{x^2}{(a+2)^2} + \frac{x^3}{(a+3)^3} + \frac{x^4}{(a+4)^4} + \ldots$$

[1]) Ce n'est que dans le § 7 qu'il apparaîtra que par n on entend ici le nombre de termes d'une série dont les coefficients, chacun en particulier, sont, pour la déduction d'une formule d'approximation, assimilés à zéro.

après substitution des valeurs de $x = 1$ et $a = 10.1$ et calcul des termes, les *six* premiers termes semblent être assez convergents; cependant ce qu'ils fournissent ne ressemble aucunement au résultat qu'on doit obtenir, lequel n'apparaît que lorsqu'on prend au moins douze termes.

Des séries dont il faudra prendre plus d'une douzaine de termes afin d'en pouvoir calculer une valeur suffisamment approximative conviennent moins à la déduction de formules d'approximation, parce qu'elles exigeraient des calculs trop étendus.

Dans cet exposé, nous admettrons toujours que les séries sur lesquelles se basent les formules d'approximation en question, convergent suffisamment après un nombre relativement petit de termes.

SECTION I.

Formules d'approximation lorsqu'on peut substituer à la fonction sous le signe intégral une série de la première classe.

Aire exacte d'une figure plane limitée par une courbe parabolique dont l'équation peut être représentée par une série de la première classe.

§ 4. Sans nuire à la généralité du problème que nous traitons dans cette section, nous pouvons admettre que l'origine des abscisses coïncide avec la première ordonnée de la figure.

A ce titre, l'axe $O' Y''$ (voir la figure à la fin de cet exposé) est déplacé du point O' au point A'; par là, à l'égard de la ligne $A' Y'$, considérée comme l'axe des y, l'équation de la vraie ligne limite de la figure sub (1) devient

$$y = f(x) \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (4)$$

et celle de la courbe parabolique sub (3)

$$\begin{aligned} y = L_0 + L_1 x + L_2 x^2 + \ldots + L_{n-1} x^{n-1} + L_n x^n + \ldots + \\ + L_{2n-2} x^{2n-2} + L_{2n-1} x^{2n-1} + L_{2n} x^{2n} + \ldots + L_{z-1} x^{z-1} \quad . . \quad (5) \end{aligned}$$

Nous représenterons l'aire de la figure limitée par la courbe, dont il est question sub (5), par I et nous l'appellerons *l'aire exacte* de la figure, parce que z, ayant une valeur aussi grande qu'on le veut, la différence entre I et *l'aire vraie*, c'est-à-dire l'aire parfaite de la figure limitée par la ligne sub (4) est inférieure, en valeur absolue, à toute grandeur donnée si petite qu'on veut.

Si l'on admet $A'B'$ comme unité de longueur, alors I est représenté par

$$I = \int_0^1 \left\{ L_0 + L_1 x + L_2 x^2 + \ldots + L_{n-1} x^{n-1} + L_n x^n + \ldots + L_{z-1} x^{z-1} \right\} dx$$

donc

$$I = L_0 + \tfrac{1}{2} L_1 + \tfrac{1}{3} L_2 + \ldots + \tfrac{1}{n} L_{n-1} + \tfrac{1}{n+1} L_n + \ldots + \tfrac{1}{z} L_{z-1} \,.\,. \quad (6)$$

Aire approximative de la figure et erreur de l'approximation.

§ 5. Etant donné, que de la vraie ligne limite $y = f(x)$, sub (4), on ne connaît les coordonnées que de n points, soit $n < z$, distribués arbitrairement sur cette ligne limite, il est possible, quelque grossière que puisse être l'approximation, de désigner une aire approximative de la figure $A' A B B'$ par

$$I_1 = R_1 y_1 + R_2 y_2 + R_3 y_3 + \ldots + R_n y_n \ldots\ldots. \quad (7)$$

où $y_1, y_2, y_3, \ldots y_n$ représentent les ordonnées des n points connus de la vraie ligne limite de la figure et $R_1, R_2, R_3, \ldots R_n$ des nombres arbitraires.

Lorsque l'erreur de l'approximation de I_1 sub (7), c'est-à-dire la différence entre l'aire exacte I et l'aire approximative I_1 est représentée par E alors

$$I - I_1 = E \qquad \text{et} \qquad I = I_1 + E \ldots\ldots\ldots \quad (8)$$

§ 6. Si les valeurs des ordonnées des n points connus, calculées de $y = f(x)$, sub (4), sont transportées dans (7), alors nous pouvons, en vertu de (5), établir:

$$\begin{aligned} I_1 = {} & R_1 \left\{ L_0 + L_1 x_1 + L_2 x_1^2 + \ldots + L_{n-1} x_1^{n-1} + L_n x_1^n + \ldots + L_{2n-1} x_1^{2n-1} + L_{2n} x_1^{2n} + \ldots + L_{z-1} x_1^{z-1} \right\} + \\ & + R_2 \left\{ L_0 + L_1 x_2 + L_2 x_2^2 + \ldots + L_{n-1} x_2^{n-1} + L_n x_2^n + \ldots + L_{2n-1} x_2^{2n-1} + L_{2n} x_2^{2n} + \ldots + L_{z-1} x_2^{z-1} \right\} + \\ & + R_3 \left\{ L_0 + L_1 x_3 + L_2 x_3^2 + \ldots + L_{n-1} x_3^{n-1} + L_n x_3^n + \ldots + L_{2n-1} x_3^{2n-1} + L_{2n} x_3^{2n} + \ldots + L_{z-1} x_3^{z-1} \right\} + \\ & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ & + R_n \left\{ L_0 + L_1 x_n + L_2 x_n^2 + \ldots + L_{n-1} x_{n-1} + L_n x_n^n + \ldots + L_{2n-1} x_n^{2n-1} + L_{2n} x_n^{2n} + \ldots + L_{z-1} x_n^{z-1} \right\} = \\ & = \Sigma \left\{ x_1^p R_1 + x_2^p R_2 + x_3^p R_3 + \ldots + x_n^p R_n \right\} L_p . \end{aligned}$$

Cette équation étant soustraite de celle sub (6), donne, en vertu de (8),

$$E = \Sigma \left\{ \tfrac{1}{p+1} - (x_1^p R_1 + x_2^p R_2 + x_3^p R_3 + \ldots + x_n^p R_n \right\} L_p \,.\,. \quad (9)$$

et

$$I = I_1 + E = I_1 + \Sigma \left\{\frac{1}{p+1} - (x_1^p R_1 + x_2^p R_2 + x_3^p R_3 + \ldots + \right.$$
$$\left. + \ldots + x_n^p R_n\right\} L_p \ldots\ldots\ldots\ldots\ldots \quad (10)$$

où p doit être remplacé successivement par 0, 1, 2, 3,...(z—1).

§ 7. Dans (9) se présentent, outre les grandeurs tout à fait inconnues L, deux groupes chacun de n grandeurs, à savoir: un groupe de n grandeurs x et un groupe de n grandeurs R, donc en tout $2n$ grandeurs. Aux grandeurs x et R que l'on veut supposer connues, peuvent s'attribuer des valeurs arbitraires, alors que les grandeurs inconnues peuvent se calculer, au moyen des équations que l'on obtient en égalant dans (9) autant de coefficients de L à 0, que contient le nombre des grandeurs inconnues de x et R. Ainsi dans (9) nous pouvons, *indépendamment des valeurs inconnues de* L, éliminer à notre gré 0, 1, 2, etc. jusqu' *au plus* $2n$ termes de E. [1])

§ 8. L'expression dans le dernier membre de (10) donne l'aire *exacte* de toute figure, qui est limitée par une courbe parabolique, dont l'équation se trouve sub (5); elle est valable pour toutes les valeurs attribuées à n, x et R. Cependant, quoique une formule dans laquelle x et R sont arbitraires, soit mathématiquement exacte, elle ne convient pourtant pas pour en obtenir une valeur utilisable de I_1. Notamment, nous ne connaissons aucune des grandeurs L et nous sommes donc obligés de négliger tous les termes de E, sub (9), pour autant qu'ils ne sont pas égalés à 0, qui pris ensemble peuvent constituer une valeur considérable.

Par conséquent on doit tâcher de développer pour I des formules, dans lesquelles les termes de E, sub (9), aussi bien quant à leur nombre que quant à leur grandeur, sont réduits à un minimum; car apparemment dans (10) I_1 diffèrera moins de I à mesure que E, sub (9), devient plus petit.

Quoique nous puissions éliminer chaque terme de chaque groupe de n termes arbitraires de E qui se suivent ou non, il convient que nous prenions à cet effet de préférence les termes dont les grandeurs L sont affectés du plus petit indice, parce que, la série sub (5) étant convergente, les termes affectés de l'indice le plus petit ont ordinairement une valeur considérablement plus grande que ceux affectés d'un indice plus grand. A ce titre, pour la déter-

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[1]) Dans chacun de ces cas, il se forme un groupe spécial de formules d'approximation dont quelques-unes, qui sont connues sous le nom de leurs auteurs, seront indiquées dans le § 10.

mination des valeurs de x et R, nous égalerons toujours, dans cet exposé, à zéro les coefficients des termes affectés de l'indice le plus petit dans (9).

Si par conséquent dans (9) les $2n$ coefficients de L_0 jusque L_{2n-1} inclusivement sont chacun en particulier égalés à zéro, si des $2n$ équations qui en résultent on déduit les n valeurs de x et les n valeurs de R, et si on transporte ces valeurs dans (7) et (10), alors on obtient pour (10) une expression pour I dans laquelle les termes de E avec L_0 jusque L_{2n-1} ne se présentent plus et on obtient donc pour (7) une expression pour I_1 qui est la plus exacte qui puisse être déduite de (9).

Si l'on suppose, par exemple, le cas où l'on veut, de n ordonnées, déterminer une valeur approximative d'une intégrale, alors on trouve les n valeurs de R, sub (7), qui pour n ordonnées, conduisent à la formule la plus exacte, au moyen des $2n$ équations qu'on obtient en égalant à 0 chacun en particulier les coefficients de L_0 jusque L_{2n-1} inclusivement. On a de la sorte (pour $p = 0, 1, 2, 3, \ldots 2n-1$):

$$\left.\begin{array}{l} R_1 + R_2 + R_3 + \ldots + R_n = 1\,, \\ x_1 R_1 + x_2 R_2 + x_3 R_3 + \ldots + x_n R_n = \frac{1}{2}\,, \\ x_1^2 R_1 + x_2^2 R_2 + x_3^2 R_3 + \ldots + x_n^2 R_n = \frac{1}{3}\,, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ x_1^{2n-1} R_1 + x_2^{2n-1} R_2 + x_3^{2n-1} R_3 + \ldots + x_n^{2n-1} R_n = \frac{1}{2n}\,. \end{array}\right\} \ldots \quad (11)$$

Avant de poursuivre, nous faisons remarquer qu'à chaque x_p, qui satisfait aux équations sub (11), s'attache en même temps une valeur $(1 - x_p)$ [1].

Pour le démontrer on prend des équations sub (11) les $(k+1)$ premières, on les multiplie en numéro d'ordre par les coefficients binomiaux du $k^{\text{ième}}$ degré affectés de signes alternativement $+$ et $-$ et additionne le tout. Alors on obtient

$$R_1 \left\{1 - \frac{k}{1} x_1 + \frac{k(k-1)}{1.\,2} x_1^2 - \ldots + (-1)^k x_1^{\,k}\right\} +$$
$$+ R_2 \left\{1 - \frac{k}{1} x_2 + \frac{k(k-1)}{1.\,2} x_2^2 - \ldots + (-1)^k x_2^{\,k}\right\} +$$
$$+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots +$$

[1] Il en résulte immédiatement que les ordonnées doivent se placer deux à deux à égale distance des deux côtés de la ligne qui, au milieu de la base de la figure, est perpendiculaire à cette base. Le nombre des ordonnées doit donc toujours être *pair*; il est vrai deux d'entre elles, notamment *les deux du milieu*, peuvent coïncider et, dans ce cas, le nombre des ordonnées *à calculer* est *impair*. Voyez l'alinéa final du § 26.

$$+R_n\{1-\frac{k}{1}x_n+\frac{k(k-1)}{1.2}x_n^2-\ldots+(-1)^k x_n^k\}=$$
$$=1-\frac{k}{1}\cdot\frac{1}{2}+\frac{k(k-1)}{1.2}\cdot\frac{1}{3}-\ldots+(-1)^k\frac{1}{k+1}.$$

Dans le premier membre de cette équation, les formes avec lesquelles R_1, R_2, R_3,...R_n ont été multipliés sont égales aux $k^{\text{ièmes}}$ puissances de $(1-x_1)$, $(1-x_2)$, $(1-x_3)\ldots(1-x_n)$; le second membre est égal à $\frac{1}{k+1}$, notamment égal au terme initial de la série des $k^{\text{ièmes}}$ différences de la série harmonique 1, $\frac{1}{2}$, $\frac{1}{3}$,... [1])

Par conséquent on obtient

$$R_1(1-x_1)^k+R_2(1-x_2)^k+\ldots+R_n(1-x_n)^k=\frac{1}{k+1}.$$

Cette équation est valable pour $k=0$, 1, 2,...$(2n-1)$. On obtient de nouveau, après substitution de ces valeurs pour k, les

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[1]) Si l'on forme d'une série de nombres arbitraires

$$A_1\ ,\ A_2\ ,\ A_3,\ldots A_n$$

les séries des 1e, 2e,...ke différences, alors on obtient pour le terme initial des ke différences, la formule

$$\Delta^k A_1=A_{k+1}-\frac{k}{1}A_k+\frac{k(k-1)}{1.2}A_{k-1}-\ldots+(-1)^k A_1.$$

Si l'on y suppose $A_1=1$, $A_2=\frac{1}{2}$, $A_3=\frac{1}{3}$,...$A^k=\frac{1}{k}$ alors, en renversant l'ordre, on a

$$\Delta^k A_1=(-1)^k\{1-\frac{k}{1}\cdot\frac{1}{2}+\frac{k(k-1)}{1.2}\cdot\frac{1}{3}-\ldots+(-1)^k\frac{1}{k+1}\}\ \ldots\ldots\ (12)$$

Par une soustraction effective, on obtient dans le cas de la série harmonique:

série donnée	$\frac{1}{1}$,	$\frac{1}{2}$,	$\frac{1}{3}$,	$\frac{1}{4}$,	$\frac{1}{5}$, ...
1e différences	$-\frac{1}{1.2}$,	$-\frac{1}{2.3}$,	$-\frac{1}{3.4}$,	$-\frac{1}{4.5}$,	....
2e différences	$\frac{1.2.}{1.2.3}$,	$\frac{1.2}{2.3.4}$,	$\frac{1.2}{3.4\ 5}$;	....	

....

Par induction, on décide que la série des ke différences peut être représentée par

$$(-1)^k\frac{k!}{(k+1)!},(-1)^k\frac{k!}{(k+2)!},\ldots\ldots\ldots\ldots\ldots$$

La justesse de cette induction apparaît, quand on déduit de ces ke différences supposées les $(k+1)$e. On obtient alors immédiatement la même chose que quand on substitue $(k+1)$ à k.

En simplifiant, on obtient

$$\Delta^k A_1=\frac{(-1)^k}{k+1}$$

En égalant cette expression à celle sub (12) et en divisant par $(-1)^k$ on a

$$\frac{1}{k+1}=1-\frac{k}{1}\cdot\frac{1}{2}+\frac{k(k-1)}{1.2}\cdot\frac{1}{3}-\ldots+(-1)^k\frac{1}{k+1}.$$

équations sub (11), seulement chaque x_p est remplacé par $(1-x_p)$, ce qu'il fallait démontrer.

Apparemment cette déduction, où il est supposé que la fonction sous le signe intégral peut être remplacée par une série de la *première* classe, ne peut être poursuivie lorsque cette fonction ne peut être représentée par une série complète. Voir § 23, 3e phrase.

Développement des formules d'approximation lorsque l'axe des ordonnées est placé au milieu de la figure.

§ 9, L'équation de la courbe limite de la figure étant généralement donnée par rapport à la ligne qui, comme axe des y, est élevée perpendiculairement *au milieu* O de la figure, nous supposerons désormais dans cette section, que l'axe $A'\,Y'$ est placé au point O au lieu du point A', de sorte que l'équation sub (4) de la vraie ligne limite de la figure, par rapport à la ligne $O\,Y$ comme axe des y, devient

$$y = \Psi(x) \ldots\ldots\ldots\ldots \quad (13)$$

et l'équation sub (5) de la courbe parabolique devient

$$y = A_0 + A_1 x + A_2 x^2 + A_3 x^3 + \ldots + A_{z-1} x^{z-1} \ldots \quad (14)$$

Si nous représentons les coordonnées d'un point, à gauche de l'axe Y, par $(-x_p, y_{-p})$ et celles d'un point qui, à droite, se trouve à la même distance de l'axe Y que le point précédent, par (x_p, y_{+p}), alors il faut, puisque z est ici toujours pair, remplacer z par $2z$, de sorte que nous avons

$$y_{-p} = A_0 - A_1 x_p + A_2 x_p^2 - A_3 x_p^3 + \ldots + A_{2z-2} x_p^{2z-2} - A_{2z-1} x_p^{2z-1}$$

et

$$y_{+p} = A_0 + A_1 x_p + A_2 x_p^2 + A_3 x_p^3 + \ldots + A_{2z-2} x_p^{2z-2} + A_{2z-1} x_p^{2z-1}$$

donc

$$\frac{y_{-p} + y_{+p}}{2} = A_0 + A_2 x_p^2 + A_4 x_p^4 + A_6 x_p^6 + \ldots + A_{2z-2} x_p^{2z-2} \ldots \quad (15)$$

et I est représenté par

$$I = \int_{-\frac{1}{2}}^{\frac{1}{2}} \{ A_0 + A_2 x^2 + A_4 x^4 + A_6 x^6 + \ldots + A_{2z-2} x^{2z-2} \} \, dx;$$

par conséquent

$$I = A_0 + \tfrac{1}{3}(\tfrac{1}{2})^2 A_2 + \tfrac{1}{5}(\tfrac{1}{2})^4 A_4 + \tfrac{1}{7}(\tfrac{1}{2})^6 A_6 + \tfrac{1}{9}(\tfrac{1}{2})^8 A_8 + \ldots +$$
$$+ \tfrac{1}{2z-1}(\tfrac{1}{2})^{2z-2} A_{2z-2} + \ldots\ldots\ldots \quad (16)$$

On peut désigner une valeur approximative de I, c'est-à-dire I_1, par

$$I_1 = R_1 . \frac{y_{-1} + y_{+1}}{2} + R_2 . \frac{y_{-2} + y_{+2}}{2} + R_3 \frac{y_{-3} + y_{+3}}{2} + \ldots + \\ + R_m \frac{y_{-m} + y_{+m}}{2} \ldots\ldots\ldots\ldots\ldots \quad (17)$$

où $(y_{-1} + y_{+1})$, $(y_{-2} + y_{+2})$, etc. représentent des couples d'ordonnées des $2m$ points connus de la vraie ligne limite de la figure sub (13) et $R_1, R_2, \ldots R_m$ des nombres arbitraires.

Si l'on représente l'erreur de l'approximation, c'est-à-dire la différence entre l'aire exacte I sub (16) et l'aire approximative I_1 sub (17) par E, alors on a

$$I - I_1 = E \text{ et } I = I_1 + E \ldots\ldots\ldots\ldots \quad (18)$$

En transportant dans (17) les demi-sommes calculées de (15) des $2m$ ordonnées connues, on obtient

$$\begin{aligned} I_1 &= R_1 \{A_0 + A_2 x_1^2 + A_4 x_1^4 + \ldots + A_{2z-2} x_1^{2z-2}\} + \\ &+ R_2 \{A_0 + A_2 x_2^2 + A_4 x_2^4 + \ldots + A_{2z-2} x_2^{2z-2}\} + \\ &+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &+ R_m \{A_0 + A_2 x_m^2 + A_4 x_m^4 + \ldots + A_{2z-2} x_m^{2z-2}\} = \\ &= \Sigma (x_1^{2p} R_1 + x_2^{2p} R_2 + x_3^{2p} R_3 + \ldots + x_m^{2p} R_m) A_{2p}. \end{aligned}$$

Cette équation soustraite de (16) donne, en vertu de (18)

$$E = \Sigma \left\{ \frac{1}{2p+1} \left(\tfrac{1}{2}\right)^{2p} - (x_1^{2p} R_1 + x_2^{2p} R_2 + x_3^{2p} R_3 + \ldots + x_m^{2p} R_m) \right\} A_{2p} \quad (19)$$

où il faut remplacer successivement p par 0, 1, 2, 3, ... $z-1$.

Au sujet des formules d'approximation les plus connues, spécialement celles de GAUSS.

§ 10. Des expressions pour I_1 et E, développées dans le § précédent on peut déduire les principales formules d'approximation connues, ainsi que les fautes qui y appartiennent comme, par exemple, les formules d'approximation selon NEWTON-COTES, STIRLING, EULER, MACLAURIN, GAUSS, CHRISTOFFEL, LOBATTO et HERMITE-TCHEBICHEFF.

Toutes les formules mentionnées ci-dessus excepté celle selon GAUSS pour un nombre *pair* d'ordonnées et celles selon HERMITE-TCHEBICHEFF, s'obtiennent en attribuant certaines valeurs fixées d'avance respectivement à toutes les abcisses $x_1, x_2, x_3, \ldots x_m$ ou à une partie seulement d'entre elles et en calculant les valeurs correspondantes de $R_1, R_2, \ldots R_m$ de (19) en y assimilant à zéro autant de coefficients de A que E comporte d'inconnues x et R.

Dans le développement des formules d'approximation selon HER-

MITE-TCHEBICHEFF, les coefficients des premiers m termes de E (19) sont égalés à 0, dans ces coefficients les R sont considérés comme égaux entre eux, c'est-à-dire qu'on prend $R_1 = R_2 = R_3 = \ldots = R_m = \frac{1}{m}$ et au moyen des m équations qui se sont formées de la sorte, on détermine les valeurs de x [1]).

Des formules mentionnées ci-dessus, nous ne discuterons que celles de GAUSS.

Les formules d'approximation de GAUSS.

§ 11. GAUSS traite deux cas [2]).

1e Dans le *premier cas*, aucune des $2m$ grandeurs x et R ne sont considérées comme connues, de sorte que le nombre de coefficients de A, c'est-à-dire le nombre de termes de E qui dans (19) peuvent être égalés à zéro, est de $2m$; le nombre des ordonnées à calculer est, dans ce cas, *pair*.

On a alors pour déterminer x et R les $2m$ équations suivantes

$$\left.\begin{array}{l} R_1 + R_2 + R_3 + \ldots + R_m = 1 \,, \\ x_1^2 R_1 + x_2^2 R_2 + x_3^2 R_3 + \ldots + x_m^2 R_m = \frac{1}{3}\left(\frac{1}{2}\right)^2 \,, \\ x_1^4 R_1 + x_2^4 R_2 + x_3^4 R_3 + \ldots + x_m^4 R_m = \frac{1}{5}\left(\frac{1}{2}\right)^4 \,, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ x_1^{4m-2} R_1 + x_2^{4m-2} R_2 + x_3^{4m-2} R_3 + \ldots + x_m^{4m-2} R_m = \frac{1}{4m-1}\left(\frac{1}{2}\right)^{4m-2}. \end{array}\right\} \ldots (20)$$

Si l'on résoud de (20) x^2 et R, on obtient les carrés des abscisses $x_1, x_2, \ldots x_m$, pour lesquelles les ordonnées $y_{\pm 1}, y_{\pm 2}, y_{\pm 3}, \ldots y_{\pm m}$ doivent être calculées de (13) et pour R les valeurs qui doivent être substituées dans (17) pour obtenir la formule pour I_1, exprimées dans $y_{\pm 1}, y_{\pm 2}, \ldots y_{\pm m}$.

A cet effet on suppose dans les §§ 26 et 27 (où la solution de x^2 et R d'équations comme celles sub (20) est donnée en général) $d = 0$, $k = 2$, $c = o$, $v = m$ et $u_p = \frac{1}{2p+1}\left(\frac{1}{2}\right)^{2p}$ et dans § 28: $r = 2m + 1$, $\Delta = 2$, $B_0 = 1$, $B_1 = 2^2 S_1$, $B_2 = 2^4 S_2, \ldots B_m = 2^{2m} S_m$, alors on trouve que les carrés $x_1^2, x_2^2, \ldots x_m^2$ de (20) sont les racines de l'équation (voir (71))

$$(x^2)^m - S_1 (x^2)^{m-1} + \ldots + (-1)^m S_m = 0$$

où, en vertu de (74)

---

[1]) Toutes les formules citées dans ce § sont développées dans notre ouvrage: *Valeur approximative d'une intégrale définie*. PARIS. GAUTHIER-VILLARS. 1905.

[2]) Voyez CARL FRIEDRICH GAUSS *Werke. Dritter Band*. p. 165 et suiv. *Herausgegeben von der Königl. Gesellschaft der Wissenschaften zu Göttingen*. 1866.

$$\left.\begin{aligned}
S_1 &= (\tfrac{1}{2})^2 \frac{m}{1} \cdot \frac{2m-1}{4m-1}, \\
S_2 &= (\tfrac{1}{2})^2 \frac{m-1}{2} \cdot \frac{2m-3}{4m-3} \cdot S_1, \\
S_3 &= (\tfrac{1}{2})^2 \frac{m-2}{3} \cdot \frac{2m-5}{4m-5} \cdot S_2, \\
&\dots\dots\dots\dots\dots\dots \\
S_{m-1} &= (\tfrac{1}{2})^2 \frac{2}{m-1} \cdot \frac{3}{2m+3} \cdot S_{m-2}, \\
S_m &= (\tfrac{1}{2})^2 \frac{1}{m} \cdot \frac{1}{2m+1} \cdot S_{m-1}.
\end{aligned}\right\} \quad \dots\dots \quad (21)$$

Les valeurs de R_p s'obtiennent de (68) après substitution de $d = 0$, etc. (voir plus haut); on obtient

$$= \frac{\frac{1}{2m-1}(\frac{1}{2})^{2m-2} - S_{1.p}\frac{1}{2m-3}(\frac{1}{2})^{2m-4} + S_{2.p}\frac{1}{2m-5}(\frac{1}{2})^{2m-6} - \dots + (-1)^{m-1}S_{m-1.p}}{x_p^{2m-2} - S_{1.p}x_p^{2m-4} + S_{2.p}x_p^{2m-6} - \dots + (-1)^{m-1}S_{m-1.p}} \quad \dots \quad (22)$$

L'erreur de I_1 sub (17) est représentée, en vertu de (19), où tous les termes avec $A_0\,(p = 0)$, jusque $A_{4m-2}\,(p = 2m-1)$ inclusivement sont égaux à 0, par

$$E = \left\{\tfrac{1}{4m+1}(\tfrac{1}{2})^{4m} - (x_1^{4m}R_1 + x_2^{4m}R_2 + x_3^{4m}R_3 + \dots + x_m^{4m}R_m)\right\} A_{4m} +$$
$$+ \left\{\tfrac{1}{4m+3}(\tfrac{1}{2})^{4m+2} - (x_1^{4m+2}R_1 + x_2^{4m+2}R_2 + x_3^{4m+2}R_3 + \dots + x_m^{4m+2}R_m)\right\} A_{4m+2} +$$
$+$ et ainsi de suite.

De (21) on peut facilement déduire, voir (56),

$$\left.\begin{aligned}
S_{1.p} &= S_1 - x_p^2, \\
S_{2.p} &= S_2 - S_{1.p}\,x_p^2, \\
S_{3.p} &= S_3 - S_{2.p}\,x_p^2, \\
&\dots\dots\dots\dots \\
S_{m-2.p} &= S_{m-2} - S_{m-3.p}\,x_p^2, \\
S_{m-1.p} &= S_{m-1} - S_{m-2.p}\,x_p^2.
\end{aligned}\right\} \quad \dots\dots\dots\dots \quad (23)$$

2°. Dans le *second cas*, une grandeur x^2 est préalablement considérée comme connue, c-à-d. $x_1 = 0$, de sorte que les deux ordonnées médianes y_{-1} et y_{+1}, coïncidant dans l'axe des y, ne forment qu'*une seule ordonnée à calculer* et que, par conséquent, le nombre d'ordonnées *à calculer* est *impair*. Les autres $(m-1)$ grandeurs x_p^2 et les m grandeurs R_p forment le nombre d'inconnues à déterminer, c'est pour cette raison que dans (19) il n'y a que $(2\,m-1)$ coefficients de A qui peuvent être égalés à zéro.

On a

$$
\begin{aligned}
R_1 + \quad R_2 + \quad R_3 + \dots + \quad R_m &= 1\ , \\
x_2^2 R_2 + x_3^2 R_3 + \dots + x_m^2 R_m &= \tfrac{1}{3}(\tfrac{1}{2})^2\ , \\
x_2^4 R_2 + x_3^4 R_3 + \dots + x_m^4 R_m &= \tfrac{1}{5}(\tfrac{1}{2})^4\ , \\
\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
x_2^{4m-4} R_2 + x_3^{4m-4} R_3 + \dots + x_m^{4m-4} R_m &= \tfrac{1}{4m-3}(\tfrac{1}{2})^{4m-4}.
\end{aligned}
$$

Les valeurs de x^2 de ces équations s'obtiennent de l'équation

$$(x^2)^{m-1} - S_{1._1}(x^2)^{m-2} + S_{2._1}(x^2)^{m-3} - S_{3._1}(x^2)^{m-4} + \dots + (-1)^{m-1} S_{m-1._1} = 0$$

où

$$
\left.
\begin{aligned}
S_{1._1} &= (\tfrac{1}{2})^2 \frac{m-1}{1} \cdot \frac{2m-1}{4m-3} \\
S_{2._1} &= (\tfrac{1}{2})^2 \frac{m-2}{2} \cdot \frac{2m-3}{4m-5} S_{1._1} \\
S_{3._1} &= (\tfrac{1}{2})^2 \frac{m-3}{3} \cdot \frac{2m-5}{4m-7} S_{2._1} \\
\dots\dots&\dots\dots\dots\dots\dots\dots \\
S_{m-2._1} &= (\tfrac{1}{2})^2 \frac{2}{m-2} \cdot \frac{5}{2m+3} \cdot S_{m-3._1} \\
S_{m-1._1} &= (\tfrac{1}{2})^2 \frac{1}{m-1} \cdot \frac{3}{2m+1} \cdot S_{m-2._1}
\end{aligned}
\right\} \quad \dots\dots\dots \quad (24)
$$

tandis que les valeurs de R s'obtiennent des deux formules suivantes, notamment la valeur de R_1 de la formule

$$R_1 = \frac{\frac{1}{2m-1}(\frac{1}{2})^{2m-2} - S_{1._1}\frac{1}{2m-3}(\frac{1}{2})^{2m-4} + S_{2._1}\frac{1}{2m-5}(\frac{1}{2})^{2m-6} - \dots + (-1)^m S_{m-2._1}\frac{1}{3}(\frac{1}{2})^2 + (-1)^{m-1} S_{m-1._1}}{(-1)^{m-1} S_{m-1._1}} \quad (2$$

et les autres valeurs de R de la formule

$$R_p = \frac{\frac{1}{2m-1}(\frac{1}{2})^{2m-2} - S_{1.p._1}\frac{1}{2m-3}(\frac{1}{2})^{2m-4} + S_{2.p._1}\frac{1}{2m-5}(\frac{1}{2})^{2m-6} - \dots + (-1)^m S_{m-2.p._1}\frac{1}{3}(\frac{1}{2})^2}{x_p^{2m-2} - S_{1.p._1} x_p^{2m-4} + S_{2.p._1} x_p^{2m-6} - \dots + (-1)^m S_{m-2.p._1} x_p^2} \ . \quad (2$$

L'erreur de l'expression pour I_1 de (17) est, en vertu de (19) représentée par

$$
\begin{aligned}
E = \Big\{ & \tfrac{1}{4m-1}(\tfrac{1}{2})^{4m-2} - (x_2^{4m-2} R_2 + x_3^{4m-2} R_3 + \dots + \\
& x_m^{4m-2} R_m)\Big\} A_{4m-2} + \Big\{\tfrac{1}{4m+1}(\tfrac{1}{2})^{4m} - (x_2^{4m} R_2 + x_3^{4m} R_3 + \dots + \\
& + x_m^{4m} R_m)\Big\} A_{4m} + \text{ et ainsi de suite.}
\end{aligned}
$$

Ensuite on trouve facilement (voir § 22)

$$\left.\begin{array}{l} S_{1.1.p} = S_{1.1} \quad - x_p^2, \\ S_{2.1.p} = S_{2.1} \quad - S_{1.1.p} \cdot x_p^2, \\ S_{3.1.p} = S_{3.1} \quad - S_{2.1.p} \cdot x_p^2, \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ S_{m-3.1.p} = S_{m-3.1} - S_{m-4.1.p} \cdot x_p^2, \\ S_{m-2.1.p} = S_{m-2.1} - S_{m-3.1.p} \cdot x_p^2. \end{array}\right\} \quad \ldots\ldots\ldots \quad (27)$$

Gauss donne les valeurs numériques de x et R, dont il est question sub (17), pour 1 à 7 ordonnées inclusivement, en 16 décimales exactes (voyez note 2, p. 12). M. R. Radau a encore calculé ces valeurs pour 8, 9 et 10 ordonnées en 10 décimales (Voyez R. Radau, *Etude sur les formules d'approximation qui servent à calculer la valeur numérique d'une intégrale définie*, dans le *Journal de Mathématiques pures et appliquées*. 3e série. Tome sixième, 1880). Dans la Table A à la fin de ce travail-ci, on les trouve pour 2 jusque 10 ordonnées en 16 décimales.

§ 12. De toutes les formules d'approximation, celles de Gauss donnent les résultats les plus exacts, pourvu qu'on puisse substituer une série de la *première* classe à la fonction sous le signe de l'intégrale en question; elles exigent cependant, même lorsqu'on applique un petit nombre d'ordonnées, des calculs extrêmement longs, où peuvent aisément se glisser des fautes de calcul [1]).

Gauss suppose qu'en appliquant ses formules, les longueurs des abscisses seront toujours exprimées en 16 décimales, puisque ce n'est que lorsque x compte un nombre suffisant de décimales, que l'on peut être assuré qu'en appliquant sa méthode les premiers $2m$, eventuellement les premiers $(2m-1)$ termes de l'erreur E disparaissent de (19). Si on applique les formules de Gauss et si, dans les calculs des y de $\varphi(x)$, $f(x)$ ou $\Psi(x)$ sub (1), (4) ou (13), les longueurs des abscisses sont exprimées en moins de 16 décimales et si pour R on admet cependant la même valeur, que celle qui s'offre dans les formules de Gauss (Table A), alors il se peut aisément que les coefficients de quelques-uns des termes avec A_2 jusque A_{4m-2} inclusivement ne soient pas assez proches de zéro, auquel cas l'inexactitude de I_1 serait sensiblement plus grande qu'on ne le déduirait de la formule.

---

[1]) Lobatto (*Calcul intégral*, p. 442) dit à ce sujet: „La méthode d'approximation d'après Gauss présente, quant au degré d'exactitude, des avantages évidents et est préférable à celle de Newton et de Cotes. Il est seulement regrettable que cette méthode plus correcte comporte des calculs assez compliqués, qui rendent assez difficile son application."

S'il y a neuf ordonnées ou plus, il sera probablement nécessaire, en appliquant la méthode de Gauss, d'employer parfois pour x plus de *seize* décimales, auquel cas le calcul de ces formules exige un temps extrêmement long.

§ 13. Si l'on calcule les valeurs des coefficients de A de (19) pour *six* ordonnées p.e., on obtient lorsqu'on prend dans la Table A les abscisses seulement en 2 décimales exactes, c'est-à-dire lorsqu'on établit

$$x_1 = 0.12\,,\qquad x_2 = 0.33 \quad \text{et} \quad x_3 = 0.47:$$

$$\begin{aligned} I = I_1 + E = {} & 0.2339\ 5696\ 7286\ 3455\ (y_{-1} + y_{+1}) + \\ & + 0.1803\ 8078\ 6524\ 0693\ (y_{-2} + y_{+2}) + 0.0856\ 6224\ 6189\ 5852\ (y_{-3} + y_{+3}) - \\ & - 0.0005\ 3714\ A_2 - 0.0002\ 3546\ A_4 - 0.0000\ 8191\ A_6 - \\ & - 0.0000\ 2468\ A_8 - 0.0000\ 0686\ A_{10} - 0.0000\ 0173\ A_{12} - \\ & - 0.0000\ 0039\ A_{14} - 0.0000\ 0008\ A_{16} - 0.0000\ 00015\ A_{18} - \text{etc.} \end{aligned} \tag{28}$$

Si l'on prend x en 5 décimales exactes, alors on a

$$x_1 = 0.11931 \qquad x_2 = 0.33060 \quad \text{et} \quad x_3 = 0.46623$$

et

$$\begin{aligned} I = I_1 + E = {} & 0.2339\ 5696\ 7286\ 3455\ (y_{-1} + y_{+1}) + \\ & + 0.1803\ 8078\ 6524\ 0693\ (y_{-2} + y_{+2}) + 0.0856\ 6224\ 6189\ 5852 (y_{-3} + y_{+3}) + \\ & + 0.0000\ 0183\ 4039\ A_2 + 0.0000\ 0057\ 3819\ A_4 + \\ & + 0.0000\ 0014\ 7821\ A_6 + 0.0000\ 0003\ 7069\ A_8 + \\ & + 0.0000\ 0000\ 9386\ A_{10} + 0.0000\ 0009\ 2415\ A_{12} + \\ & + 0.0000\ 0007\ 3335\ A_{14} + 0.0000\ 0003\ 6299\ A_{16} + \\ & + 0.0000\ 0001\ 4553\ A_{18} + \text{etc.}\ ^{1)} \ldots\ldots\ldots\ldots \end{aligned} \tag{29}$$

Pour x en 10 décimales, on trouve

$$x_1 = 0.1193\ 0959\ 30\,,\quad x_2 = 0.3306\ 0469\ 32 \quad \text{et} \quad x_3 = 0.4662\ 3475\ 71$$

et

$$\begin{aligned} I = I_1 + E = {} & 0.2339\ 5696\ 7286\ 3455\ (y_{-1} + y_{+1}) + \\ & + 0.1803\ 8078\ 6524\ 0693\ (y_{-2} + y_{+2}) + 0.0856\ 6224\ 6189\ 5852\ (y_{-3} + y_{+3}) + \\ & + 0.0000\ 0000\ 0012\ 7996\ A_2 + 0.0000\ 0000\ 0001\ 9693\ A_4 + \\ & + 0.0000\ 0000\ 0000\ 3218\ A_6 + 0.0000\ 0000\ 0000\ 0511\ A_8 + \\ & + 0.0000\ 0000\ 0000\ 0122\ A_{10} + 0.0000\ 0009\ 0097\ 4940\ A_{12} + \\ & + 0.0000\ 0007\ 2760\ 5382\ A_{14} + 0.0000\ 0003\ 6157\ 2569\ A_{16} + \\ & + 0.0000\ 0001\ 4316\ 9388\ A_{18} + \text{etc.} \ldots\ldots\ldots\ldots \end{aligned} \tag{30}$$

[1]) Le coefficient de A_{12} (qui en comparaison du coefficient du terme immédiatement antérieur est remarquablement grand) est à peu près égal au coefficient de A_{12} dans la formule pour *six* ordonnées de la Table A. Voyez aussi les formules sub (30), (31), (32) et (33).

Pour x en 16 décimales, on trouve

$$x_1 = 0.1193\ 0959\ 3041\ 5985 \quad , \quad x_2 = 0.3306\ 0469\ 3233\ 1323$$
$$\text{et } x_3 = 0.4662\ 3475\ 7101\ 5760$$

$$\text{et } I = I_1 + E = 0.2339\ 5696\ 7286\ 3455\ (y_{-1} + y_{-1}) + \\ + 0.1803\ 8078\ 6524\ 0693\ (y_{-2} + y_{+2}) + 0.0856\ 6224\ 6189 \\ 5852\ (y_{-3} + y_{+3}) + 0.0000\ 0009\ 0097\ 4927\ A_{12} + 0.0000 \\ 0007\ 2760\ 5509\ A_{14} + 0.0000\ 0003\ 6157\ 2582\ A_{16} + \\ + 0.0000\ 0001\ 4316\ 9388\ A_{18} + \text{etc.} \ldots\ldots \quad (31)$$

Formules d'approximation dans lesquelles les longueurs des abscisses selon Gauss sont exprimées en moins de seize décimales.

§ 14. Voici comment on obtient des formules d'approximation qui, avec un nombre égal d'ordonnées et un nombre égal de décimales pour x, sont plus exactes que celles qui ont été développées dans le § précédent.

On prend de la Table A, pour chaque formule en particulier, x en nombre égal, inférieur à seize décimales exactes, et on calcule, suivant ces longueurs arrondies, les valeurs de R de (22), éventuellement de (25) et (26), l'expression pour I_1 de (17) et enfin l'expression pour E de (19).

De cette manière on trouve, pour un nombre égal d'ordonnées et pour les mêmes longueurs des abscisses que celles dont on s'est servi respectivement dans (28), (29) et (30) les formules suivantes:

Pour $x_1 = 0.12 \quad , \quad x_2 = 0.33 \quad$ et $\quad x_3 = 0.47$:

$$I = I_1 + E = 0.2324\ 6285\ 8338\ 5646\ (y_{-1} + y_{+1}) + \\ + 0.1855\ 3350\ 9700\ 1764\ (y_{-2} + y_{+2})\ 0.0820\ 0363\ 1961\ 2591\ (y_{-3} + y_{+3}) - \\ - 0.0000\ 1633\ 6188\ A_6 - 0.0000\ 0870\ 1796\ A_8 - \\ - 0.0000\ 0317\ 1460\ A_{10} - 0.0000\ 0089\ 5102\ A_{12} - \\ - 0.0000\ 0020\ 7914\ A_{14} - 0.0000\ 0003\ 9648\ A_{16} - \\ - 0.0000\ 0000\ 6437\ A_{18} - \text{etc} \ldots\ldots \quad (32)$$

Pour $x_1 = 0.11931 \quad , \quad x_2 = 0.33060 \quad$ et $\quad x_3 = 0.46623$

$$I = I_1 + E = 0.2339\ 5631\ 2171\ 0420\ (y_{-1} + y_{+1}) + \\ + 0.1803\ 7353\ 2773\ 4742\ (y_{-2} + y_{+2}) + 0.0856\ 7015\ 5055\ 4838\ (y_{-3} + y_{+3}) + \\ + 0.0000\ 0000\ 4306\ A_6 - 0.0000\ 0000\ 3825\ A_8 + \\ + 0.0000\ 0000\ 1833\ A_{10} + 0.0000\ 0009\ 0771\ A_{12} - \\ - 0.0000\ 0007\ 2975\ A_{14} + 0.0000\ 0003\ 6220\ A_{16} + \\ + 0.0000\ 0001\ 4335\ A_{18} + \text{etc} \ldots\ldots \quad (33)$$

Pour $x_1 = 0.1193\ 0959\ 30$, $x_2 = 0.3306\ 0469\ 32$ et $x_3 = 0.4662\ 3475\ 71$:

$$I = I_1 + E = 0.2339\ 5696\ 7227\ 0741\,(y_{-1} + y_{+1} +$$
$$+ 0.1803\ 8078\ 6572\ 5009\,(y_{-2} + y_{+2}) + 0.0856\ 6224\ 6198\ 4250\,(y_{-3} + y_{+3}) +$$
$$+ 0.0000\ 0000\ 0000\ 0140\ A_6 - 0.0000\ 0000\ 0000\ 0016\ A_8 -$$
$$- 0.0000\ 0000\ 0000\ 0016\ A_{10} + 0.0000\ 0009\ 0097\ 4921\ A_{12} +$$
$$+ 0.0000\ 0007\ 2760\ 5508\ A_{14} + 0.0000\ 0003\ 6157\ 2582\ A_{16} +$$
$$+ 0.0000\ 0001\ 4316\ 9388\ A_{18} + \text{etc.} \ldots\ldots \quad (34)$$

En comparant les formules sub (28), (29) et (30) avec celles sub (32), (33) et (34), on voit immédiatement que, dans les trois premières, I_1 diffère plus de I que dans les trois dernières.

§ 15. Pour montrer, numériquement, la plus grande exactitude des formules du § précédent, lorsque les calculs pour I_1 ont la même étendue, en comparaison de celles du § 13, nous inscrivons ci-dessous, pour chacune de ces sept formules, le résultat du calcul d'une valeur approximative de l'intégrale $\int_8^9 \frac{dx^1}{x^1}$, c'est-à-dire du logarithme *néperien* de $^9/_8$, valeur qui, en 24 décimales, est exactement équivalente à

0. 1177 8303 5656 3834 5453 8794.

Attendu que l'axe des y est placé au milieu de la figure, nous substituons, pour la commodité, dans l'intégrale ci-dessus $8.5 + x$ à x^1, de sorte qu'on obtient

$$\int_{-\frac{1}{2}}^{+\frac{1}{2}} \frac{dx}{8.5 + x} \qquad \text{et} \qquad y = \frac{1}{8.5 + x},$$

c'est-à-dire pour *six* ordonnées, ou $m = 3$:

$$\text{pour } x = -x_1 : y_1 = y_{-1} = \frac{1}{8.5 - x_1} \text{ et pour } x = x_1 : y = y_{+1} = \frac{1}{8.5 + x_1},$$
$$\text{,,} \quad x = -x_2 : y_2 = y_{-2} = \frac{1}{8.5 - x_2} \text{ ,,} \quad \text{,,} \quad x = x_2 : y = y_{+2} = \frac{1}{8.5 + x_2},$$
$$\text{,,} \quad x = -x_3 : y_3 = y_{-3} = \frac{1}{8.5 - x_3} \text{ ,,} \quad \text{,,} \quad x = x_3 : y = y_{+3} = \frac{1}{8.5 + x_3}.$$

On obtient, lorsque les abscisses sont données en deux décimales, notamment lorsque $x = 0.12$, $x_2 = 0.33$ et $x_3 = 0.47$, de la formule

sub (28) $\quad I_1 = 0.\ 1177\ 8392\ldots$
,, (32) $\quad I_1 = 0.\ 1177\ 8303\ 5688 \ldots\ldots\ldots$ (35)

Lorsque les abscisses sont données en 5 décimales, on obtient de la formule

sub (29) $I_1 = 0.\ 1177\ 8303\ 27$.................. (36)

„ (33) $I_1 = 0.\ 1177\ 8303\ 5656\ 3820$.......... (37)

Si les abscisses sont prises en 10 décimales, alors on obtient de la formule

sub (30) $I_1 = 0.\ 1177\ 8303\ 5656\ 363$........... (38)

„ (34) $I_1 = 0.\ 1177\ 8303\ 5656\ 3834\ 5445\ 91$.. (39)

Si les abscisses sont exprimées en 16 décimales, on obtient de la formule

sub (31) $I_1 = 0.1177\ 8303\ 5656\ 3834\ 5446\ 29$.. (40)

Il apparaît, chaque fois, pour six ordonnées:

en comparant (35) avec (36), que le résultat pour x en 2 décimales, suivant la formule (32), sera plus exact que celui pour x en 5 décimales, suivant la formule de la Table B;

en comparant (37) avec (38), que le résultat pour x en 5 décimales, suivant la formule (33), sera plus exact que celui pour x en 10 décimales, suivant la formule de la Table susdite.

Etc.

Sur le nombre de décimales, dans lesquelles il convient d'exprimer x et R.

§ 16. Dans la Table A nous avons, à la suite de Gauss, aussi pour l'application d'un petit nombre d'ordonnées, exprimé les abscisses en *seize* décimales. Ce nombre est pris arbitrairement; il doit toujours être relativement grand. Mais, dans les deux §§ précédents, il est apparu que, même lorsque la série sub (14) converge assez fortement, on peut, en appliquant *six* ordonnées, se contenter de moins de seize décimales pour x, parce que, dans le cas où la convergence n'est pas trop forte, l'erreur de l'approximation reste assez constante, soit qu'on prenne pour x dix ou seize et plus de décimales [1]).

Si, par exemple, on compare la formule (31) où x a seize décimales, avec celle sub (34), où x n'est donné qu'en dix décimales, il apparaît que, dans les deux formules — qui chacune sont destinées pour six ordonnées — les coefficients des A homonymes, de A_{12} jusque A_{18} inclusivement, sont semblables jusqu'à la 15e décimale ou plus. Dans (34), les termes avec A_6, A_8 et A_{10} se présen-

[1]) En cas de faible convergence, on pourra probablement pour *six* ordonnées se contenter d'encore moins de dix décimales.

tent aussi il est vrai, tandis que dans (31) ils ne paraissent plus; mais ces termes ont dans (34) des coefficients numériques si petits, que leur valeur totale n'a pas d'importance en comparaison de la valeur totale des termes avec A_{12}, A_{14} et A_{16}, dont les coefficients numériques sont respectivement plus de 6, 45 et 22 millions de fois plus grands que ceux des termes avec A_6, A_8 et A_{10} et ce n'est que dans les séries qui convergent extrêmement fort que les relations $\frac{A_6}{A_{12}}$, $\frac{A_8}{A_{14}}$ et $\frac{A_{10}}{A_{16}}$ peuvent se rapprocher de celles dont il est question ici. On obtient alors aussi, comme il résulte de (39) et (40), pour l'intégrale donnée, selon les deux formules (31) et (34) une même approximation, concordante jusque dans *vingt* décimales. Par conséquent, pour des séries qui ne sont pas très fortement convergentes et, à plus forte raison, pour des séries faiblement convergentes, l'exactitude maximum, pour autant que celle-ci puisse être atteinte avec *six* ordonnées, s'obtiendra déjà, et cela à une minime différence près, si x est donné en dix décimales au maximum, lorsque la formule pour I_1 est établie conformément au § 14. Le *surplus d'au moins six décimales*, que Gauss propose d'employer *toujours*, est donc souvent sans utilité, parce que, malgré la grande augmentation des calculs, l'exactitude du résultat n'en est pas appréciablement accrue.

La chose devient cependant tout autre, quand il s'agit de séries très fortement convergentes. Alors en effet la valeur totale des termes affectés des plus petits indices, peut avoir une influence appréciable sur la dernière décimale dans laquelle I_1 doit être exprimé. Dans ce cas, en appliquant la méthode de Gauss, on doit, lorsque le nombre des abscisses est assez considérable, exprimer les abscisses en plus de dix, peut-être en seize ou plus de décimales.

Si l'on veut donc appliquer dans tous les cas les formules de la Table A, pour chaque fonction dont on ne sait pas d'avance si elle peut être exprimée par une série faiblement ou fortement convergente, il faut, pour toute sûreté, que pour chaque fonction, chaque x soit exprimé dans un grand nombre de décimales, mais alors encore il y a lieu de se demander si *seize* décimales pour x sont bien toujours suffisantes si on met en compte, par exemple plus de neuf ordonnées.

Le fait, qu'on ne peut pas d'avance juger définitivement combien de décimales il faut prendre pour x pour réduire les termes avec A_2 jusqu'à A_{4m-2} inclusivement, de telle façon que leur valeur totale ne puisse plus avoir d'influence appréciable sur la valeur de I_1, ce fait n'est pas un des moindres inconvénients de la méthode de Gauss et on est par conséquent, pour toute sûreté, obligé en

appliquant cette méthode, de prendre pour x un nombre plus grand de décimales que peut-être il n'est nécessaire.

D'ailleurs dix décimales pour x donnent déjà lieu à des calculs très longs et très ennuyeux.

C'est pour ces différentes raisons, que dans la Table B, placée à la suite de cet exposé, on n'a pris pour x que deux décimales, par là aucun chiffre n'est de trop et cependant les termes dans (19) avec A_0 jusque A_{2m-2} inclusivement sont absolument sans influence, tandis que les termes homonymes avec A_m etc. sont plus petits que ceux qui appartiennent, entre autre, aux formules selon NEWTON-COTES et MACLAURIN pour un même nombre d'ordonnées. On ne cherche pas alors l'exactitude dans un grand nombre de décimales pour x, mais dans un grand nombre d'ordonnées, qui sont faciles à calculer.

Les nombreuses applications de formules des Tables A et B m'ont prouvé d'une façon évidente, que le calcul des formules pour m ordonnées de la Table A prenait beaucoup plus de temps que le calcul de formules pour $(2m—1)$ ordonnées de la Table B, alors que, en comparant l'exactitude relative des deux Tables, il apparaît encore que les formules pour $(2m—1)$ ordonnées de la Table A sont notablement moins exactes que celles pour le double d'ordonnées moins une, c'est-à-dire pour $(4m—3)$ ordonnées de la Table B.

§ 17. Les valeurs de R sub (22), (25) et (26) consistent également de groupes, formés de suites infinies de décimales, qui ne peuvent donc être inscrites qu'en nombre restreint dans les formules pour I_1.

Le fait de ne mettre en compte qu'un nombre restreint de décimales pour R, ne produit d'ailleurs aucune erreur, si R est exprimé dans un nombre tellement grand de décimales que même si on en prenait davantage encore cela ne pourrait avoir aucune influence appréciable sur le calcul du résultat. C'est pourquoi, dans la Table B, les valeurs pour R sont indiquées par un nombre de décimales plus grand qu'il ne sera jamais nécessaire; cependant ce plus grand nombre de décimales ne complique pas les calculs, parce que la personne qui fait ursage des formules de la table B est elle-même juge du nombre de décimales qu'il lui convient, pour toute sûreté, de prendre dans chaque cas particulier, en rapport avec le degré d'exactitude qu'elle veut atteindre.

Du nombre d'ordonnées à appliquer.

§ 18. La formule (19) indique — conformément à ce qui se

trouve dans la 2e phrase du § 14 — que si toutes les m grandeurs x, donc aussi leurs carrés, sont admises d'avance comme connues, le premier terme de E, dont le coefficient numérique n'est pas égal à 0, sera

$$\left\{\frac{1}{2m+1}\left(\tfrac{1}{2}\right)^{2m} - \left(x_1^{2m} R_1 + x_2^{2m} R_2 + x_3^{2m} R_3 + \ldots + x_m^{2m} R_m\right)\right\} A_{2m}$$

indifféremment si x_1 est égal ou plus grand que zéro.

Il en résulte que, soit que les deux ordonnées médianes coïncident dans l'axe des y, soit qu'elles soient à quelque distance de cet axe, dans les deux cas l'expression algébrique pour le premier terme de l'erreur E, sub (19), donc le *rang* de l'erreur de l'approximation, est la même; seule la grandeur, c'est-a-dire la valeur du coefficient numérique de ce terme diffèrera, étant une fonction des valeurs de x.

J'ai fait expressément dans ce but un grand nombre de calculs de formules pour deux jusque dix ordonnées inclusivement et pour 2 et plus de décimales pour x et j'ai constaté que la différence dans la grandeur de la faute entre l'approximation pour $(2m-1)$ et celle pour $2m$ ordonnées est généralement tout à fait insignifiante. [1])

Si les longueurs d'abscisses sont données en deux décimales corrigées, comme il est indiqué dans le § suivant, alors l'erreur en question pour $(2m-1)$ ordonnées est même quelquefois un peu plus petite que celle pour le nombre pair suivant; jamais cependant il ne me parut que le calcul plus long pour $2m$ ordonnées au lieu de $(2m-1)$ présentât des avantages suffisants. C'est pourquoi j'ai calculé pour la Table B seulement des formules pour $(2m-1)$ ordonnées, comme STIRLING l'a fait aussi (LOBATTO, *Calcul Intégral*, p. 411).

Correction des longueurs d'abscisses de GAUSS, *lorsqu'elles sont exprimées en deux décimales seulement.*

§ 19. Lorsqu'on applique les abscisses de GAUSS (pourvu qu'elles soient exprimées dans un nombre suffisant de décimales) pour le calcul de la valeur de I_1, alors les premiers $2m$ termes de l'erreur E, sub (19), sont chacun en particulier égaux à 0. Ceci n'a plus lieu lorsqu'on a posé comme condition, que toutes les abscisses

[1]) A cet égard, il convient de faire remarquer, que lorsqu'il s'agit de mesurer un nombre impair d'ordonnées, les deux ordonnées médianes, qui dans ce cas coïncident, sont toujours placées exactement, ce qui, lorsqu'il y a un nombre pair d'ordonnées n'est le cas pour aucune d'entre elles.

seront indiquées dans seulement deux décimales, comme il est question dans § 16, alors les grandeurs x de (19) ne sont pas déduites des équations égalées à 0, mais elles sont déterminées d'avance et les m grandeurs R sont alors les seules inconnues qui restent à déterminer.

Suivant (19), on trouve alors pour le premier terme de l'erreur E qui appartient à ce groupe d'abscisses réduites à 2 décimales (que nous nommons le premier groupe d'abscisses)

$$\left\{\tfrac{1}{2m+1}(\tfrac{1}{2})^{2m} - (x_1^{2m} R_1 + x_2^{2m} R_2 + x_3^{2m} R_3 + \ldots + x_m^{2m} R_m)\right\} A_{2m} = \beta . A_{2m} \ldots$$

La valeur de β, notamment

$$\beta = \tfrac{1}{2m+1}(\tfrac{1}{2})^{2m} - (x_1^{2m} R_1 + x_2^{2m} R_2 + x_3^{2m} R_3 + \ldots + x_m^{2m} R_m$$

peut le plus souvent être réduite, en allongeant ou en raccourcissant d'un ou de deux centièmes de l'unité de longueur, une ou plusieurs des longueurs du premier groupe de longueurs d'abscisses, de telle manière qu'il en résulte un nouveau groupe d'abscisses (le deuxième groupe) dans lequel le coefficient du terme avec A_{2m} devient un peu plus petit que celui du premier groupe.

Il ne faut pas perdre de vue ici, que cet allongement ou ce raccourcissement ne peut pas comporter plus de deux centièmes de l'unité de longueur, alors que, en cas d'allongement ou de raccourcissement plus considérable, quelques abscisses pourraient s'écarter trop des abscisses de Gauss et que par là les coefficients avec A_{2m+2} etc. pourraient sensiblement augmenter en valeur.

Si nous représentons les longueurs des abscisses arrondies et non corrigées (ainsi celles du premier groupe) respectivement pas z_1, z_2, $z_3, \ldots z_m$, celles du deuxième groupe par x_1, x_2, $x_3, \ldots x_m$ et si nous remplaçons x_p par $z_p + \alpha_p\delta$ et si $\delta = 0.01$ et $\alpha_p = 1$ ou 2, alors on peut écrire, suivant (19)

$$\left.\begin{array}{l}
R_1 + R_2 + R_3 + \ldots + R_m = 1 , \\
+ \alpha_1\delta)^2 R_1 + (z_2 + \alpha_2\delta)^2 R_2 + (z_3 + \alpha_3\delta)^2 R_3 + \ldots + (z_m + \alpha_m\delta)^2 R_m = \tfrac{1}{3}\ (\tfrac{1}{2})^2 , \\
+ \alpha_1\delta)^4 R_1 + (z_2 + \alpha_2\delta)^4 R_2 + (z_3 + \alpha_3\delta)^4 R_3 + \ldots + (z_m + \alpha_m\delta)^4 R_m = \tfrac{1}{5}\ (\tfrac{1}{2})^4 , \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
+ \alpha_1\delta)^{2m} R_1 + (z_2 + \alpha_2\delta)^{2m} R_2 + (z_3 + \alpha_3\delta)^{2m} R_3 + \ldots + (z_m + \alpha_m\delta)^{2m} R_m = \tfrac{1}{2m+1}(\tfrac{1}{2})^{2m} + \beta
\end{array}\right\} \quad (41)$$

Si nous introduisons les notations suivantes

$S_1 =$ la somme des m carrés $(z_1 + \alpha_1\delta)^2$, $(z_2 + \alpha_2\delta)^2, \ldots (z_m + \alpha_m\delta)^2$,
$S_2 =$ la somme des produits deux à deux de ces m carrés,
$S_3 =$ la somme des produits trois à trois de ces m carrés,
et ainsi de suite,
$S_m =$ le produit (continu) de ces m carrés;

alors ces carrés $(z_1 + \alpha_1 \delta)^2$ etc. sont les racines de l'équation du $2m^{\text{ième}}$ degré

$$(z + \alpha\delta)^{2m} - S_1 (z + \alpha\delta)^{2m-2} + S_2 (z + \alpha\delta)^{2m-4} + \ldots + (-1)^m S_m = 0.$$

Si nous additionnons les équations sub (41) après les avoir multipliées successivement, en commençant par la dernière, par 1, $-S_1$, $+S_2$, $-S_3, \ldots (-1)^m\ S_m$, nous obtenons l'égalité suivante:

$$\frac{1}{2m+1}\left(\frac{1}{2}\right)^{2m} - S_1 \cdot \frac{1}{2m-1}\left(\frac{1}{2}\right)^{2m-2} + S_2 \frac{1}{2m-3}\left(\frac{1}{2}\right)^{2m-4} - \ldots + (-1)^m S_m = \beta \quad (42)$$

Pour pouvoir déterminer maintenant les valeurs de $\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_m$, qui rendent β le plus petit possible, nous remplaçons dans (42) S_1 par la somme de tous les carrés $(z_p + \alpha_p \delta)^2$, S_2 par la somme des produits deux à deux de tous ces carrés, et ainsi de suite, S_m par le produit (continu) de ces carrés, nous développons les puissances et les produits et négligeons tous les termes dans lesquels se présentent des puissances du $2^{\text{ième}}$ et plus haut degré de α_p et de δ; alors s'établit une équation dans laquelle les grandeurs α_p, élevées seulement au premier degré, sont les seules inconnues.

Après l'introduction encore des notations suivantes:

$S_{1.p}$ = la somme des carrés $z_1^2, z_2^2, \ldots z_m^2$, sauf de z_p^2,
$S_{2.p}$ = la somme des produits deux à deux de ces mêmes $(m-1)$ carrés,
$S_{3.p}$ = la somme des produits trois à trois de ces mêmes $(m-1)$ carrés, et ainsi de suite,
$S_{m-1.p}$ = le produit (continu) de ces mêmes $(m-1)$ carrés; et
S_1 = la somme de tous les carrés $z_1^2, z_2^2, z_3^2, \ldots z_m^2$,
S_2 = la somme de leurs produits deux à deux,
S_3 = la somme de leurs produits trois à trois, et ainsi de suite
S_m = leur produit (continu), (42) devient

$$\begin{aligned}
&\Big\{\frac{1}{2m+1}\left(\tfrac{1}{2}\right)^{2m} - S_1 \frac{1}{2m-1}\left(\tfrac{1}{2}\right)^{2m-2} + S_2 \frac{1}{2m-3}\left(\tfrac{1}{2}\right)^{2m-4} - \ldots + (-1)^m S_m \Big\} - \\
&- 2 z_1 \delta \Big\{\frac{1}{2m-1}\left(\tfrac{1}{2}\right)^{2m-2} - S_{1.1} \frac{1}{2m-3}\left(\tfrac{1}{2}\right)^{2m-4} + S_{2.1} \frac{1}{2m-5}\left(\tfrac{1}{2}\right)^{2m-6} - \ldots + (-1)^{m-1} S_{m-1.1}\Big\}\alpha_1 - \\
&- 2 z_2 \delta \Big\{\frac{1}{2m-1}\left(\tfrac{1}{2}\right)^{2m-2} - S_{1.2} \frac{1}{2m-3}\left(\tfrac{1}{2}\right)^{2m-4} + S_{2.2} \frac{1}{2m-5}\left(\tfrac{1}{2}\right)^{2m-6} - \ldots + (-1)^{m-1} S_{m-1.2}\Big\}\alpha_2 - \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&- 2 z_m \delta \Big\{\frac{1}{2m-1}\left(\tfrac{1}{2}\right)^{2m-2} - S_{1.m} \frac{1}{2m-3}\left(\tfrac{1}{2}\right)^{2m-4} + S_{2.m} \frac{1}{2m-5}\left(\tfrac{1}{2}\right)^{2m-6} - \ldots + (-1)^{m-1} S_{m-1.m}\Big\}\alpha_m = \min (43
\end{aligned}$$

Valable pour $2m$, c'est-à-dire pour un nombre pair d'ordonnées.

Pour $(2m-1)$, c'est-à-dire pour un nombre impair d'ordonnées, lorsque $z_1 = 0$, on trouve la formule

$$\left\{\frac{1}{2m+1}\left(\frac{1}{2}\right)^{2m} - S_{1.1}\cdot\frac{1}{2m-1}\left(\frac{1}{2}\right)^{2m-2} + S_{2.1}\frac{1}{2m-3}\left(\frac{1}{2}\right)^{2m-4} - \ldots +\right.$$
$$\left. + (-1)^{m-1} S_{m-1.1}\cdot\frac{1}{3}\left(\frac{1}{2}\right)^2\right\} - 2z_2\delta\left\{\frac{1}{2m-1}\left(\frac{1}{2}\right)^{2m-2} - S_{1.1.2}\frac{1}{2m-3}\left(\frac{1}{2}\right)^{2m-4} +\right.$$
$$\left. + S_{2_{1.2}}\frac{1}{2m-5}\left(\frac{1}{2}\right)^{2m-6} - \ldots + (-1)^m S_{m-2.1.2}\cdot\frac{1}{3}\left(\frac{1}{2}\right)^2\right\}\alpha_2 -$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$- 2z_m\delta\left\{\frac{1}{2m-1}\left(\frac{1}{2}\right)^{2m-2} - S_{1.1.m}\cdot\frac{1}{2m-3}\left(\frac{1}{2}\right)^{2m-4} + S_{2.1.m}\frac{1}{2m\ \ 5}\left(\frac{1}{2}\right)^{2m-6} - \ldots +\right.$$
$$\left. + (-1)^m S_{m-2.1.m}\cdot\frac{1}{3}\left(\frac{1}{2}\right)^2\right\}\alpha_m = \min \ldots\ldots\ldots \quad (44)$$

§ 20. Comme application des formules du § précédent, nous posons le cas de $m = 3$, c'est-à-dire qu'il faut calculer $2m = 6$ ordonnées. De la Table A nous obtenons alors pour *le premier groupe* de longueurs d'abscisses

$$z_1 = 0.12 \quad , \quad z_2 = 0.33 \quad \text{et} \quad z_3 = 0.47$$

et de (19) pour le premier terme de l'erreur E, faisant partie de ce groupe

$$\left\{\frac{1}{7}\left(\frac{1}{2}\right)^6 - (z_1^6 R_1 + z_2^6 R_2 + z_3^6 R_3)\right\} A_6 = -0.0000\ 1633\ 6188\ A_6,$$

tandis que, d'après (43), on trouve pour le premier terme de l'erreur, appartenant au second groupe de longueurs d'abscisses

$$\begin{aligned}
&\Big[\left\{\tfrac{1}{7}\left(\tfrac{1}{2}\right)^6 - S_1\cdot\tfrac{1}{5}\left(\tfrac{1}{2}\right)^4 + S_2\cdot\tfrac{1}{3}\left(\tfrac{1}{2}\right)^2 - S_3\right\} - \\
&- 2z_1\delta\left\{\tfrac{1}{5}\left(\tfrac{1}{2}\right)^4 - S_{1.1}\cdot\tfrac{1}{3}\left(\tfrac{1}{2}\right)^2 + S_{2.1}\right\}\alpha_1 - \\
&- 2z_2\delta\left\{\tfrac{1}{5}\left(\tfrac{1}{2}\right)^4 - S_{1.2}\cdot\tfrac{1}{3}\left(\tfrac{1}{2}\right)^2 + S_{2.2}\right\}\alpha_2 - \\
&- 2z_3\delta\left\{\tfrac{1}{5}\left(\tfrac{1}{2}\right)^4 - S_{1.3}\cdot\tfrac{1}{3}\left(\tfrac{1}{2}\right)^2 + S_{2.3}\right\}\alpha_3 \quad \Big] A_6 = \\
&= \Big[-0.0000\ 1633\ 6188 - \\
&\quad - 0.0000\ 2177\ 4424\ \alpha_1 + \\
&\quad + 0.0000\ 2592\ 0664\ \alpha_2 - \\
&\quad - 0.0000\ 3565\ 5694\ \alpha_3 \quad \Big] A_6;
\end{aligned}$$

expression, qui devient la plus petite pour $\alpha_1 = 1$, $\alpha_2 = 0$ et $\alpha_3 = -1$, de sorte que les abscisses du second groupe sont

$$x_1 = 0.13 \quad , \quad x_2 = 0.33 \quad \text{et} \quad x_3 = 0.46 \ldots\ldots \quad (45)$$

En effet, on trouve pour ces longueurs de x

$$\begin{aligned}
I = I_1 + E = {}& 0.2466\ 1426\ 3490\,(y_{-1} + y_{+1}) + \\
&+ 0.1569\ 3803\ 5364\,(y_{-2} + y_{+2}) + 0.0964\ 4770\ 1146\,(y_{-3} + y_{+3}) - \\
&- 0.0000\ 0314\ 7065\ A_6 + 0.0000\ 0313\ 4520\ A_8 + \\
&+ 0.0000\ 0214\ 2798\ A_{10} + 0.0000\ 0094\ 1791\ A_{12} + \\
&+ 0.0000\ 0034\ 8202\ A_{14} + \text{etc.} \ldots\ldots\ldots\ldots \quad (46)
\end{aligned}$$

Si on compare cette formule à celle sub (32) on constate, que dans (46) non seulement le premier, mais aussi quelques-uns des termes de l'erreur E qui suivent immédiatement le premier terme sont plus petits que les termes homonymes dans (32).

De (46) on trouve pour une valeur approximative de l'intégrale $\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{dx}{8.5+x}$

$$I_1 = 0.1177\ 8303\ 5657\ 4\ldots$$

contre

$$I_1 = 0.1177\ 8303\ 5688\ \ldots\ \text{de (32), voir (35).}$$

Comme seconde application, nous posons que le nombre des ordonnées à calculer sera de $(2m-1)=5$, tandis qu'ici x sera aussi exprimé en deux décimales, et que maintenant aussi $\delta = 0.01$; dans ce cas, la Table A donne pour le *premier* groupe de longueurs d'abscisses

$$z_1 = 0 \quad , \quad z_2 = 0.27 \quad \text{et} \quad z_3 = 0.45,$$

et on obtient, d'après (44), pour le premier terme de l'erreur, qui fait partie du *second* groupe

$$\begin{aligned}&\left[\left\{\tfrac{1}{7}\left(\tfrac{1}{2}\right)^6 - \tfrac{1}{5}\left(\tfrac{1}{2}\right)^4 (z_2^2 + z_3^2) + \tfrac{1}{3}\left(\tfrac{1}{2}\right)^2 z_2^2 z_3^2\right\} - \right.\\ &\left. - 0.02\, z_2 \left\{\tfrac{1}{5}\left(\tfrac{1}{2}\right)^4 - \tfrac{1}{3}\left(\tfrac{1}{2}\right)^2 z_3^2\right\} \alpha_2 - 0.02\, z_3 \left\{\tfrac{1}{5}\left(\tfrac{1}{2}\right)^4 - \tfrac{1}{3}\left(\tfrac{1}{2}\right)^2 z_2^2\right\} \alpha_3\right] A_6 = \\ &= [0.0000\ 1983 + \\ &+ 0.0000\ 23625\, \alpha_2 - \\ &- 0.0000\ 57825\, \alpha_3 \quad]\, A_6\, ;\end{aligned}$$

expression qui devient la plus petite pour $\alpha_2 = -1$ et $\alpha_3 = 0$; par conséquent il faut prendre $x_2 = 0.26$ et $x_3 = 0.45$.

Abstraction faite du signe, on trouve ici pour le premier terme de l'erreur E de la formule pour cinq ordonnées, $0.0000\ 0336\ A_6$ contre $0.0000\ 1988\ A_6$ pour celui qui appartient à la formule pour les longueurs d'abscisses en deux décimales empruntées à la Table A.

La formule pour I pour les longueurs d'abscisses corrigées est

$$\begin{aligned}I = I_1 + E = 0.2688\ 7768\ 7681\, y_1 + \\ +0.2398\ 7744\ 5928\,(y_{-2}+y_{+2}) + 0.1256\ 8371\ 0232\,(y_{-3}+y_{+3}) - \\ -0.0000\ 0336\, A_6 + 0.0000\ 0133\, A_8 + 0.0000\ 0251\, A_{10} + \\ +0.0000\ 0140\, A_{12} + 0.0000\ 0056\, A_{14} + 0.0000\ 0019\, A_{16} + \text{etc.}\end{aligned} \quad (47)$$

De (47) on trouve pour une valeur approximative de l'intégrale $\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{dx}{8.5+x}$

$$I_1 = 0.\ 1177\ 8303\ 5657\ 4$$

pour *cinq* ordonnées, tout comme plus haut pour *six* ordonnées. [1])

Les formules pour I_1 de la Table B ont été déduites de la façon exposée dans ce §-ci et dans le § précédent.

[1]) Voir la remarque dans la dernière phrase du § 18.

Détermination de l'exactitude d'une valeur approximative.

§ 21. Dans le calcul de la valeur approximative d'une intégrale définie, il est généralement nécessaire de savoir jusqu'à quelle figure (chiffre ou zéro) cette valeur est exacte.

A cet effet, on peut recourir à deux formules, qui dans la Table A (ou dans la Table B) se suivent immédiatement. Nous représentons ces formules, pour les distinguer l'une de l'autre, par I_1 et I_2; la dernière étant arrangée pour plus d'ordonnées que la première, est par conséquent plus exacte que l'autre, de sorte que, à la série ininterrompue de figures égales, comptées à partir du premier chiffre, que les résultats calculés d'après les deux formules pour I_1 et I_2 ont en commun, on peut conclure à une valeur pour I_1 qui est exacte jusqu'à la dernière de ces figures égales. Cette détermination de l'exactitude exige il est vrai le calcul de deux formules avec des ordonnées toutes différentes. C'est d'ailleurs là la voie qu'on doit suivre, lorsqu'on fait usage des formules de Gauss; pour les formules de la Table B, on peut exécuter un calcul plus commode.

Notamment il est plus simple de calculer pour I_1 une des formules de la Table B et de rendre plus faciles les calculs pour I_2 en employant pour I_2 les mêmes ordonnées que pour I_1 en y adjoignant deux nouvelles.

Si nous admettons que, dans la formule à développer pour I_2, on attribue aux $(m-1)$ abscisses $x_1, x_2, x_3, \ldots x_{m-1}$, les mêmes valeurs que dans la formule pour I_1, il faut alors encore déterminer les valeurs des m grandeurs R et l'abscisse inconnue x_m, donc en tout $(m+1)$ inconnues. A cet effet nous pouvons dans (19) égaler à zéro les coefficients des premiers $(m+1)$ termes de E, chacun en particulier, et, des équations ainsi formées, résoudre les $(m+1)$ inconnues qui en même temps satisfont à ces $(m+1)$ équations. Nous avons

$$\left.\begin{array}{l} R_1 + R_2 + R_3 + \ldots + R_m = 1 \\ x_1^2 R_1 + x_2^2 R_2 + x_3^2 R_3 + \ldots + x_m^2 R_m = \frac{1}{3}\left(\frac{1}{2}\right)^2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ x_1^{2m} R_1 + x_2^{2m} R_2 + x_3^{2m} R_3 + \ldots + x_m^{2m} R_m = \frac{1}{2m+1}\left(\frac{1}{2}\right)^{2m}. \end{array}\right| \quad \ldots (48)$$

Si, dans (66) et (69), nous posons $d = 0, k = 2, c = 0, v = m$ et $u_p = \frac{1}{2p+1}\left(\frac{1}{2}\right)^{2p}$, on trouve de (66) le même groupe d'équations, que celui sub (48) et par conséquent pour ces équations vaut, d'après (68) l'égalité

$$R_p = \frac{\frac{1}{2m-1}\left(\frac{1}{2}\right)^{2m-2} - S_{1.p}.\frac{1}{2m-3}\left(\frac{1}{2}\right)^{2m-4} + S_{2.p}\frac{1}{2m-5}\left(\frac{1}{2}\right)^{2m-6} - \ldots + (-1)^{m-1}S_{m-1.p}}{x_p^{\,2m-2} - S_{1.p}.x_p^{\,2m-4} + S_{2.p}.x_p^{\,2m-6} - \ldots + (-1)^{m-1}S_{m-1.p}} \quad \ldots . \quad (49)$$

et d'après la première équation sub (72) l'égalité

$$\frac{1}{2m+1}\left(\frac{1}{2}\right)^{2m} - S_1.\frac{1}{2m-1}\left(\frac{1}{2}\right)^{2m-2} + S_2.\frac{1}{2m-3}\left(\frac{1}{2}\right)^{2m-4} - \ldots + (-1)^{m-1}S_{m-1}.\frac{1}{3}\left(\frac{1}{2}\right)^2 + (-1)^m S_m = 0$$

La dernière égalité devient (voir (56)), après substitution de $S_1 = S_{1.m} + x_m^{\,2}$, $S_2 = S_{2.m} + S_{1.m}\,.\,x_m^{\,2}$, etc.

$$\frac{1}{2m+1}\left(\frac{1}{2}\right)^{2m} - (S_{1.m} + x_m^{\,2}).\frac{1}{2m-1}\left(\frac{1}{2}\right)^{2m-2} + (S_{2.m} + S_{1.m}.x_m^{\,2})\frac{1}{2m-3}\left(\frac{1}{2}\right)^{2m-4} - \ldots$$
$$\ldots + (-1)^{m-1}(S_{m-1.m} + S_{m-2.m}.x_m^{\,2}) + (-1)^m S_{m-1.m}.x_m^{\,2} = 0$$

d'où

$$x_m^{\,2} = \frac{\frac{1}{2m+1}\left(\frac{1}{2}\right)^{2m} - S_{1.m}\frac{1}{2m-1}\left(\frac{1}{2}\right)^{2m-2} + S_{2.m}.\frac{1}{2m-3}\left(\frac{1}{2}\right)^{2m-4} - \ldots + (-1)^{m-1}S_{m-1.m}}{\frac{1}{2m-1}\left(\frac{1}{2}\right)^{2m-2} - S_{1.m}\frac{1}{2m-3}\left(\frac{1}{2}\right)^{2m-4} + S_{2.m}\frac{1}{2m-5}\left(\frac{1}{2}\right)^{2m-6} - \ldots + (-1)^m S_{m-2.m} + (-1)^{m-1}S_{m-1.m}}$$

En appliquant cette formule au cas, par exemple, de $m = 4$, lorsqu' auparavant on pose (voir (45))

$$x_1 = 0.13 \quad , \quad x_2 = 0.33 \quad \text{et} \quad x_3 = 0.46,$$

on trouve

$$x_4^{\,2} = -\tfrac{4}{3}.$$

Cette valeur négative pour $x_4^{\,2}$ montre, que s'il faut attribuer à x_1, x_2 et x_3 les valeurs que nous venons de mentionner, il est impossible d'y joindre une quatrième qui puisse, avec les trois précédentes, satisfaire aux $(m+1) = 5$ équations sub (48).

Pour rester maintenant aussi près que possible de la valeur calculée de $x_4^{\,2}$, on peut poser $x_4 = 0$. Cette valeur de x_4 jointe aux longueurs prescrites des autres abscisses, par conséquent

$$x_1 = 0.13 \quad , \quad x_2 = 0.33 \quad , \quad x_3 = 0.46 \quad \text{et} \quad x_4 = 0,$$

conduite à la formule auxiliaire suivante,

$$I_2 = 0.2414\,1627\,5088\,(y_{-1} + y_{+1}) +$$
$$+ 0.1584\,6732\,4625\,(y_{-2} + y_{+2}) + 0.0960\,7580\,3242\,(y_{-3} + y_{+3}) +$$
$$+ 0.0080\,8119\,4089\,y_4,$$

tandis que, sub (46) on a trouvé pour I_1

$$I_1 = 0.2466\,1426\,3490\,(y_{-1} + y_{+1}) +$$
$$+ 0.1569\,3803\,5364\,(y_{-2} + y_{+2}) + 0.0964\,4770\,1146\,(y_{-3} + y_{+3})$$

Si nous entendons, comme STIRLING [1]) par „correction" ou „corr."

[1]) STIRLING. *Methodus Differentialis.* Londres 1730.

la différence entre la valeur approximative de l'intégrale calculée pour le cas de $2m$ ou de $(2m-1)$ ordonnées et la valeur plus exactement approximative, que l'on obtient en adjoignant au nombre des ordonnées deux ordonnées nouvelles, alors corr. $= I_2 - I_1$ et donc, dans le cas envisagé plus haut,

$$\begin{aligned}\text{corr.}: = &- 0.0051\,9798\,8402\,(y_{-1}+y_{+1}) + \\ &+ 0.0015\,2928\,9261\,(y_{-2}+y_{+2}) - 0.0003\,7189\,7904\,(y_{-3}+y_{+3}) + \\ &0.0080\,8119\,4089\,y_4 \ldots\ldots\ldots\ldots \quad (50)\end{aligned}$$

De (46) on a trouvé pour une valeur approximative de $\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{dx}{8.5+x}$:

$$I_1 = 0.\ 1177\ 8303\ 5657\ 4$$

De (50) on trouve corr.: $= -\ 0.0000\ 0000\ 0000\ 53$, d'où l'on peut conclure que la valeur trouvée pour I_1 est exacte jusqu'à la onzième décimale inclusivement, ce qui est d'accord avec l'énoncé de la valeur approximative de ladite intégrale dans le § 15.

Les corrections qui ont été admises dans la Table B ont le sens donné ci-dessus à „corr.": et ont été développées d'une façon identique à celle que nous venons d'indiquer.

Prenons encore la formule pour 5 ordonnées de cette Table. A la fin du § 20, on a trouvé de cette formule pour une valeur approximative de l'intégrale $\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{dx}{8.5+x}$

$$I_1 = 0.\ 1177\ 8303\ 5657\ 4$$

tandis que l'on trouve de la formule pour „corr."

$$\text{corr.}: = -\ 0.\ 0000\ 0000\ 0001\ 3$$

d'où il résulte que la valeur calculée de I_1 est exacte jusqu'à la onzième décimale inclusivement et est plus exactement représentée par

$$I_1 = 0.\ 1177\ 8303\ 5656.$$

Calcul des valeurs de S_p, $S_{p.q}$, etc.

§ 22. Les longueurs d'abscisses corrigées, exprimées seulement en deux décimales et dont nous représentons les carrés par

$$z_1^2\ ,\ z_2^2\ ,\ z_3^2, \ldots z_m^2 \ldots\ldots\ldots\ldots \quad (51)$$

étant connues, il est facile, pour de petites valeurs de m, de calculer assez rapidement les valeurs correspondantes des grandeurs S_p, $S_{p.q}$ etc., au moyen des sommes des produits deux à deux, trois à trois, etc. desdits carrés; mais lorsque m est grand, ces calculs deviennent extrêmement compliqués à cause du grand nombre

de produits, qui alors doivent être déterminés. Cependant, on peut donner aux expressions pour S_p, etc. une autre forme, qui est quelquefois plus commode.

1^e. Nous représentons en général la somme des $p^{\text{ièmes}}$ puissanses des carrés sub (51), à l'exception des carrés x_q^2 et x_r^2, par $\Sigma^p{}_{q.r}$, et la somme des produits p à p de ces mêmes $(m-2)$ carrés par $S_{p.q.r}$, alors les carrés sub (51), sauf les carrés z_q^2 et z_r^2, sont les racines de l'équation

$$f(z^2) = (z^2 - z_1^2)(z^2 - z_2^2)\ldots(z^2 - z_m^2) \ldots\ldots\ldots \quad (52)$$

Dans (52), les facteurs $(z^2 - z_q^2)$ et $(z^2 - z_r^2)$ ne se présentent pas; pour (52) on peut écrire aussi

$$f(z^2) = (z^2)^{m-2} - S_{1.q.r}(z^2)^{m-3} + S_{2.q.r}(z^2)^{m-4} - \ldots + (-1)^{m-2} S_{m-2.q.r} = 0$$

la dérivée de cette dernière équation est

$$\begin{aligned} f^1(z^2) = (m-2)(z^2)^{m-3} - (m-3)\, S_{1.q.r}(z^2)^{m-4} + \ldots + \\ + (-1)^{m-1} S_{m-3.q.r} = 0 \ \ldots\ldots\ldots\ldots \quad (53) \end{aligned}$$

Le rapport entre $f^1(z^2)$ et $f(z^2)$ s'exprime par [1])

$$\frac{f^1(z^2)}{f(z^2)} = \frac{1}{z^2 - z_1^2} + \frac{1}{z^2 - z_2^2} + \frac{1}{z^2 - z_3^2} + \ldots + \frac{1}{z^2 - z_m^2};$$

donc

$$f^1(z^2) = \frac{f(z^2)}{z^2 - z_1^2} + \frac{f(z^2)}{z^2 - z_2^2} + \frac{f(z^2)}{z^2 - z_3^2} + \ldots + \frac{f(z^2)}{z^2 - z_m^2};$$

dans la dernière équation les termes $\dfrac{f(z^2)}{z^2 - z_q^2}$ et $\dfrac{f(z^2)}{z^2 - z_r^2}$ ne se présentent pas.

Si l'on divise $f(z^2)$ par $(z^2 - z_1^2)$ on obtient

$$\frac{f(z^2)}{z^2 - z_1^2} = \begin{array}{l|l|l} (z^2)^{m-3} - S_{1.q.r} & (z^2)^{m-4} + S_{2.q.r} & (z^2)^{m-5} - \ldots \\ \qquad\quad + z_1^2 & \qquad\quad - S_{1.q.r}.z_1^2 & \\ & \qquad\quad + z_1^4 & \end{array}$$

Si dans cette dernière équation on remplace z_1^2 successivement par chacune des $(m-3)$ autres racines, on aura en additionnant tous les résultats, l'équation suivante, qui est identique à celle sub (53)

$$\begin{aligned} f^1(z^2) = (m-2)(z^2)^{m-3} - \{(m-2)\, S_{1.q.r} - \Sigma^1{}_{q.r}\}(z^2)^{m-4} + \\ + \{(m-2)\, S_{2.q.r} - \Sigma^1{}_{q.r} \cdot S_{1.q.r} + \Sigma^2{}_{q.r}\}(z^2)^{m-5} + \ldots\ldots \end{aligned}$$

1) J. A. Serret, *Cours d'Algèbre Supérieure* I. p. 111 et 377—379.

La comparaison de cette expression pour $f^1(z^2)$ avec celle sub (53), fournit les relations que nous donnons ici:

$$(m-3)\,S_{1.q.r} = (m-2)\,S_{1.q.r} - \Sigma^1_{q.r} \quad ,$$
$$(m-4)\,S_{2.q.r} = (m-2)\,S_{2.q.r} - \Sigma^1_{q.r} . S_{1.q.r} + \Sigma^2_{q.r} \quad ,$$
$$(m-5)\,S_{3.q.r} = (m-2)\,S_{3.q.r} - \Sigma^1_{q.r}\, S_{2.q.r} + \Sigma^2_{q.r} . S_{1.q.r} - \Sigma^3_{q.r} \quad ,$$

et ainsi de suite.

Après quelques réductions on obtient les équations suivantes, qui ont été données, pour la première fois, par Newton:

$$\left.\begin{array}{l} S_{1.q.r} = \Sigma^1_{q.r} \quad , \\ 2\,S_{2.q.r} = \Sigma^1_{q.r} . S_{1.q.r} - \Sigma^2_{q.r} \quad , \\ 3\,S_{3.q.r} = \Sigma^1_{q.r} . S_{2.q.r} - \Sigma^2_{q.r} . S_{1.q.r} + \Sigma^3_{q.r} \quad , \\ 4\,S_{4.q.r} = \Sigma^1_{q.r} . S_{3.q.r} - \Sigma^2_{q.r} . S_{2.q.r} + \Sigma^3_{q.r} . S_{1.q.r} - \Sigma^4_{q.r} \quad , \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ (m-2)\,S_{m-2.q.r} = \Sigma^1_{q.r} . S_{m-3.q.r} - \Sigma^2_{q.r} . S_{m-4.q.r} + \dots + (-1)^{m-1}\Sigma_{q.r}^{m-2} \end{array}\right\} \dots (54)$$

2e. Les carrés sub (51) sont les racines de l'équation

$$f(z_2) = (z^2)^m - S_1\,(z^2)^{m-1} + \dots + (-1)^m S_m = 0 \dots \quad (55)$$

Si l'on isole des carrés sub (51) le carré $x_p{}^2$ et qu'on mette $S_{1.p}$ = la somme des carrés sub (51), excepté le carré $x_p{}^2$, $S_{2.p}$ = la somme des produits deux à deux de ces mêmes $(m-1)$ carrés, et ainsi de suite, alors les $(m-1)$ carrés restants sont les racines de l'équation

$$(z^2)^{m-1} - S_{1.p}\,(z^2)^{m-2} + \text{etc.}$$

En multipliant cette dernière équation avec $(x^2 - x_p{}^2)$, alors, puisque ce produit est identique à l'équation $f(z^2)$ sub (55), le $(q+1)^{\text{ième}}$ terme du produit sera égal au $(q+1)^{\text{ième}}$ terme de (55), par conséquent on a en général

$$S_m = S_{m.p} + S_{m-1.p} . x_p{}^2 \dots\dots\dots\dots \quad (56)$$

SECTION II.

Formules d'approximation lorsque la fonction sous le signe intégral ne peut pas être remplacé par une série de la première classe.

§ 23. Les formules des Tables A et B ont été établies dans la supposition que la fonction sous le signe intégral peut être représenté par une série de la *première* classe. Si ces formules sont appliquées à une intégrale dont la fonction ne peut pas être remplacée par une série de puissances complète, les calculs n'atteindront pas à la plus grande exactitude accessible.

Si auparavant on découvre une lacune dans la série, on peut en tenir compte dans le cas où l'on veut connaître les abscisses les plus avantageuses pour l'intégrale, dont on veut calculer une valeur approximative.

Pour la question qui nous occupera dans cette Section, nous introduirons (voir l'alinéa final du § 8) de nouveau les lignes A^1X et A^1Y^1 comme axes des coordonnées et nous considérerons les intégrales de $x^d F(x)\,dx$ et $\frac{1}{a \pm b\,x^k}\cdot dx$ entre les limites 0 et 1. Les lettres d et k représentent des nombres entiers plus grands que 1.

Dans ces deux fonctions, on voit immédiatement:

Dans la première, que la ligne représentée par l'équation $y = x^d F(x)$ passe par le point d'intersection A^1 des axes A^1X et A^1Y^1 et que par conséquent dans (5) $L = 0$, et, dans la deuxième, que, dans la série, qui peut remplacer $\frac{1}{a \pm bx^k}$, il manque plusieurs termes, et qu'ainsi ces deux séries n'appartiennent pas à celles de la *première classe*.

§ 24. 1°. Lorsque dans $x^d F(x)$, la fonction $F(x)$ elle-même peut être représentée par la série de la *première* classe

$$F(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \text{etc.},$$

alors la valeur de l'intégrale de $x^d F(x) dx$, entre les limites 0 et 1, est

$$I = \int_0^1 x^d F(x)\,dx = \tfrac{1}{d+1} a_0 + \tfrac{1}{d+2} a_1 + \tfrac{1}{d+3} a_2 + \tfrac{1}{d+4} a_3 + \text{etc.} \quad (57)$$

Si l'on représente les coordonnées des n points connus de la

vraie ligne limite de la figure par (x_1, y_1), (x_2, y_2), $(x_3, y_3) \ldots (x_n, y_n)$ alors une aire approximative de la figure est

$$I_1 = R_1 y_1 + R_2 y_2 + R_3 y_3 + \ldots + R_n y_n \ldots\ldots \quad (58)$$

et $I = I_1 + E$.

Si l'on substitue dans (58) les n ordonnées calculées, alors on obtient

$$\begin{aligned} I = & R_1 (a_0 x_1^{\;d} + a_1 x_1^{\;d+1} + a_2 x_1^{\;d+2} + a_3 x_1^{\;d+3} + \text{etc.}) + \\ & + R_2 (a_0 x_2^{\;d} + a_1 x_2^{\;d+1} + a_2 x_2^{\;d+2} + a_3 x_2^{\;d+3} + \text{etc.}) + \\ & + R_3 (a_0 x_3^{\;d} + a_1 a_3^{\;d+1} + a_2 x_3^{\;d+2} + a_3 x_3^{\;d+3} + \text{etc.}) + \\ & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ & + R_n (a_0 x_n^{\;d} + a_1 x_n^{\;d+1} + a_2 x_n^{\;d+2} + a_3 x_n^{\;d+3} + \text{etc.}) = \\ & = \Sigma (x_1^{\;d+p} R_1 + x_2^{\;d+p} R_2 + x_3^{\;d+p} R_3 + \ldots + x_n^{\;d+p} R_n). \end{aligned}$$

La dernière équation soustraite de (57) procure l'équation

$$E = \Sigma \left\{ \frac{1}{d+(p+1)} - (x_1^{\;d+p} R_1 + x_2^{\;d+p} R_2 + x_3^{\;d+p} R_3 + \ldots + x_n^{\;d+p} R_n) \right\}$$

dans laquelle il faut poser successivement $p = 0,\ 1,\ 2$, etc.

Si dans la dernière expression on égale à zéro chacun des $2n$ premiers termes, alors les valeurs de x de

$$\left.\begin{array}{l} x_1^{\;d} R_1 + x_2^{\;d} R_2 + x_3^{\;d} R_3 + \ldots + x_n^{\;d} R_n = \frac{1}{d+1} \\ x_1^{\;d+1} R_1 + x_2^{\;d+1} R_2 + x_3^{\;d+1} R_3 + \ldots + x_n^{\;d+1} R_n = \frac{1}{d+2} \\ x_1^{\;d+2} R_1 + x_2^{\;d+2} R_2 + x_3^{\;d+2} R_3 + \ldots + x_n^{\;d+2} R_n = \frac{1}{d+3} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ x_1^{\;d+2n-1} R_1 + x_2^{\;d+2n-1} R_2 + x_3^{\;d+2n-1} R_3 + \ldots + x_n^{\;d+2n-1} R_n = \frac{1}{d+2n} \end{array}\right\} \ldots (59)$$

donnent les longueurs d'abcisses pour les formules d'approximation les plus exactes pour n ordonnées. Si dans (66) et (69) nous posons $c = 0$, $k = 1$, $v = n$ et $u_p = \frac{1}{d+p+1}$, on trouve facilement que dans (59) les grandeurs x sont les racines de l'équation

$$x^n - S_1 x^{n-1} + S_2 x^{n-2} - S_3^{n-3} + \ldots + (-1)^n S_n = 0$$

tandis qu'en posant dans (72) et (73) $r = d + n + 1$, $v = n$, $\Delta = 1$, $B_0 = 1, B_1 = S_1, \ldots B_n = S_n$, on obtient

$$S_1 = \frac{n}{1} \cdot \frac{d+n}{d+2n},$$

$$S_2 = \frac{n-1}{2} \cdot \frac{d+n-1}{d+2n-1} S_1,$$

$$S_3 = \frac{n-2}{3} \cdot \frac{d+n-2}{d+2n-2} S_2,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

$$S_{n-1} = \frac{2}{n-1} \cdot \frac{d+2}{d+n+2} S_{n-2},$$
$$S_n \quad = \frac{1}{n} \cdot \frac{d+1}{d+n+1} \cdot S_{n-1}.$$

On trouve, par exemple, pour $d = 1$ et $n = 4$ les grandeurs x de l'équation suivante

$$x^4 - S_1 . x^3 + S_2 x^2 - S_3 x + S_4 = 0$$

où

$$S_1 = \tfrac{4}{1} \cdot \tfrac{5}{9} = \tfrac{20}{9},$$
$$S_2 = \tfrac{3}{2} \cdot \tfrac{4}{8} \cdot \tfrac{20}{9} = \tfrac{5}{3},$$
$$S_3 = \tfrac{2}{3} \cdot \tfrac{3}{7} \cdot \tfrac{5}{3} = \tfrac{10}{21} \text{ et}$$
$$S_4 = \tfrac{1}{4} \cdot \tfrac{2}{6} \cdot \tfrac{10}{21} = \tfrac{5}{126}$$

donc

$$x^4 - \tfrac{20}{9} x^3 + \tfrac{5}{3} x^2 - \tfrac{10}{21} x + \tfrac{5}{126} = 0$$

d'où

$$\left.\begin{aligned} x_1 &= 0.1397\ 5986 \\ x_2 &= 0.4164\ 0957 \\ x_3 &= 0.7231\ 5699 \\ x_4 &= 0.9428\ 9580 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (60)$$

Remarquons ici que la série pour $F(x)$ a été supposée de la *première* classe, ce qui n'est pas toujours facile à constater, il s'en faut, et si, parmi les premiers termes de cette série, il en manquait un ou plusieurs, alors les valeurs sub (60) ne désignent non plus les longueurs d'abscisses les plus favorables.

Pour $d = 1$ et $n = 3$, on trouve

$$\left.\begin{aligned} x_1 &= 0.2123\ 4054 \\ x_2 &= 0.5905\ 3314 \\ \text{et } x_3 &= 0.9114\ 1204 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (61)$$

Pour $d = 1$ et $n = 2$:

$$x_1 = 0.3550\ 5103$$
$$\text{et } x_2 = 0.8448\ 4897$$

et pour $d = 1$ et $n = 1$:

$$x_1 = \tfrac{2}{3}. \quad \ldots\ldots\ldots\ldots \quad (62)$$

2°. Remplaçons l'expression $y = \frac{1}{a \pm bx} k$, dans laquelle on suppose $\frac{b}{a}$ 1, par l'équation suivante

$$y = \frac{1}{a} \mp \frac{b}{a^2} . x^k + \frac{b^2}{a^3} . x^{2k} \mp \frac{b^3}{a^4} . x^{3k} + \text{etc.} \ldots\ldots\ldots \quad (63)$$

dans laquelle, des signes doubles, les signes supérieurs vont ensemble, ainsi que les signes inférieurs.

Nous posons $\frac{1}{a} = a_0$, $\mp \frac{b}{a^2} = a_1$; $\frac{b^2}{a^3} = a_2$, $\mp \frac{b^3}{a^4} = a^3$, etc., d'où (63) devient

$$y = a_0 + a_1 x^k + a_2 . x^{2k} + a_3 . x^{3k} + \text{etc.}$$

Si les abscisses des n points connus de la vraie ligne limite de la figure sont $x_1, x_2, x_3, \ldots x_n$, alors

$$\begin{aligned} I_1 = {} & R_1 (a_0 + a_1 x_1^k + a_2 x_1^{2k} + a_3 x_1^{3k} + \text{etc.}) + \\ & + R_2 (a_0 + a_1 x_2^k + a_2 x_2^{2k} + a_3 x_2^{3k} + \text{etc.}) + \\ & + R_3 (a_0 + a_1 x_3^k + a_2 x_3^{2k} + a_3 x_3^{3k} + \text{etc.}) + \\ & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ & + R_n (a_0 + a_1 x_n^k + a_2 x_n^{2k} + a_3 x_n^{3k} + \text{etc.}) = \\ & = (R_1 + R_2 + R_3 + \ldots + R_n) a_0 + \\ & + (x_1^k R_1 + x_2^k R_2 + x_3^k R_3 + \ldots + x_n^k R_n) a_1 + \\ & + (x_1^{2k} R_1 + x_2^{2k} R_2 + x_3^{2k} R_3 + \ldots + x_n^{2k} R_n) a_2 + \end{aligned}$$

$+$ et ainsi de suite.

La dernière égalité ayant été soustraite de $I = \int_0^1 \frac{1}{a \pm bx^k} dx$, notamment

$$I = a_0 + \frac{1}{k+1} a_1 + \frac{1}{2k+1} a_2 + \frac{1}{3k+1} a_3 + \text{etc.}$$

donne, pour le calcul des n longueurs d'abscisses, les équations suivantes

$$\begin{aligned} & R_1 + R_2 + R_3 + \ldots + R_n = 1 , \\ & x_1^k R_1 + x_2^k R_2 + x_3^k R_3 + \ldots + x_n^k R_n = \frac{1}{k+1} , \\ & x_1^{2k} R_1 + x_2^{2k} R_2 + x_3^{2k} R_3 + \ldots + x_n^{2k} R_n = \frac{1}{2k+1} , \\ & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ & x_1^{(2n-1)k} R_1 + x_2^{(2n-1)k} R_2 + x_3^{(2n-1)k} R_3 + \ldots + x_n^{(2k-1)k} R_n = \frac{1}{(2n-1)k+1} . \end{aligned}$$

Pour $k = 2$ et $n = 3$, par exemple, on trouve les longueurs d'abscisses de l'équation

$$(x^2)^3 - S_1 (x^2)^2 + S_2 (x^2) - S_3 = 0$$

où

$$S_1 = \tfrac{3}{1} \cdot \tfrac{5}{11} = \tfrac{15}{11},$$
$$S_2 = \tfrac{2}{2} \cdot \tfrac{3}{9} \cdot \tfrac{15}{11} = \tfrac{5}{11},$$
$$\text{et } S_3 = \tfrac{1}{3} \cdot \tfrac{1}{7} \cdot \tfrac{5}{11} = \tfrac{5}{231},$$

par conséquent

$$\left.\begin{aligned} x_1 &= 0.2386\ 1920 \\ x_2 &= 0.6612\ 0943 \\ \text{et } x_3 &= 0.9324\ 6953 \end{aligned}\right\} \dots\dots\dots\dots\dots \quad (64)$$

Conclusion.

§ 25. Les distances des pieds des trois ordonnées pour $(2m - 1) = 3$ de la Table A, calculées à partir du point A^1, comportent

$$\left.\begin{aligned} x_1 &= 0.5 - 0.3872\ 9833 = 0.1127\ 0167 \\ x_2 &= 0.5 \\ \text{et } x_3 &= 0.5 + 0.3872\ 9833 = 0.8872\ 9833 \end{aligned}\right\} \dots \quad (65)$$

Si l'on compare ces distances avec les longueurs des abscisses sub (61) et (64) et ces longueurs-ci entre elles, on constate, que toute lacune dans le commencement de la série, qui peut remplacer la fonction sous le signe intégral, peut avoir une très grande influence sur les longueurs des abscisses les plus favorables. C'est ainsi que, par exemple, pour $n = 3$, l'abscisse x_1 dans (64) est plus de deux fois aussi grande que celle dans (65).

Si l'on compare ensuite les $2m = 4$ longueurs d'abscisses sub (60) avec celles correspondantes de la Table A, celles-ci étant comptées à partir du point A^1, notamment

$$x_1 = 0.0694\ 3184,$$
$$x_2 = 0.3300\ 0948,$$
$$x_3 = 0.6699\ 9052,$$
$$\text{et } x_4 = 0.9305\ 6816,$$

on voit, que lorsqu'on applique les longueurs d'après Gauss à des fonctions dont les séries ne sont pas de la *première* classe, *les 14 ou 15 dernières des 16 décimales*, dans lesquelles x est exprimé dans la Table A n'ont *absolument aucune valeur*.

La comparaison des longueurs d'abscisses sub (62) avec celles correspondantes de la Table A donne lieu à une remarque analogue.

De ces différents faits on peut déduire, que les formules de la Table A ne peuvent être employées avec avantage, que pour un nombre restreint de fonctions, notamment pour celles dont on sait avec certitude qu'elles peuvent être remplacées par des séries de

la *première* classe; pour toutes les autres il faudra parvenir à l'exactitude dans le calcul de la valeur de l'intégrale en prenant un nombre considérable d'ordonnées.

Il en résulte, en rapport avec les observations dans les §§ 12, 16 et 21, qu'en général, pour le calcul de la valeur approximative d'une intégrale définie, les formules de la Table B méritent d'être préférées à celles de la Table A.

SECTION III.

Développement de formules sur lesquelles reposent celles destinées au calcul d'une intégrale définie.

§ 26. Des $(c+v)$ équations [1])

$$\left.\begin{array}{l}
x_1^{\,d}R_1+\quad x_2^{\,d}.R_2+\quad x_3^{\,d}.R_3+\ldots+\quad x_{c+v}{}^{d}R_{c+v}=u_0\;, \\
x_1^{\,k+d}R_1+\quad x_2^{\,k+d}R_2+\quad x_3^{\,k+d}R_3+\ldots+\quad x_{c+v}{}^{k+d}R_{c+v}=u_1\;, \\
x_1^{\,2k+d}R_1+\quad x_2^{\,2k+d}R_2+\quad x_3^{\,2k+d}R_3+\ldots+\quad x_{c+v}{}^{2k+d}R_{c+v}=u_2\;, \\
x_1^{\,3k+d}R_1+\quad x_2^{\,3k+d}R_2+\quad x_3^{\,3k+d}R_3+\ldots+\quad x_{c+v}{}^{3k+d}R_{c+v}=u_3\;, \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
x_1^{\,(c+v-1)k+d}R_1+x_2^{\,(c+v-1)k+d}R_2+x_3^{\,(c+v-1)k+d}R_3+\ldots+x_{c+v}{}^{(c+v-1)k+d}R_{c+v}=u_{c+v-1},
\end{array}\right\}\ldots\;(66)$$

où c, v, d et k représentent des nombres positifs arbitraires et les $(c+v)$ grandeurs u, de même les nombres x sont connus et les nombres R sont inconnus, déterminer la valeur d'un R arbitraire [2]).

Solution. Remarquons que toutes les $(c+v)$ grandeurs x se présentent tout à fait de la même façon dans les équations sub (66). C'est pourquoi il ne faut pas résoudre immédiatement chaque R séparément, mais il faut tout d'abord tâcher de déterminer les coefficients de l'équation dont les $(c+v)$ grandeurs x^k sont les racines. C'est à cet effet que nous introduisons les notations suivantes:

$S_{1.p}=$ la somme des puissances $x_1^{\,k},\ldots x_{c+v}{}^{k}$, à l'exception de $x_p^{\,k}$,

$S_{2.p}=$ la somme des produits deux à deux de ces mêmes $(c+v-1)$ puissances,

$S_{3.p}=$ la somme des produits trois à trois de ces mêmes $(c+v-1)$ puissances, et ainsi de suite.

$S_{c+v-1.p}=$ le produit (continu) de ces mêmes $(c+v-1)$ puissances.

---

[1]) Ici la somme $(c+v)$ pourrait être remplacée par une seule lettre; cependant il est préférable, à cause de la clarté et en rapport avec la solution des équations sub (69) et avec les formules qui doivent en être déduites, de garder la somme $(c+v)$.

[2]) Voir *Mémoires de l'Académie de Berlin.* Recherches par M. DE LA GRANGE. 1775, pag. 183 et 1792, pag. 247.

Les puissances $x_1^k, \ldots x_{c+v}^k$, à l'exception de x_p^k, sont alors les racines de l'équation

$$x^{(c+v-1)k} - S_{1.p} \,.\, x^{(c+v-2)k} + S_{2.p} \,.\, x^{(c+v-3)k} - \ldots + (-1)^{c+v-1} S_{c+v-1.p} = 0.$$

Nous représentons aussi le premier membre de cette équation par

$$\varphi_p(x^k) = (x^k - x_1^k)(x^k - x_2^k) \ldots (x^k - x_{c+v}^k) = 0 \ldots . \quad (67)$$

dans ce produit, il est entendu que le facteur $(x^k - x_p^k)$ ne se présente pas.

Pour résoudre les équations sub (66), nous les additionnons, après les avoir multipliées, la dernière par $+1$, l'avant-dernière par $-S_{1.p}$, la précédente par $+S_{2.p}$, et ainsi de suite, et enfin la première par $(-1)^{c+v-1} S_{c+v-1.p}$.

Si l'on réunit les termes avec le même R, on obtient

$$\begin{aligned}
&R_1 \left\{ x_1^{(c+v-1)k+d} - S_{1.p} x_1^{(c+v-2)k+d} + S_{2.p} x_1^{(c+v-3)k+d} - \ldots + (-1)^{c+v-1} S_{c+v-1.p}\, x_1^d \right\} + \\
&+ R_2 \left\{ x_2^{(c+v-1)k+d} - S_{1.p} x_2^{(c+v-2)k+d} + S_{2.p} x_2^{(c+v-3)k+d} - \ldots + (-1)^{c+v-1} S_{c+v-1.p}\, x_2^d \right\} + \\
&+ R_3 \left\{ x_3^{(c+v-1)k+d} - S_{1.p} x_3^{(c+v-2)k+d} + S_{2.p} x_3^{(c+v-3)k+d} - \ldots + (-1)^{c+v-1} S_{c+v-1.p}\, x_3^d \right\} + \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&+ R_{c+v} \left\{ x_{c+v}^{(c+v-1)k+d} - S_{1.p} x_{c+v}^{(c+v-2)k+d} + S_{2.p} x_{c+v}^{(c+v-3)k+d} - \ldots + (-1)^{c+v-1} S_{c+v-1.p}\, x_{c+v}^k \right\} = \\
&= u_{c+v-1} - S_{1.p} \,.\, u_{c+v-2} + S_{2.p} \,.\, u_{c+v-3} - \ldots + (-1)^{c+v-1} S_{c+v-1.p} u_0.
\end{aligned}$$

Le premier membre de cette équation peut être également représenté par

$$R_1 \,.\, \varphi_p(x_1^k) x_1^d + R_2 \,.\, \varphi_p(x_2^k) x_2^d + R_3 \,.\, \varphi_p(x_3^k) x_3^d + \ldots + R_{c+v} \,.\, \varphi_p(x_{c+v}^k) x_{c+v}^d$$

où, en vertu de la supposition au sujet de $\varphi_p(x^k)$ sub (67), les coefficients de toutes les grandeurs R, excepté le coefficient de la grandeur R_p, sont égaux à zéro, par conséquent on obtient, après avoir divisé par le coefficient de R_p:

$${}_p = \frac{u_{c+v-1} - S_{1.p} \,.\, u_{v+c-2} + S_{2.p} \,.\, u_{v+c-3} - \ldots + (-1)^{c+v-1} S_{c+v-1.p} u_0}{x_p^{(c+v-1)k+d} - S_{1.p} \,.\, x_p^{(c+v-2)k+d} + S_{2.p} \,.\, x_p^{(c+v-3)k+d} - \ldots + (-1)^{c+v-1} S_{c+v-1.p} \,.\, x_p^d} \ldots (68)$$

Dans les applications, que dans cet exposé, nous faisons du groupe d'équations sub (66) on a toujours $x_1 < x_2 < x_3 < \ldots < x_{c+v}$ et donc, en vertu de (67), le dénominateur dans (68), abstraction faite du signe, est plus grand que zéro; par conséquent R_p reçoit dans (68) toujours une valeur unique et déterminée. On ne peut pas admettre pour x deux valeurs identiques, par exemple $x_q = x_{q+1}$, parce que, dans ce cas, le dénominateur de (68) serait égal à zéro et que par conséquent la valeur de R_p deviendrait infiniment grande. Il est clair qu'il est bien permis de prendre une des valeurs de x notamment $x_1 = 0$ (voir la note à la p. 8)

§ 27. Des $(c+v)+v=(c+2v)$ équations

$$\left.\begin{array}{llll}
x_1^{\,d}\,R_1+ & x_2^{\,d}\,R_2+\ldots+ & x_v^{\,d}\,R_v+\ldots+ & x_{c+v}^{\,d}\,R_{c+v}=u_0\quad, \\
x_1^{\,k+d}\,R_1+ & x_2^{\,k+d}\,R_2+\ldots+ & x_v^{\,k+d}\,R_v+\ldots+ & x_{c+v}^{\,k+d}\,R_{c+v}=u_1\quad, \\
x_1^{\,2k+d}\,R_1+ & x_2^{\,2k+d}\,R_2+\ldots+ & x_v^{\,2k+d}\,R_v+\ldots+ & x_{c+v}^{\,2k+d}\,R_{c+v}=u_2\quad, \\
\ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\
x_1^{(c+2v-1)k+d}\,R_1+ & x_2^{(c+2v-1)k+d}\,R_2+\ldots+ & x_v^{(c+2v-1)k+d}\,R_v+\ldots+ & x_{c+v}^{(c+2v-1)k+d}\,R_{c+v}=u_{c+2v-1}.
\end{array}\right\}\ldots(69)$$

dans lesquelles les $(c+2v)$ grandeurs u, de même que les c grandeurs $x_1,\ x_2,\ x_3,\ldots x_c$ sont connues et dans lesquelles les autres v grandeurs x, ainsi que toutes les $(c+v)$ grandeurs R sont inconnues, résoudre les $(c+2v)$ inconnues.

Solution. Du § précédent est apparu comment on peut trouver les valeurs de R au moyen des premières $(c+v)$ équations de (69), quand toutes les grandeurs $x_1,\ x_2,\ x_3,\ldots x_{c+v}$ sont connues. Pour pouvoir exprimer ici R_p dans les données, il nous importe donc d'abord de définir les v grandeurs inconnues x.

A cet effet, nous introduisons les notations suivantes:

$$\left.\begin{array}{l}
S_1=\text{la somme de toutes les } (c+v) \text{ grandeurs } x_1^{\,k},\ x_2^{\,k},\ldots x_{c+v}^{\,k}, \\
S_2=\text{la somme de leurs produits deux à deux,} \\
\qquad\text{et ainsi de suite,} \\
S_{c+v}=\text{leurs produit (continu)}
\end{array}\right\}\ldots(70)$$

Les $(c+v)$ grandeurs (x^k) sont alors les racines de l'équation

$$x^{(c+v)k}-S_1x^{(c+v-1)k}+S_2x^{(c+v-2)k}-\ldots+(-1)^{c+v}S_{c+v}=0\ldots(71)$$

Le premier membre de cette égalité peut être également représenté par

$$f(x^k) \text{ ou } (x^k-x_1^{\,k})\,(x^k-x_2^{\,k})\,(x^k-x_3^{\,k})\ldots(x^k-x_{c+v}^{\,k})\ ^{1})$$

Qu'on prenne maintenant, des $(c+2v)$ équations, sub (69), $(c+v+1)$ qui se suivent immédiatement (ce qui peut se faire de v manières) et qu'on les additionne, après les avoir multipliées, la dernière par $+1$, l'avant-dernière par $-S_1$, celle qui précède par $+S_2$, et ainsi de suite, et enfin la première par $(-1)^{c+v}S_{c+v}$. Réunissant les termes avec le même R, on obtient

$$\begin{array}{l}
R_1x_1^{\,zk+d}\{x_1^{\,(c+v)k}-S_1x_1^{\,(c+v-1)k}+S_2x_1^{\,(c+v-2)k}-\ldots+(-1)^{c+v}S_{c+v}\}+ \\
+R_2x_2^{\,zk+d}\{x_2^{\,(c+v)k}-S_1x_2^{\,(c+v-1)k}+S_2x_2^{\,(c+v-2)k}-\ldots+(-1)^{c+v}S_{c+v}\}+ \\
+R_3x_3^{\,zk+d}\{x_3^{\,(c+v)k}-S_1x_3^{\,(c+v-1)k}+S_2x_3^{\,(c+v-2)k}-\ldots+(-1)^{c+v}S_{c+v}\}+ \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
+R_{c+v}x_{c+v}^{\,zk+d}\{x_{c+v}^{\,(c+v)k}-S_1x_{c+v}^{\,(c+v-1)k}+S_2x_{c+v}^{\,(c+v-2)k}-\ldots+(-1)^{c+v}S_{c+v}\}= \\
=u_{c+v+z}-S_1\,.\,u_{c+v+z-1}+S_2\,.\,u_{c+v+z-2}-\ldots+(-1)^{c+v}\,S_{c+v}\,.\,u_z.
\end{array}$$

---

[1]) Le rapport de cette expression avec celle pour $\Phi_p(x\ )$ sub (67) est désigné par $f(x^k)=(x^k-x_p^{\,k})\,.\,\Phi_p(x^k)$.

Le premier membre de cette égalité peut être également représenté par

$$R_1 x_1^{zk+d}.f(x_1^k)+R_2 x_2^{zk+d}.f(x_2^k)+R_3 x_3^{zk+d}.f(x_3^k)+\ldots+R_{c+v}x_{c+v}^{zk+d}.f(x_{c+v}^k).$$

Puisque, dans cette expression, à cause de la supposition qui a été faite au sujet de $f(x^k)$, tous les termes sont égaux à zéro, nous avons

$$u_{c+v+z}-S_1.u_{c+v+z-1}+S_2.u_{c+v+z-2}-\ldots+(-1)^{c+v}S_{c+v}.u_z=0.$$

Pour z on peut poser successivement les valeurs 0, 1, 2,... $(v-1)$ de sorte qu'on obtient les v équations suivantes:

$$\left.\begin{array}{l}
u_{c+v}-S_1.u_{c+v-1}+S_2.u_{c+v-2}-S_3.u_{c+v-3}+\ldots+(-1)^{c+v}S_{c+v}.u_0=0\\
u_{c+v+1}-S_1.u_{c+v}+S_2.u_{c+v-1}-S_3.u_{c+v-2}+\ldots+(-1)^{c+v}S_{c+v}.u_1=0\\
u_{c+v+2}-S_1.u_{c+v+1}+S_2.u_{c+v}-S_3.u_{c+v-1}+\ldots+(-1)^{c+v}S_{c+v}.u_2=0\\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
u_{c+2v-1}-S_1.u_{c+2v-2}+S_2.u_{c+2v-3}-S_3.u_{c+2v-4}+\ldots+(-1)^{c+v}S_{c+v}.u_{v-1}=0
\end{array}\right\}\ldots(72)$$

Etablissons encore

$\mathbf{s}_1$ = la somme des c grandeurs connues x_1^k, x_2^k, x_3^k,...x_c^k,

$\mathbf{s}_2$ = la somme de leurs produits deux à deux; etc.

et

$\boldsymbol{S}_1$ = la somme des v grandeurs inconnues x_{c+1}^k, x_{c+2}^k,...x_{c+v}^k,

$\boldsymbol{S}_2$ = la somme de leurs produits deux à deux;

et ainsi de suite.

Si nous substituons ensuite, dans (72) à S_1, S_2, S_3, etc. ses expressions dans $\mathbf{s}_1$, $\mathbf{s}_2$, $\mathbf{s}_3$,...$\mathbf{s}_c$, $\boldsymbol{S}_1$, $\boldsymbol{S}_2$, $\boldsymbol{S}_3$,..,$\boldsymbol{S}_v$, alors nous obtenons, après quelque réduction

$$\begin{array}{l}
\{u_{c+v}-\mathbf{s}_1.u_{c+v-1}+\mathbf{s}_2.u_{c+v-2}-\mathbf{s}_3.u_{c+v-3}+\ldots+(-1)^c.u_v\}-\\
-\{u_{c+v-1}-\mathbf{s}_1.u_{c+v-2}+\mathbf{s}_2.u_{c+v-3}-\mathbf{s}_3.u_{c+v-4}+\ldots+(-1)^c.u_{v-1}\}\boldsymbol{S}_1+\\
+\{u_{c+v-2}-\mathbf{s}_1.u_{c+v-3}+\mathbf{s}_2.u_{c+v-4}-\mathbf{s}_3.u_{c+v-5}+\ldots+(-1)^c.u_{v-2}\}\boldsymbol{S}_2-\\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
+(-1)^v\{u_c-\mathbf{s}_1.u_{c-1}+\mathbf{s}_2\,u_{c-2}-\mathbf{s}_3.u_{c-3}+\ldots+(-1)^c\,u_0\}\boldsymbol{S}_v=0.
\end{array}$$

$$\begin{array}{l}
\{u_{c+v+1}-\mathbf{s}_1.u_{c+v}+\mathbf{s}_2\,u_{c+v-1}-\mathbf{s}_3.u_{c+v-2}+\ldots+(-1)^c.u_{v+1}\}-\\
-\{u_{c+v}-\mathbf{s}_1.u_{c+v-1}+\mathbf{s}_2\,u_{c+v-2}-\mathbf{s}_3.u_{c+v-3}+\ldots+(-1)^c.u_v\}\boldsymbol{S}_1+\\
+\{u_{c+v-1}-\mathbf{s}_1.u_{c+v-2}+\mathbf{s}_2\,u_{c+v-3}-\mathbf{s}_3.u_{c+v-4}+\ldots+(-1)^c.u_{v-1}\}\boldsymbol{S}_2-\\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
+(-1)^v\{u_{c+1}-\mathbf{s}_1.u_c+\mathbf{s}_2\,u_{c-1}-\mathbf{s}_3.u_{c-2}+\ldots+(-1)^c.u_1\}\boldsymbol{S}_v=0.
\end{array}$$

. . . .

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$$\begin{array}{l}
\{u_{c+2v-1}-\mathbf{s}_1.u_{c+2v-2}+\mathbf{s}_2.u_{c+2v-3}-\mathbf{s}_3.u_{c+2v-4}+\ldots+(-1)^c.u_{2v-1}\}-\\
-\{u_{c+2v-2}-\mathbf{s}_1.u_{c+2v-3}+\mathbf{s}_2.u_{c+2v-4}-\mathbf{s}_3.u_{c+2v-5}+\ldots+(-1)^c.u_{2v-2}\}\boldsymbol{S}_1+\\
+\{u_{c+2v-3}-\mathbf{s}_1.u_{c+2v-4}+\mathbf{s}_2.u_{c+2v-5}-\mathbf{s}_3\,u_{c+2v-6}+\ldots+(-1)^c\,u_{2v-3}\}\boldsymbol{S}_2-\\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
+(-1)^v\{u_{c+v-1}-\mathbf{s}_1.u_{c+v-2}+\mathbf{s}_2u_{c+v-3}-\mathbf{s}_3.u_{c+v-4}+\ldots+(-1)^c.u_{v-1}\}\boldsymbol{S}_v=0.
\end{array}$$

De ce groupe de v équations linéaires, on peut résoudre les v grandeurs $\boldsymbol{S}_1, \boldsymbol{S}_2, \ldots \boldsymbol{S}_v$, qui en sont les seules inconnues. Les v inconnues x_k sont alors les racines de l'équation

$$(x^k)^v - \boldsymbol{S}_1 (x^k)^{v-1} + \boldsymbol{S}_2 (x^k)^{v-2} - \boldsymbol{S}_3 (x^k)^{v-3} + \ldots + (-1)^v \boldsymbol{S}_v = 0.$$

Dans les équations sub (69), toutes les lettres u représentent des nombres arbitraires. Cependant dans chacune des formules d'approximation, qui seront développées dans ce travail, il existe un certain rapport entre ces nombres. Si l'on tient compte de ce rapport, alors, comme il apparaîtra dans le prochain §, la relation des grandeurs $\boldsymbol{S}$ devient beaucoup plus simple.

§ 28. Des v équations

$$\left.\begin{aligned}
&\frac{1}{r}B_0 - \frac{1}{r-\Delta}B_1 + \frac{1}{r-2\Delta}\cdot B_2 - \ldots + (-1)^{t-1}\frac{1}{r-(t-1)\Delta}B_{t-1} + (-1)^t\cdot\frac{1}{r-t\Delta}\cdot B_t + \\
&\qquad + \ldots + (-1)^v\frac{1}{r-v\Delta}B_v = 0, \\
&\frac{1}{r+\Delta}\cdot B_0 - \frac{1}{r}B_1 + \frac{1}{r-\Delta}B_2 - \ldots + (-1)^{t-1}\cdot\frac{1}{r-(t-2)}B_{t-1} + (-1)^t\frac{1}{r-(t-1)\Delta}B_t + \\
&\qquad + \ldots + (-1)^v\frac{1}{r-(v-1)\Delta}B_v = 0, \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&\frac{1}{r+(v-1)\Delta}\cdot B_0 - \frac{1}{r+(v-2)\Delta}B_1 + \frac{1}{r+(v-3)\Delta}B_2 - \ldots + (-1)^{t-1}\frac{1}{r-(t-v)\Delta}B_{t-1} + \\
&\qquad + (-1)^t\frac{1}{r-\{t-(v-1)\}\Delta}B_t + \ldots + (-1)^v\frac{1}{r-\Delta}B_v = 0;
\end{aligned}\right\} \quad (73)$$

dans lesquelles r, Δ et v sont connus et dans lesquelles on peut attribuer à B_0 une valeur arbitraire, résoudre les v grandeurs inconnues $B_1, B_2, B_3, \ldots B_v$.

Solution. On pose $(-1)^t B_t = \beta_t$ et on transfère les termes avec B_0 du premier au second membre, alors on a:

$$\frac{1}{r-\Delta}\cdot\frac{\beta_1}{\beta_0} + \frac{1}{r-2\Delta}\cdot\frac{\beta_2}{\beta_0} + \frac{1}{r-3\Delta}\cdot\frac{\beta_3}{\beta_0} + \ldots + \frac{1}{r-(t-1)\Delta}\cdot\frac{\beta_{t-1}}{\beta_0} + \frac{1}{r-t\Delta}\cdot\frac{\beta_t}{\beta_0} + \ldots +$$
$$+ \frac{1}{r-v\Delta}\cdot\frac{\beta_v}{\beta_0} = -\frac{1}{r},$$

$$\frac{1}{r}\cdot\frac{\beta_1}{\beta_0} + \frac{1}{r-\Delta}\cdot\frac{\beta_2}{\beta_0} + \frac{1}{r-2\Delta}\cdot\frac{\beta_3}{\beta_9} + \ldots + \frac{1}{r-(t-2)\Delta}\cdot\frac{\beta_{t-1}}{\beta_0} + \frac{1}{r-(t-1)\Delta}\cdot\frac{\beta_t}{\beta_0} + \ldots +$$
$$+ \frac{1}{r-(v-1)\Delta}\cdot\frac{\beta_v}{\beta_0} = -\frac{1}{r+\Delta},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\frac{1}{r+(v-2)\Delta}\cdot\frac{\beta_1}{\beta_0}+\frac{1}{r+(v-3)\Delta}\cdot\frac{\beta_2}{\beta_0}+\frac{1}{r+(v-4)\Delta}\cdot\frac{\beta_3}{\beta_0}+\dots+\frac{1}{r-(t-v)\Delta}\cdot\frac{\beta_{t-1}}{\beta_0}+$$
$$+\frac{1}{r-\{t-(v-1)\}\Delta}\cdot\frac{\beta_t}{\beta_0}+\dots+\frac{1}{r-\Delta}\cdot\frac{\beta_v}{\beta_0}=-\frac{1}{r+(v-1)\Delta}.$$

Les fractions $\frac{\beta_p}{\beta_0}$ sont proportionnelles aux déterminants obtenus de la matrice des coefficients en remplaçant la $p^{\text{ième}}$ colonne par la colonne des valeurs connues et les signes négatifs par $+$.

Alors à $\frac{\beta_t}{\beta_0}$ correspond

$$-\begin{vmatrix}
\frac{1}{r-\Delta} & \frac{1}{r-2\Delta} & \dots & \frac{1}{r-(t-1)\Delta} & \frac{1}{r} & \frac{1}{r-(t+1)\Delta} & \frac{1}{r-(t+r)\Delta} & \dots & \frac{1}{r-v\Delta} \\
\frac{1}{r} & \frac{1}{r-\Delta} & \dots & \frac{1}{r-(t-2)\Delta} & \frac{1}{r+\Delta} & \frac{1}{r-t\Delta} & \frac{1}{r-(t+1)\Delta} & \dots & \frac{1}{r-(v-1)\Delta} \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
\frac{1}{r+(v-2)\Delta} & \frac{1}{r+(v-3)\Delta} & \dots & \frac{1}{r-(t-v)\Delta} & \frac{1}{r+(v-1)\Delta} & \frac{1}{r-\{t-(v-2)\}\Delta} & \frac{1}{r-\{t-(v-3)\}\Delta} & \dots & \frac{1}{r-\Delta}
\end{vmatrix}$$

Avec $\frac{\beta_{t+1}}{\beta_0}$:

$$\begin{vmatrix}
\frac{1}{r-\Delta} & \frac{1}{r-2\Delta} & \dots & \frac{1}{r-(t-1)\Delta} & \frac{1}{r-t\Delta} & \frac{1}{r} & \frac{1}{r-(t+2)\Delta} & \dots & \frac{1}{r-v\Delta} \\
\frac{1}{r} & \frac{1}{r-\Delta} & \dots & \frac{1}{r-(t-2)\Delta} & \frac{1}{r-(t-1)\Delta} & \frac{1}{r+\Delta} & \frac{1}{r-(t+1)\Delta} & \dots & \frac{1}{r-(v-1)\Delta} \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
\frac{1}{r+(v-2)\Delta} & \frac{1}{r-(v-3)\Delta} & \dots & \frac{1}{r-(t-v)\Delta} & \frac{1}{r-\{t-(v-1)\}\Delta} & \frac{1}{r+(v-1)\Delta} & \frac{1}{r-\{t-(v-3)\}\Delta} & \frac{1}{r-\{t-(v-4)\}\Delta} \dots & \frac{1}{r-\Delta}
\end{vmatrix}$$

Qu'on soustraie, dans le premier déterminant, la $(t+1)^{\text{ième}}$ colonne et, dans le deuxième déterminant, la $(t-1)^{\text{ième}}$ colonne, de toutes les autres colonnes; les éléments deviennent alors des fractions, celles qui se trouvent dans la même colonne ont toutes les numérateurs égaux, celles qui se trouvent dans la même ligne horizontale ont toutes un facteur commun dans le dénominateur. On peut ainsi, du premier déterminant, isoler le facteur

$$\frac{-(-1)^t.(t+1)!\,(v-t-1)!\,\Delta^{v-1}}{[r-(t+1)\Delta].[r-t\Delta]\dots[r-(t-v+2)\Delta]}$$

et, du deuxième déterminant, le facteur

$$\frac{(-1)^t . t! (v-t)!}{[r-t\Delta].[r-(t-1)\Delta]\ldots[r-(t-v+1)\Delta]}.$$

Dans ces deux cas, le déterminant restant est le même. Le rapport des grandeurs B_t et B_{t+1} est par conséquent égal à celui des facteurs isolés. Après simplification, il reste

$$\frac{B_{t+1}}{B_t} = \frac{v-t}{t+1} \cdot \frac{r-(t+1)\Delta}{r-(t-v+1)\Delta}.$$

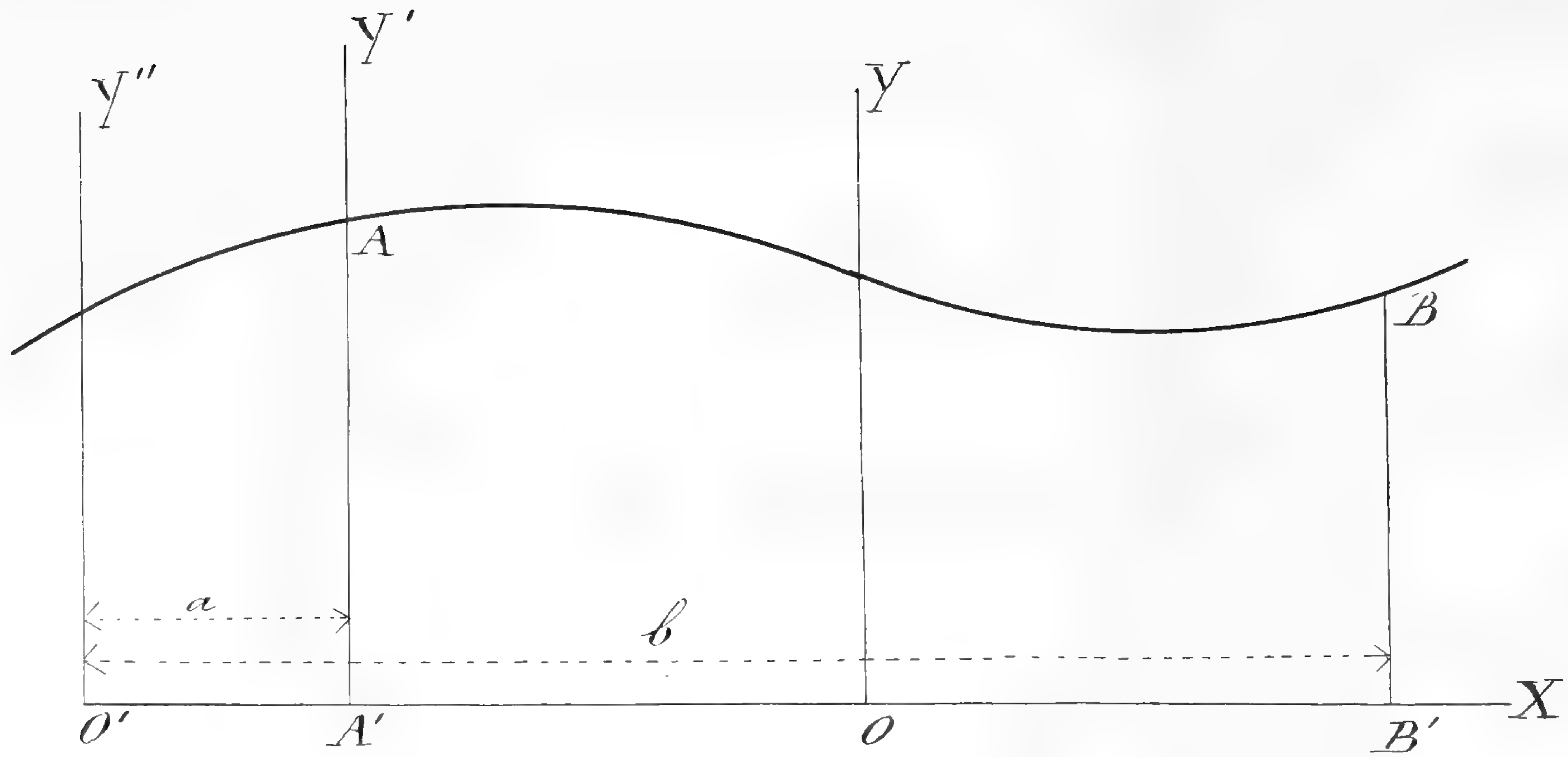
Si dans cette expression on remplace successivement t par 0, 1, 2, 3,...$(v-1)$, alors on obtient

$$\left.\begin{array}{lll}
B_1 & = \dfrac{r-\Delta}{r+(v-1)\Delta} \cdot \dfrac{v}{1} \cdot B_0 & , \\[2ex]
B_2 & = \dfrac{r-2\Delta}{r+(v-2)\Delta} \cdot \dfrac{v-1}{2} \cdot B_1 & , \\[2ex]
B_3 & = \dfrac{r-3\Delta}{r+(v-3)\Delta} \cdot \dfrac{v-2}{3} \cdot B_2 & , \\[2ex]
\multicolumn{3}{l}{\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots} \\[2ex]
B_{v-1} & = \dfrac{r-(v-1)\Delta}{r-v} \cdot \dfrac{2}{v-1} \cdot B_{v-2} & , \\[2ex]
B_v & = \dfrac{r-v\Delta}{r} \cdot \dfrac{1}{v} \cdot B_{v-1} & .
\end{array}\right\} \ldots\ldots (74)$$

TABLE A. Nombre d'ordonnées.		
	$x_1 = 0{,}2886\ 7513\ 4594\ 8129.$	$I = \frac{1}{2}(y_{-1} + y_{+1})$ $E = 0{,}0055\ 5556\ A_4 + 0{,}0016\ 5289\ A_6 + \ldots\ldots$
3.	$x_1 = 0,$ $x_2 = 0{,}3872\ 9833\ 4620\ 7414.$	$I_1 = 0{,}4\ y_1 + 0{,}27\ (y_{-2} + y_{+2}).$ $E = 0{,}0003\ 5714\ A_6 + 0{,}0001\ 5278\ A_8 + \ldots\ldots$
4.	$x_1 = 0{,}1699\ 9052\ 1792\ 4281,$ $x_2 = 0{,}4305\ 6815\ 5797\ 0263.$	$I_1 = 0{,}3260\ 7257\ 7431\ 2731\ (y_{-1} + y_{+1}) + 0{,}1739\ 2742\ 2568\ 7269\ (y_{-2} + y_{+2}).$ $E = 0{,}0000\ 2268\ A_8 + \ldots\ldots$
5.	$x_1 = 0,$ $x_2 = 0{,}2692\ 3465\ 5052\ 8415,$ $x_3 = 0{,}4530\ 8992\ 2969\ 3320.$	$I_1 = 0{,}284\ y_1 + 0{,}2393\ 1433\ 5249\ 6832\ (y_{-2} + y_{+2}) + 0{,}1184\ 6344\ 2528\ 0945\ (y_{-3} + y_{+3}).$ $E = 0{,}0000\ 0143\ 1549\ A_{10} + \ldots\ldots$
6.	$x_1 = 0{,}1193\ 0959\ 3041\ 5985,$ $x_2 = 0{,}3306\ 0469\ 3233\ 1323,$ $x_3 = 0{,}4662\ 3475\ 7101\ 5760.$	$I_1 = 0{,}2339\ 5696\ 7286\ 3455\ (y_{-1} + y_{+1}) + 0{,}1803\ 8078\ 6524\ 0693\ (y_{-2} + y_{+2}) +$ $+ 0{,}0856\ 6224\ 6189\ 5852\ (y_{-3} + y_{+3}).$ $E = 0.0000\ 0009\ 0097\ A_{12} + \ldots\ldots$
7.	$x_1 = 0,$ $x_2 = 0{,}2029\ 2257\ 5688\ 6986,$ $x_3 = 0{,}3707\ 6559\ 2799\ 6972,$ $x_4 = 0{,}4745\ 5395\ 6171\ 3793.$	$I_1 = 0{,}2089\ 7959\ 1836\ 7347\ y_1 + 0.1909\ 1502\ 5252\ 5595\ (y_{-2} + y_{+2}) +$ $+ 0{,}1398\ 5269\ 5744\ 6383\ (y_{-3} + y_{+3}) + 0{,}0647\ 4248\ 3084\ 4348\ (y_{-4} + y_{+4}).$ $E = 0.0000\ 0000\ 5660\ A_{14} + \ldots\ldots$
8.	$x_1 = 0{,}0917\ 1732\ 1247\ 8249,$ $x_2 = 0{,}2627\ 6620\ 4958\ 1645,$ $x_3 = 0{,}3983\ 3323\ 8706\ 8134,$ $x_4 = 0{,}4801\ 4492\ 8248\ 7681.$	$I_1 = 0.1813\ 4189\ 1689\ 1810\ (y_{-1} + y_{+1}) + 0.1568\ 5332\ 2938\ 9436\ (y_{-2} + y_{+2}) +$ $+ 0{,}1111\ 9051\ 7226\ 6872\ (y_{-3} + y_{+3}) + 0{,}0506\ 1426\ 8145\ 1881\ (y_{-4} + y_{+4}).$ $E = 0{,}0000\ 0000\ 0355\ A_{16} + \ldots\ldots$
9.	$x\ \ = 0,$ $x_2 = 0{,}1621\ 2671\ 1701\ 9045,$ $x_3 = 0{,}3066\ 8571\ 6350\ 2952,$ $x_4 = 0{,}4180\ 1555\ 3663\ 3179,$ $x_5 = 0{,}4840\ 8011\ 9753\ 8131.$	$I_1 = 0.1651\ 1967\ 7500\ 6324\ y_1 + 0.1561\ 7353\ 8520\ 0001\ (y_{-2} + y_{+2}) +$ $+ 0{,}1303\ 0534\ 8201\ 4678\ (y_{-3} + y_{+3}) + 0{,}0903\ 2408\ 0347\ 4287\ (y_{-4} + y_{+4}) +$ $+ 0.0406\ 3719\ 4180\ 7872\ (y_{-5} + y_{+5}).$ $E = 0{,}0000\ 0000\ 0022\ A_{18} + \ldots\ldots$
10.	$x_1 = 0{,}0744\ 3716\ 9490\ 8156,$ $x_2 = 0{,}2166\ 9769\ 7064\ 6236,$ $x_3 = 0{,}3397\ 0478\ 4149\ 5122,$ $x_4 = 0{,}4325\ 3168\ 3344\ 4923,$ $x_5 = 0{,}4869\ 5326\ 4258\ 5859.$	$I_1 = 0.1477\ 6211\ 2357\ 3764\ (y_{-1} + y_{+1}) + 0{,}1346\ 3335\ 9654\ 9982\ (y_{-2} + y_{+2}) +$ $+ 0{,}1095\ 4318\ 1257\ 9910\ (y_{-3} + y_{+3}) + 0{,}0747\ 2567\ 4575\ 2903\ (y_{-4} + y_{+4}) +$ $+ 0.0333\ 3567\ 2154\ 3441\ (y_{-5} + y_{+5}).$ $E = 0{,}0000\ 0000\ 0001\ 3950\ A_{20} + \ldots\ldots$

TABLE B.

Nombre d'ordonnées à mesurer.		
3.	$x_1 = 0,$ $x_2 = 0{,}39.$	$I_1 = 0{,}4521\;1483\;6730\;2213\,y_1 + 0{,}2739\;4258\;1634\;8893\,(y_{-2} + y_{+2}).$ $E = -0{,}0001\;1500\,A_4 + 0{,}0003\;0427\,A_6 + 0{,}0001\;4080\,A_8 + \dots\dots$
5.	$x_1 = 0,\ x_2 = 0{,}39,$ $x_3 = 0.01$	corr: $= -11{,}5055\;8842\;8665\,y_1 - 0{,}0037\;8473\;3036\,(y_{-2} + y_{+2}) + 5{,}7565\;7894\;7368\,(y_{-3} + y_{+3}).$
5.	$x_1 = 0,\ x_2 = 0{,}26,$ $x_3 = 0.45$	$I_1 = 0{,}2688\;7768\;7681\;1064\;8453\,y_1 + 0{,}2398\;7744\;5927\;5115\;0315\,(y_{-2} + y_{+2}) + 0{,}1256\;8371\;0231\;9352\;5459\,(y_{-3} + y_{+3}).$ $E = -0{,}0000\;0335\;7143\,A_6 + 0{,}0000\;0133\;1728\,A_8 + 0{,}0000\;0250\;8965\,A_{10} + 0{,}0000\;0140\;1848\,A_{12} + 0{,}0000\;0055\,A_{14} + \dots$
7.	$x_1 = 0,\ x_2 = 0.26,$ $x_3 = 0{,}45,\ x_4 = 0.5.$	corr: $= 0{,}0009\;8087\;5340\,y_1 - 0.0010\;0915\;1447\,(y_{-2} + y_{+2}) + 0{,}0012\;9362\;7501\,(y_{-3} + y_{+3}) - 0{,}0007\;7496\;3725\,(y_{-4} + y_{-4}).$
7.	$x_1 = 0,\ x_2 = 0{,}20,$ $x_3 = 0.36,\ x_4 = 0.47.$	$I_1 = 0{,}2097\;5447\;0980\;6416\;9640\;0461\,y_1 + 0{,}1824\;7432\;9387\;6422\;8742\;9068\,(y_{-2} + y_{+2}) +$ $+ 0{,}1380\;7607\;6969\;2986\;6560\;7711\,(y_{-3} + y_{+3}) + 0{,}0745\;7235\;8152\;7381\;9876\;2990\,(y_{-4} + y_{+4}).$ $E = 0{,}0000\;0005\;5192\,A_8 + 0{,}0000\;0019\;5652\,A_{10} + 0{,}0000\;0014\;0693\,A_{12} + 0{,}0000\;0007\;1315\,A_{14} + 0{,}0000\;0003\,A_{16} + \dots\dots$
9.	$x_1 = 0,\ x_2 = 0{,}14,$ $x_3 = 0{,}31,\ x_4 = 0{,}42,$ $x_5 = 0{,}48.$	$I_1 = 0.1108\;0125\;4553\;8909\;4319\;1036\;1769\,y_1 +$ $+ 0{,}1744\;8969\;5686\;6362\;9922\;8778\;4231\,(y_{-2} + y_{+2}) + 0{,}1485\;6536\;6367\;4303\;3946\;3273\;5653\,(y_{-3} + y_{+3}) +$ $+ 0{,}0747\;0874\;5459\;9309\;2982\;1885\;8212\,(y_{-4} + y_{+4}) + 0{,}0468\;3556\;5209\;0569\;5989\;0544\;1020\,(y_{-5} + y_{+5}).$ $E = 0{,}0000\;0000\;5188\,A_{10} + 0{,}0000\;0003\;2005\,A_{12} + 0{,}0000\;0002\;5372\,A_{14} + \dots\dots$
11.	$x_1 = 0,\ x_2 = 0.14,$ $x_3 = 0{,}25,\ x_4 = 0.37,$ $x_5 = 0.44,\ x_6 = 0.49.$	$I_1 = 0{,}0828\;1895\;6609\;8793\;2657\;4313\;4813\;8734\;3572\,y_1 +$ $+ 0.2074\;9069\;2760\;1711\;3256\;6405\;2835\;9730\;1815\,(y_{-2} + y_{+2}) +$ $+ 0.2624\;6970\;4997\;3907\;5235\;9300\;5662\;0412\;8537\,(y_{-3} + y_{+3}) +$ $+ 0.1829\;1472\;5915\;8427\;5919\;8224\;1585\;0777\;1593\,(y_{-4} + y_{+4}) +$ $+ 0.1237\;3949\;7353\;4044\;9347\;3558\;1538\;7965\;5414\,(y_{-5} + y_{+5}) +$ $+ 0.0577\;4746\;5753\;4322\;0925\;3884\;8750\;3645\;5497\,(y_{-6} + y_{+6}).$ $E = 0{,}0000\;0098\;1552\,A_{12} + \dots\dots$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